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# Optimal Coordinated Resource Allocation in *Ad Hoc* Network Systems: A Sequential Two-Stage Approach

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***Abstract*— Motivated by various problems such as distributed computation and multiagent coordination, an optimal coordinated resource allocation problem under dynamically changing environment has been solved by means of a sequential, two-stage, optimal semistable control approach. Technically we formulate this resource allocation problem into a linear, time-varying quadratic semistabilization problem with topologically changing, distributed iterative algorithms for resource allocation in peer-to-peer networks. To solve this problem, we propose a novel, sequential two-stage design. The first stage is to guarantee the convergence of the optimal policy while the second stage is to derive the explicit recursive formulas for optimal strategies under a finite set of convergence-guaranteed candidate policies.**

## I. Introduction

Motivated by various problems such as distributed computation in computer science [1], multiagent coordination in control and automation [2], mobile sensing and detection in defense industry [3], and resource allocation in the military [4], we consider an optimal coordinated resource allocation problem in *ad hoc* or peer-to-peer network systems. More specifically, the objective here is to develop some optimal strategies for a simplified optimal coordinated resource allocation problem in [5].

Mathematically, this problem can be formulated into *optimal semistable control* [6]–[9]–an optimal search algorithm in a topologically graph-related network so that available resources in different locations of the network can be relocated via some linear, iterative laws to optimize certain cost criterion. The cost functional here is defined to balance both fast convergence of the state and relocating control effort at each location, that is, we want to achieve the fastest convergence of the resource allocation process by using the least effort we need. The time-invariant optimal semistable control problem has been solved in [6]–[9]. However, optimal semistable control over network topologies undergoing switching or time-dependent communication links still remains an open problem. One may argue that this problem looks quite similar to the classical LQR problem. However, we argue that there are some big differences between the proposed optimal control problem and LQR that we shall address as follows.

First, we point out that unlike the classical LQR problem, the steady-state value of the optimal policy in our problem is a *Nash-type equilibrium* [10], which implies that the equilibrium status depends on both initial conditions and the choice of the optimal policy, and hence, is not fixed and is unknown *a priori*. Consequently, neither can one shift the equilibrium state to turn in into a classical LQR problem, nor can formulate our problem into a nonzero set point regulation or a signal tracking problem. Moreover, the overall optimal task cannot be directly decomposed into several optimal subtasks by use of dynamic programming since these subtasks are generally coupled with each other due to this nondeterministic equilibrium state.

Secondly, since we consider an optimal policy under a dynamically changing environment, that is, the network graph topology is not necessarily fixed, the whole problem becomes time-dependent, which leads to a time-dependent optimal control law. In such a scenario, generally there does not exist a time-independent matrix transformation to convert the problem into quotation space to discuss its solution, which however, is quite common for the time-invariant case [11], [12].

Thus, all the existing methods cannot be applied to our proposed problem. In this paper, we propose a new, sequential two-stage approach to obtain a set of optimal solutions for the proposed optimization problem by decoupling the intrinsic link between Nash-type equilibria and designed update laws. More specifically, in the first stage we guarantee the convergence of the state to its equilibrium by restricting the designed update law into a set of candidate update laws after certain time instant. This restriction is natural since network resource is always limited for resource allocation. Then in the next stage we solve the optimization problem by using the equilibrium obtained in the first stage and a sequential optimization technique.

## II. Problem Formulation

In this section, first we introduce the notation we use in this paper. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n$-dimensional real vectors, $\mathbb{R}^{m\times n}$ denotes the set of $m$ by $n$ real matrices, $\overline{\mathbb{Z}}_+$ denotes the set of nonnegative integers, $\mathbb{Z}_+$ denotes the set of positive integers, and $\|\cdot\|$ denotes the Euclidean vector norm or the induced 2-norm of a matrix. Furthermore, for $A \in \mathbb{R}^{m\times n}$, $A_{(i,j)}$ denotes the $(i,j)$th entry of $A$, $\mathcal{R}(A)$ denotes the range space of $A$, $\mathcal{N}(A)$ denotes the null space of $A$, and $\mathrm{rank}(A)$ denotes the rank of $A$. Finally, for $A \in \mathbb{R}^{n\times n}$, $\mathrm{spec}(A)$ denotes the spectrum of $A$, $\rho(A)$ denotes the spectrum radius of $A$, and $A^{\#}$ denotes the group inverse of $A$[1].

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[1]For $A \in \mathbb{R}^{n\times n}$, the *group inverse* is a matrix $X \in \mathbb{R}^{n\times n}$ satisfying $XAX = X$, $AX = XA$, and $AXA = A$.

An *ad hoc* network is characterized by a strongly connected digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of the set of nodes $\mathcal{V} = \{1, \ldots, q\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where each edge $(i, j) \in \mathcal{E}$ is an ordered pair of distinct nodes. The set of neighbors of node $i$ is denoted by $\mathbb{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Finally, we denote the value of the node $i \in \{1, \ldots, q\}$ at time $t$ by $x_i(t) \in \mathbb{R}$.

