2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Robust $L_2 - L_\infty$ Consensus Control of the Second-order Multi-agent Systems with Time-delay

Yan Cui, Yingmin Jia, Junping Du and Fashan Yu

***Abstract*— Robust $L_2 - L_\infty$ consensus control is studied for the second-order multi-agent systems with external disturbances and parameter uncertainties. By defining an appropriate controlled output, the consensus problem of the systems is transformed into a normal $L_2 - L_\infty$ control problem, and distributed state feedback protocols with time-delay are proposed. Sufficient conditions are established for the convergence to consensus of the network under fixed or switching topology. Numerical simulations are provided to demonstrate the effectiveness of our theoretical results.**

I. INTRODUCTION

Coordination control of a group of agents has received compelling attentions within the control community. Its broad applications involve satellite clusters [1], unmanned air vehicles [2], formation control [3], distributed sensor network [4], rendezvous in space [5] and so forth. As one of the most important issues in the coordination control, consensus for the multi-agent systems means that all the agents could reach an agreement on certain quantities of interest by employing the appropriate control protocols based on local information.

In the past decades, consensus problems for multi-agent system have been studied by many researchers [6]-[15]. However, the multi-agent systems are often subject to external disturbances and parameter uncertainties in practical applications, such as actuator bias, measurement or calculation errors and the variation of communication links, which might usually destroy the convergence performance of multi-agent systems. For such cases, several research has been done. For example, Ren considered the actuator saturation that might cause time-delay or uncertainties and proposed several consensus algorithms for second-order multi-agent systems in the absence or presence of a group reference velocity, respectively [12]. In [13], Lin *et al*. introduced H_∞ method into the consensus problem of multi-agent systems with external disturbances and parameter uncertainties for directed networks with zero and nonzero time-delay on fixed and switching topologies. In [15], Liu *et al*. designed the H_∞ controller and obtained consensus condition with the desired performance to impair the external disturbances for the multi-agent system with first-order, second-order, high-order and linear coupling dynamics. [13] and [15] both employed the H_∞ control method to attenuate the external disturbance signal. Whereas, the peak value of the controlled output in many projects is required to be within a certain range, when the impact of external disturbances and time-delays to the performance of the system is taken into account. Here we try to solve these problem by $L_2 - L_\infty$ control method. $L_2 - L_\infty$ control not only resembles H_∞ control that can attenuate the external disturbance signal but also minimizes the controlled output value for a multi-agent system.

In this paper, the $L_2 - L_\infty$ control method is adopted to solve the consensus problem of second-order multi-agent systems with external disturbances and parameter uncertainties in presence of time-delay. By defining an appropriate controlled output, the consensus problem of multi-agent systems is transformed into the $L_2 - L_\infty$ control problem. In doing the analysis, we turn the original system with a singular Laplacian matrix into a reduced order system that can be stabilized. Then we derive sufficient conditions in terms of bilinear matrix inequalities (BMIs) with the desired $L_2 - L_\infty$ performance.

Throughout this paper, $\mathbf{1}_n$ denotes the column vector of n dimension whose elements are all ones. $*$ denotes the symmetric part of a symmetric matrix. $\mathrm{diag}\{m_1, \cdots, m_n\}$ denotes a block-diagonal matrix whose diagonal blocks are given by $m_1, \cdots, m_n$. The symmetric matrix $X > 0$ means that X is positive definite.

II. PRELIMINARIES

Let $\mathscr{G} = (\mathscr{V}, \varepsilon, \mathscr{A})$ be a directed graph of order n with the set of nodes $\mathscr{V} = \{s_1, \cdots, s_n\}$, set of edges $\varepsilon \subseteq \mathscr{V} \times \mathscr{V}$, and a weighted adjacency matrix $\mathscr{A} = [a_{ij}]$ with nonnegtive adjacency elements a_{ij}. The node indexes belong to a finite index set $\mathscr{I} = \{1, 2, \cdots, n\}$. In particular, it is assumed that $a_{ii} = 0$ for $\forall i \in \mathscr{I}$. In graph $\mathscr{G}$, node s_i represents the ith agent, and the set of neighbors of node s_i is denoted by $N_i = \{s_j \in \mathscr{V} : (s_i, s_j) \in \varepsilon\}$. The out-degree is defined as $d_o(s_i) = \sum_{j=1}^{n} a_{ij}$. The Laplacian with the directed graph is defined as $L = \Delta - \mathscr{A}$, where $\Delta = [\Delta_{ij}]$ is a diagonal matrix with $\Delta_{ii} = d_o(s_i)$. A directed path is a sequence of ordered edges of the form $(s_{i_1}, s_{i_2}), (s_{i_2}, s_{i_3}), \cdots$, where $s_{i_j} \in \mathscr{V}