In this paper, we consider distributed linear iterations given by the form

$$x_i(t+1) = W_{(i,i)}(t+1)x_i(t) + \sum_{j \in \mathbb{N}_i} W_{(i,j)}(t+1)x_j(t),$$
$$x_i(0) = x_{i0}, \quad i = 1, \ldots, q, \quad t \in \overline{\mathbb{Z}}_+, \tag{1}$$

where $W_{(i,j)} : \mathbb{Z}_+ \to \mathbb{R}$ denotes the weight on $x_j$ at node $i$. Letting $W_{(i,j)}(\cdot) = 0$ for $j \neq \mathbb{N}_i$, this iteration can be rewritten as a compact form

$$x(t+1) = W(t+1)x(t), \quad x(0) = x_0, \quad t \in \overline{\mathbb{Z}}_+, \tag{2}$$

where $x(t) = [x_1(t), \ldots, x_q(t)]^{\mathrm{T}} \in \mathbb{R}^q$. The constraint on the matrix $W : \mathbb{Z}_+ \to \mathbb{R}^{q \times q}$ can be expressed as $W(\cdot) \in \mathcal{W}$, where $\mathcal{W} = \{W \in \mathbb{R}^{q \times q} : W_{(i,j)} = 0 \text{ if } (i, j) \notin \mathcal{E} \text{ and } i \neq j\}$. In this paper, we relax this restriction on $\mathcal{W}$ by considering all possible topological graphs, not necessarily some predetermined connected ones.

The aim of this paper is to *design* the weight matrix $W(\cdot)$ so that for any initial value $x_0$, $\lim_{t\to\infty} x(t) = x_{\mathrm{e}}$, $x_{\mathrm{e}} \neq 0$, $x_{\mathrm{e}} \neq x_0$, and the cost functional

$$\begin{aligned} J_N \;\; = \;\; & \sum_{t=0}^{N} [(x(t) - x_{\mathrm{e}})^{\mathrm{T}} Q(t)(x(t) - x_{\mathrm{e}}) \\ & + (x(t+1) - x_{\mathrm{e}})^{\mathrm{T}} R(t+1)(x(t+1) - x_{\mathrm{e}})] \end{aligned} \tag{3}$$

is minimized, where $N \geq 1$ is a given positive integer, $Q(t) = Q^{\mathrm{T}}(t) > 0$, and $R(t+1) = R^{\mathrm{T}}(t+1) > 0$, $t \in \{0, \ldots, N\}$. Here $x_{\mathrm{e}}$ can be viewed as a generalized *Nash equilibrium* [10].

The constant matrix case where $W(t+1) \equiv W$, $Q(t) \equiv Q$, and $R(t+1) \equiv R$ has been solved in [6], [7]. In this case, the property that $x(t) = W^t x(0)$ has been heavily used to simplify the form of the cost functional. Unfortunately, this property is not true for the time-varying case. Hence, it is quite challenging to solve the time-varying matrix case given by (2) and (3). In this paper, we propose a sequential two-stage approach to obtain a set of optimal solutions for this optimization problem by decoupling the intrinsic link between generalized Nash equilibria and $W(\cdot)$. More specifically, in the first stage we guarantee the convergence constraint $\lim_{t\to\infty} x(t) = x_{\mathrm{e}}$ by restricting $W(\cdot)$ into a set of special matrices after the time instant $t \geq N+1$. This restriction is natural since network resource is always limited for resource allocation. Then in the next stage we solve the minimization of $J_N$ over the finite horizon $\{0, 1, \ldots, N\}$ in a sequential way. The key difficulty here is that $x_{\mathrm{e}}$ is not a fixed point, but a function of $W(\cdot)$, depending on the choices of the sequence $\{W(t)\}_{t=1}^{\infty}$ and $x(0)$.

It is important to note that we do not assume $W(\cdot)$ is a *stochastic* or *nonnegative matrix* [13]. Hence, many prevalent methods based on nonnegative matrix theory cannot be used to solve our problem. Here we use the recently developed semistability theory for linear time-invariant systems [14] and switched linear systems [15] to obtain the convergence result for $W(\cdot)$ in the first stage.

## III. The First Stage: Convergence

In this section, we first introduce the notion of *discrete-time semistability* [14], [16] which is needed in the paper.

*Definition 3.1:* A matrix $A \in \mathbb{R}^{q \times q}$ is called *discrete-time semistable* if $\operatorname{spec}(A) \subseteq \{s \in \mathbb{C} : |s| < 1\} \cup \{1\}$, and if $1 \in \operatorname{spec}(A)$, then 1 is semisimple. A matrix $A \in \mathbb{R}^{q \times q}$ is called *nontrivially discrete-time semistable* if $A$ is semistable and $A \neq I_q$.

Since a nontrivially discrete-time matrix is a discrete-time matrix, all the properties for discrete-time semistable matrices hold for nontrivially discrete-time matrices. Note that the notion of semistable matrices is much weaker than the notion of stochastic matrices [1], [2], [17] in the literature since no nonnegativity constraint is required for the elements of a semistable matrix. There are a lot of useful properties of discrete-time semistable matrices [14], [16]. In particular, we have the following two results.

*Lemma 3.1 ([14]):* If $W \in \mathbb{R}^{q \times q}$ is discrete-time semistable, then $I_q - W$ is group invertible, that is, $(I_q - W)^{\#}$ exists. Alternatively, $W$ is discrete-time semistable if and only if $\lim_{k\to\infty} W^k$ exists. Furthermore, if $W$ is discrete-time semistable, then $\lim_{k\to\infty} W^k = I_q - (I_q - W)(I_q - W)^{\#}$.

*Lemma 3.2 ([14]):* If $W \in \mathbb{R}^{q \times q}$ is discrete-time semistable, then $\mathcal{N}(I_q - W) = \mathcal{R}(I_q - (I_q - W)(I_q - W)^{\#})$ and $\mathcal{R}(I_q - W) = \mathcal{N}(I_q - (I_q - W)(I_q - W)^{\#})$.

The next result gives a necessary and sufficient condition to characterize discrete-time semistable matrices using eigenvectors.

*Lemma 3.3:* Let $W \neq I_q$. Then $W \in \mathbb{R}^{q \times q}$ is discrete-time semistable if and only if for any $x \in \mathbb{R}^q$, $Wx \neq x$ is equivalent to $\|Wx\| < \|x\|$.

Alternatively, we have additional necessary and sufficient conditions to guarantee a matrix to be discrete-time semistable. To state this result, first, we need the following technical lemma.

*Lemma 3.4:* Assume $A, B \in \mathbb{R}^{q \times q}$ are both idempotent matrices. If $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(B)$, then $A = B$.

Now we present a necessary and sufficient condition in terms of null and range spaces for a discrete-time semistable matrix.

*Proposition 3.1:* For any $x \in \mathbb{R}^q$, there exists a unit vector $\alpha \in \mathbb{R}^q$ such that $\lim_{t\to\infty} W^t x = \alpha\alpha^{\mathrm{T}} x$ if and only if $W$ is discrete-time semistable, $\mathcal{R}(W - I_q) \subseteq \mathcal{N}(\alpha\alpha^{\mathrm{T}})$, and $\mathcal{N}(W - I_q) \subseteq \mathcal{R}(\alpha\alpha^{\mathrm{T}})$.

Proposition 3.1 states a relationship between the limiting state of (2), that is, $\lim_{t\to\infty} x(t) = x_{\mathrm{e}}$ and the constant discrete-time semistable matrix $W(t) = W$. This result can be used to design the desired steady-state pattern $\alpha\alpha^{\mathrm{T}}$ for the constant iteration $x(t+1) = Wx(t)$ by focusing on the null and range spaces of $W - I_q$. Note that if $\alpha = (1/\sqrt{q})[1, \ldots, 1]^{\mathrm{T}}$, then Proposition 3.1 becomes agreement algorithms [1] or linear averaging [18] in distributed systems and consensus [2] in multiagent coordination.

Let $M_1, \ldots, M_n$ be finite, nontrivially discrete-time semistable matrices that are nonzero and mutually different and let $\mathcal{M} = \{M_1, \ldots, M_n\}$. From now on, we make the following standing assumption as our design requirement in the paper.

*Assumption 3.1:* There exist $T \in \mathbb{Z}_+$ and $K \in \mathbb{Z}_+$ such that $\prod_{k=1}^{K} W(T + nK - k) \in \mathcal{M}$ for every $n \in \mathbb{Z}_+$.

Assumption 3.1 is related to the notion of *joint spectral radius* [19]. Specifically, Assumption 3.1 assumes that the joint spectral radius of $\prod_{k=1}^{K} W(T+nK-k)$ has an upper bound less than or equal to 1 for every $n \geq 1$. Note that $T$ is not specified here. In fact, $T$ needs to be designed in this paper. By introducing this $T$, we relax the restriction on $W$ so that each $W$ need not be a nontrivially discrete-time semistable matrix at every instant of time to guarantee the convergence of (2) as in [20], but the combination of series of matrices over a time range needs be nontrivially discrete-time semistable. Such a weaker assumption is widely adopted in distributed systems [1], [21] as well as multiagent coordination [2].

Let $\mathcal{I}$ denote the index set of nontrivially discrete-time semistable matrices $M_i$ that appear infinitely often in the sequence $\{\prod_{k=1}^{K} W(T+nK-k)\}_{n=1}^{\infty}$. Define $L_i = I_q - (I_q - M_i)(I_q - M_i)^{\#}$ for every $i = 1, \ldots, n$. Clearly, $L_i^2 = L_i$, that is, $L_i$ is a projection for every $i = 1, \ldots, n$. Assumption 3.1 itself is not enough to guarantee the convergence of (2). The following assumption provides additional requirements on the null space and boundedness of each $W(t)$, $t \geq T$.

*Assumption 3.2:* $\bigcap_{i \in \mathcal{I}} \mathcal{N}(I_q - M_i) \subseteq \mathcal{N}(W(t))$ and $W(t)$ is bounded for all $t \geq T$.

The following result is the main convergence result for (2) with weak conclusions.

*Theorem 3.1:* Consider (2). Assume that Assumptions 3.1 and 3.2 hold. Then for any $x_0 \in \mathbb{R}^q$, $\lim_{t \to \infty} x(t) = x_\infty \in \bigcap_{i \in \mathcal{I}} \mathcal{N}(I_q - M_i) = \bigcap_{i \in \mathcal{I}} \mathcal{R}(L_i)$.

Theorem 3.1 is a general but weak result about the convergence of (2) which is not easy to be used in our problem. This is due to the fact that there is no strong relationship between $x_\infty$ and $M_i$ although $x_\infty \in \bigcap_{i \in \mathcal{I}} \mathcal{N}(I_q - M_i)$ gives a soft connection. To explore a stronger result, let $M = \bigcap_{i=1}^{n} \mathcal{N}(I_q - M_i)$, $M^c = \mathrm{span}\{\mathcal{R}(I_q - M_i) : 1 \leq i \leq n\}$, and $P_{M,M^c}$ be the projection on $M$ along $M^c$. The following recurrence assumption is used to further characterize the structure of $\bigcap_{i \in \mathcal{I}} \mathcal{R}(L_i)$ in Theorem 3.1.

*Assumption 3.3:* Each matrix $M_i$ in $\mathcal{M}$ appears infinitely often in the sequence $\{\prod_{k=1}^{K} W(T + nK - k)\}_{n=1}^{\infty}$.

Under this assumption, $\bigcap_{i \in \mathcal{I}} \mathcal{R}(L_i)$ can be simplified into a projection.

*Theorem 3.2:* Consider (2). Assume that Assumptions 3.1–3.3 hold. Then $x_\infty = P_{M,M^c} x(T - 1) = P_{M,M^c}[\prod_{k=1}^{T-1} W(T-k)]x_0$.

For the case where each $W(t)$, $t \geq T$, is a constant nontrivially discrete-time semistable matrix, we end up with the known result.

*Corollary 3.1:* Consider (2). Assume that Assumption 3.1 holds with $n = 1$ and $K = 1$. Then $x_\infty = L_1 x(T-1) = L_1[\prod_{k=1}^{T-1} W(T-k)]x_0$.

To explicitly derive an optimal matrix sequence $\{W(t)\}_{t=1}^{\infty}$, a clear relationship between $x_\infty$ and $W(t)$ is needed. However, $P_{M,M^c}$ in Theorem 3.2 is still not very convenient due to its implicit connection with $M$. Next, we introduce the following assumption to overcome this difficulty.

*Assumption 3.4:* There exists $j \in \{1, \ldots, n\}$ such that $M = \mathcal{N}(I_q - M_j)$ and $\mathcal{N}(I_q - M_j) \cap \mathcal{R}(I_q - M_j) = \{0\}$.

Using Assumption 3.4, one can alleviate the complexity of $P_{M,M^c}$ in Theorem 3.2 by a much simpler form, which will be given in Theorem 3.3. At the same time, this simplified problem is still quite challenging without losing its physical and technical significance. Before proceeding, next, we give three sufficient conditions to guarantee that $\mathcal{N}(I_q - M_j) \cap \mathcal{R}(I_q - M_j) = \{0\}$ so that Assumption 3.4 is technically feasible. To state these results, however, we need the following definition first.

*Definition 3.2 ([6], [9], [14]):* Let $A \in \mathbb{R}^{q \times q}$ and $C \in \mathbb{R}^{l \times q}$. The pair $(A, C)$ is *semiobservable* if $\bigcap_{i=1}^{q} \mathcal{N}\left(C(I_q - A)^{i-1}\right) = \mathcal{N}(I_q - A)$. The pair $(A, C)$ is *weakly semiobservable* if $\bigcap_{i=1}^{q} \mathcal{N}\left(C(I_q - A)^{i}\right) = \mathcal{N}(I_q - A)$.

Semiobservability and weak semiobservability are extensions of observability and detectability. In particular, semiobservability is an extension of zero-state observability to *equilibrium observability*, whereas weak semiobservability is an extension of detectability to *equilibrium detectability*. The following two results give sufficient conditions to guarantee $\mathcal{N}(I_q - M_j) \cap \mathcal{R}(I_q - M_j) = \{0\}$.

*Lemma 3.5 ([14]):* Let $A \in \mathbb{R}^{q \times q}$. If there exist a $q \times q$ matrix $P \geq 0$ and an $l \times q$ matrix $C$ such that $(A, C)$ is semiobservable and $P = A^{\mathrm{T}} P A + R$, where $R \triangleq C^{\mathrm{T}} C$, then $\mathcal{N}(I_q - A) \cap \mathcal{R}(I_q - A) = \{0\}$.

*Lemma 3.6 ([9]):* Let $A \in \mathbb{R}^{q \times q}$. If there exist a $q \times q$ matrix $P \geq 0$ and an $l \times q$ matrix $C$ such that $(A, C)$ is weakly semiobservable and $(I_q - A)^{\mathrm{T}} P (I_q - A) = (I_q - A)^{\mathrm{T}} (A^{\mathrm{T}} P A + R)(I_q - A)$, where $R \triangleq C^{\mathrm{T}} C$, then $\mathcal{N}(I_q - A) \cap \mathcal{R}(I_q - A) = \{0\}$.

*Theorem 3.3:* Consider (2). Assume that Assumptions 3.1–3.4 hold. Then $x_\infty = L_j x(T-1) = L_j[\prod_{k=1}^{T-1} W(T-k)]x_0$.

To illustrate Theorem 3.3, we provide the following example to end the discussion on convergence part of the design.

*Example 3.1:* Consider (2) with

$$W(t+1) = \frac{1}{8} \begin{bmatrix} 6 + (-1)^t & 2 - (-1)^t \\ 2 - (-1)^t & 6 + (-1)^t \end{bmatrix}. \tag{4}$$

Clearly, $W(t+1) \in \{M_1, M_2\}$ for all $t \in \overline{\mathbb{Z}}_+$, where

$$M_1 = \frac{1}{8} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}, \quad M_2 = \frac{1}{8} \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}. \tag{5}$$

Moreover, $M_1$ and $M_2$ are nontrivially discrete-time semistable. Note that $\mathcal{N}(I_2 - M_i) = \{\alpha[1, 1]^{\mathrm{T}} : \alpha \in \mathbb{R}\}$

and $\mathcal{N}(I_2 - M_i) \cap \mathcal{R}(I_2 - M_i) = \{0\}$ for every $i = 1, 2$. Hence, Assumptions 3.1–3.4 hold with $K = T = 1$. Now it follows from Theorem 3.3 that $\lim_{\infty} x(t) = L_i x(0)$ for every $i = 1, 2$, where $L_1 = L_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

## IV. The Second Stage: Optimality

In this section, we derive explicit necessary conditions for optimality. The most famous optimality technique is the Bellman principle or dynamic programming [22]. In this method, the minimization of $J_N$ can be viewed as an $N+1$-step decision process in which the $N+1$ decisions $W(1), W(2), \ldots, W(N+1)$ are to be made such that the given quadratic cost functional (3) is minimized. Rather than attempting to make the $N+1$ decisions simultaneously, it would be desirable to develop a procedure for making the decisions one at a time, that is, to reduce the $N+1$ step problem to $N+1$ one-step problems [23], [24].

However, this idea is generally not valid in this problem due to the fact that these $N+1$ one-step problems are coupled with each other. For example, at each step, $x_\infty$ is not a fixed point, but a function of the sequence $\{W(t)\}_{t=1}^{\infty}$ and the initial value $x(0)$. Hence, one cannot simply treat the steady-state value as constant and use the dynamic programming to obtain the optimal solution.

Our idea here is to weaken this coupling by taking $x_\infty$ to be a function of $\{W(t)\}_{t=T}^{\infty}$, where $T > N$. In this case, we can take the $N+1$ decisions sequentially. Here we consider a simplified two-substage optimal policy process, that is, $T = N+1$ and $T = N+2$. The case where $T \geq N+3$ can be discussed in a similar way as $T = N+1$ and $T = N+2$. Hence, without loss of generality, we assume $T = N+1$ or $T = N+2$.

### A. Single-Step

We begin by assuming that Assumptions 3.1–3.4 hold. Define

$$\begin{aligned} V_1 \quad = \quad & \min_{W(N+1)} [(W(N+1)x(N) - x_\infty)^{\mathrm{T}} \\ & R(N+1)(W(N+1)x(N) - x_\infty) \\ & +(x(N) - x_\infty)^{\mathrm{T}} Q(N)(x(N) - x_\infty)]. \end{aligned} \tag{6}$$

Next, we consider two choices on $T$, that is, $T = N+1$ and $T = N+2$.

**Case 1.** $T = N+1$.

Under such a circumstance, it follows from Theorem 3.3 that $x_\infty = L_j x(N)$ for some $j \in \{1, \ldots, n\}$. In this case,

$$\begin{aligned} V_1 = & \min_{W(N+1)} [x^{\mathrm{T}}(N)(W(N+1) - L_j)^{\mathrm{T}} R(N+1) \\ & \times (W(N+1) - L_j)x(N) + x^{\mathrm{T}}(N)(I_q - L_j)^{\mathrm{T}} \\ & \times Q(N)(I_q - L_j)x(N)]. \end{aligned} \tag{7}$$

Hence, if we drop the time index, then $V_1$ can be written as

$$\begin{aligned} V_1 &= \min_W [x^{\mathrm{T}}(W - L_j)^{\mathrm{T}} R(W - L_j)x \\ & \quad + x^{\mathrm{T}}(I_q - L_j)^{\mathrm{T}} Q(I_q - L_j)x] \\ &= x^{\mathrm{T}}(I_q - L_j)^{\mathrm{T}} Q(I_q - L_j)x. \end{aligned} \tag{8}$$

Thus,

$$W_{\min}(N+1) = L_j. \tag{9}$$

**Case 2.** $T = N+2$.

For the case where $T = N+2$, it follows from Theorem 3.3 that $x_\infty = L_j x(N+1) = L_j W(N+1)x(N)$ for some $j \in \{1, \ldots, n\}$. In this case,

$$\begin{aligned} V_1 = & \min_{W(N+1)} [x^{\mathrm{T}}(N)W^{\mathrm{T}}(N+1)(I_q - L_j)^{\mathrm{T}} \\ & R(N+1)(I_q - L_j)W(N+1)x(N) \\ & + x^{\mathrm{T}}(N)(I_q - L_j W(N+1))^{\mathrm{T}} Q(N) \\ & (I_q - L_j W(N+1))x(N)]. \end{aligned} \tag{10}$$

Hence, if we drop the time index, then $V_1$ can be written as

$$\begin{aligned} V_1 = & \min_W [x^{\mathrm{T}} W^{\mathrm{T}}(I_q - L_j)^{\mathrm{T}} R(I_q - L_j)Wx \\ & + x^{\mathrm{T}}(I_q - L_j W)^{\mathrm{T}} Q(I_q - L_j W)x]. \end{aligned} \tag{11}$$

*Lemma 4.1:* Consider $V_1$ given by (11). Then $(I_q - L_j)^{\mathrm{T}} R(I_q - L_j) + L_j^{\mathrm{T}} Q L_j > 0$ and

$$\begin{aligned} W_{\min} &= \arg V_1 \\ &= [(I_q - L_j)^{\mathrm{T}} R(I_q - L_j) + L_j^{\mathrm{T}} Q L_j]^{-1} L_j^{\mathrm{T}} Q. \end{aligned}$$

Thus,

$$\begin{aligned} W_{\min}(N+1) \quad = \quad & [(I_q - L_j)^{\mathrm{T}} R(N+1)(I_q - L_j) \\ & + L_j^{\mathrm{T}} Q(N) L_j]^{-1} L_j^{\mathrm{T}} Q(N). \end{aligned} \tag{12}$$

Note that in both cases, $V_1$ can be rewritten as

$$\begin{aligned} V_1 \quad = \quad & x^{\mathrm{T}}(N) W_{\min}^{\mathrm{T}}(N+1)(I_q - L_j)^{\mathrm{T}} \\ & R(N+1)(I_q - L_j) W_{\min}(N+1) x(N) \\ & + x^{\mathrm{T}}(N)(I_q - L_j W_{\min}(N+1))^{\mathrm{T}} Q(N) \\ & (I_q - L_j W_{\min}(N+1)) x(N). \end{aligned} \tag{13}$$

Defining

$$L(N+1) = L_j, \tag{14}$$

$$\begin{aligned} S(N+1) = & (I_q - L(N+1))^{\mathrm{T}} R(N+1) \\ & (I_q - L(N+1)), \end{aligned} \tag{15}$$

$$U(N+1) = 0, \tag{16}$$

$$\begin{aligned} U(N) = & W_{\min}^{\mathrm{T}}(N+1) S(N+1) W_{\min}(N+1) \\ & + (I_q - L(N+1) W_{\min}(N+1))^{\mathrm{T}} Q(N) \\ & (I_q - L(N+1) W_{\min}(N+1)), \end{aligned} \tag{17}$$

we have

$$W_{\min}(N+1) = \begin{cases} L(N+1), & T = N+1, \\ [S(N+1) + L^{\mathrm{T}}(N+1) Q(N) \\ L(N+1)]^{-1} L^{\mathrm{T}}(N+1) Q(N), & T = N+2. \end{cases} \tag{18}$$

and $V_1 = x^{\mathrm{T}}(N) U(N) x(N)$.

### B. Double-Step

Now we turn to the question of optimal policy for a two-step process. Define

$$\begin{aligned} V_2 \;=\; & \min_{W(N)} \min_{W(N+1)} \{[(W(N)x(N-1)-x_\infty)^{\mathrm{T}} \\ & R(N)(W(N)x(N-1)-x_\infty) \\ & +(x(N-1)-x_\infty)^{\mathrm{T}}Q(N-1)(x(N-1)-x_\infty)] \\ & +[(W(N+1)x(N)-x_\infty)^{\mathrm{T}}R(N+1) \\ & (W(N+1)x(N)-x_\infty)+(x(N)-x_\infty)^{\mathrm{T}} \\ & Q(N)(x(N)-x_\infty)]\}. \end{aligned} \tag{19}$$

Then it follows from the principle of dynamic programming that

$$\begin{aligned} V_2 \;=\; & \min_{W(N)} \{[(W(N)x(N-1)-x_\infty)^{\mathrm{T}} \\ & R(N)(W(N)x(N-1)-x_\infty) \\ & +(x(N-1)-x_\infty)^{\mathrm{T}}Q(N-1) \\ & (x(N-1)-x_\infty)]+V_1\}. \end{aligned} \tag{20}$$

**Case 1.** $T=N+1$.

Define $L(N)=L_j=L(N+1)W_{\min}(N+1)$. Since $V_1=x^{\mathrm{T}}(N-1)W^{\mathrm{T}}(N)U(N)W(N)x(N-1)$ and $x_\infty=L_jx(N)=L(N)W(N)x(N-1)$, it follows that

$$\begin{aligned} V_2 \;=\; & \min_{W(N)} \{[x^{\mathrm{T}}(N-1)W^{\mathrm{T}}(N)(I_q-L(N))^{\mathrm{T}} \\ & R(N)(I_q-L(N))W(N)x(N-1) \\ & +x^{\mathrm{T}}(N-1)[I_q-L(N)W(N)]Q(N-1) \\ & [I_q-L(N)W(N)]x(N-1)] \\ & +x^{\mathrm{T}}(N-1)W^{\mathrm{T}}(N)U(N)W(N)x(N-1)\} \\ =\; & \min_{W(N)} \{[x^{\mathrm{T}}(N-1)W^{\mathrm{T}}(N)S(N)W(N)x(N-1) \\ & +x^{\mathrm{T}}(N-1)[I_q-L(N)W(N)]Q(N-1) \\ & [I_q-L(N)W(N)]x(N-1)]\}, \end{aligned} \tag{21}$$

where we defined $S(N)=U(N)+(I_q-L(N))^{\mathrm{T}}R(N)(I_q-L(N))$.

Hence, if we drop the time index, then $V_2$ can be rewritten as

$$\begin{aligned} V_2 \;=\; & \min_{W}\{x^{\mathrm{T}}W^{\mathrm{T}}SWx \\ & +x^{\mathrm{T}}(I_q-LW)Q(I_q-LW)x\}. \end{aligned} \tag{22}$$

*Lemma 4.2:* Consider $V_2$ given by (22). Then $S+L^{\mathrm{T}}QL>0$ and

$$W_{\min}=\arg V_2=[S+L^{\mathrm{T}}QL]^{-1}L^{\mathrm{T}}Q.$$

Thus,

$$\begin{aligned} W_{\min}(N) \;=\; & [S(N)+L^{\mathrm{T}}(N)Q(N-1)L(N)]^{-1} \\ & L^{\mathrm{T}}(N)Q(N-1). \end{aligned} \tag{23}$$

**Case 2.** $T=N+2$.

Define $L(N)=L_jW_{\min}(N+1)=L(N+1)W_{\min}(N+1)$. Then $x_\infty=L_jx(N+1)=L(N)W(N)x(N-1)$. In this case, $V_2$ is given by the same form as (21) and $W_{\min}(N)$ is given by the same form as (23). Finally, we have $V_2=x^{\mathrm{T}}(N-1)U(N-1)x(N-1)$, where

$$\begin{aligned} U(N-1) \;=\; & W_{\min}^{\mathrm{T}}(N)S(N)W_{\min}(N) \\ & +(I_q-L(N)W_{\min}(N))^{\mathrm{T}}Q(N-1) \\ & (I_q-L(N)W_{\min}(N)). \end{aligned} \tag{24}$$

### C. $m-1$ Steps

We derive the general formula by using mathematical induction. Consider the optimal policy at time $N-m+1$ for a process involving $m-1$ steps for $m\geq 3$. Define $L(N-m+2)=L(N-m+3)W_{\min}(N-m+3)$. Then for both $T=N+1$ and $T=N+2$, we have $x_\infty=L(N-m+2)W(N-m+2)x(N-m+1)$. Thus, by mathematical induction, one can assume that the optimal policy is characterized by a set of coupled recursive equations

$$\begin{aligned} & W_{\min}(N-m+2)= \\ & \quad [S(N-m+2)+L^{\mathrm{T}}(N-m+2) \\ & \quad Q(N-m+1)L(N-m+2)]^{-1} \\ & \quad L^{\mathrm{T}}(N-m+2)Q(N-m+1), \end{aligned} \tag{25}$$

$$\begin{aligned} & S(N-m+2)= \\ & \quad U(N-m+2)+(I_q-L(N-m+2))^{\mathrm{T}} \\ & \quad R(N-m+2)(I_q-L(N-m+2)), \end{aligned} \tag{26}$$

$$\begin{aligned} & U(N-m+1)= \\ & \quad W_{\min}^{\mathrm{T}}(N-m+2)S(N-m+2)W_{\min}(N-m+2) \\ & \quad +(I_q-L(N-m+2)W_{\min}(N-m+2))^{\mathrm{T}} \\ & \quad Q(N-m+1) \\ & \quad (I_q-L(N-m+2)W_{\min}(N-m+2)), \end{aligned} \tag{27}$$

$$\begin{aligned} & V_{m-1}= \\ & \quad x^{\mathrm{T}}(N-m+1)U(N-m+1)x(N-m+1). \end{aligned} \tag{28}$$

### D. $m$ Steps

Now we need to prove that for $m$ steps of optimal policy, we should have the similar forms as the case of $m-1$ steps except for the time index. For $m$ steps of optimal policy, it follows from the Bellman principle that

$$\begin{aligned} V_m \;=\; & \min_{W(N-m+1)} \{[(W(N-m+1)x(N-m)-x_\infty)^{\mathrm{T}} \\ & R(N-m+1)(W(N-m+1)x(N-m) \\ & -x_\infty)+(x(N-m)-x_\infty)^{\mathrm{T}}Q(N-m) \\ & (x(N-m)-x_\infty)]+V_{m-1}\} \\ =\; & \min_{W(N-m+1)} \{[x^{\mathrm{T}}(N-m)W^{\mathrm{T}}(N-m+1) \\ & (I_q-L(N-m+1))^{\mathrm{T}}R(N-m+1) \\ & (I_q-L(N-m+1))W(N-m+1)x(N-m) \\ & +x^{\mathrm{T}}(N-m)[I_q-L(N-m+1) \\ & W(N-m+1)]Q(N-m)[I_q-L(N-m+1) \\ & W(N-m+1)]x(N-m)]+x^{\mathrm{T}}(N-m) \\ & W^{\mathrm{T}}(N-m+1)U(N-m+1) \\ & W(N-m+1)x(N-m)\} \end{aligned}$$

$$
\begin{aligned}
= \quad & \min_{W(N-m+1)} \{[x^{\mathrm{T}}(N-m)W^{\mathrm{T}}(N-m+1) \\
& S(N-m+1)W(N-m+1)x(N-m) \\
& +x^{\mathrm{T}}(N-m)[I_q - L(N-m+1) \\
& W(N-m+1)]Q(N-m)[I_q - L(N-m+1) \\
& W(N-m+1)]x(N-m)]\}, \qquad (29)
\end{aligned}
$$

where

$$
\begin{aligned}
S(N-m+1) &= U(N-m+1) \\
&\quad +(I_q - L(N-m+1))^{\mathrm{T}} \\
&\quad R(N-m+1)(I_q - L(N-m+1)), \qquad (30) \\
L(N-m+1) &= L(N-m+2) \\
&\quad W_{\min}(N-m+2). \qquad (31)
\end{aligned}
$$

Hence, this case is identical to the double-step case. Using the similar arguments, one can obtain

$$
\begin{aligned}
& W_{\min}(N-m+1) = \\
&\quad [S(N-m+1) + L^{\mathrm{T}}(N-m+1) \\
&\quad Q(N-m)L(N-m+1)]^{-1} \\
&\quad L^{\mathrm{T}}(N-m+1)Q(N-m), \qquad (32) \\
& U(N-m) = \\
&\quad W_{\min}^{\mathrm{T}}(N-m+1)S(N-m+1)W_{\min}(N-m+1) \\
&\quad +(I_q - L(N-m+1)W_{\min}(N-m+1))^{\mathrm{T}} \\
&\quad Q(N-m)(I_q - L(N-m+1)W_{\min}(N-m+1)), \qquad (33) \\
& V_m = x^{\mathrm{T}}(N-m)U(N-m)x(N-m). \qquad (34)
\end{aligned}
$$

Thus, by mathematical induction, we have the following result for optimality.

*Theorem 4.1:* Assume that Assumptions 3.1–3.4 hold. Then for $k = N, N-1, \ldots, 1$, the optimal policy for (2) is given by (14)–(16), (18), and

$$
\begin{aligned}
W_{\min}(k) &= [S(k) + L^{\mathrm{T}}(k)Q(k-1) \\
&\quad L(k)]^{-1}L^{\mathrm{T}}(k)Q(k-1), \qquad (35) \\
S(k) &= U(k) + (I_q - L(k))^{\mathrm{T}}R(k)(I_q - L(k)), \qquad (36) \\
L(k) &= L(k+1)W_{\min}(k+1), \qquad (37) \\
U(k) &= W_{\min}^{\mathrm{T}}(k+1)S(k+1)W_{\min}(k+1) \\
&\quad +(I_q - L(k+1)W_{\min}(k+1))^{\mathrm{T}} \\
&\quad Q(k)(I_q - L(k+1)W_{\min}(k+1)). \qquad (38)
\end{aligned}
$$

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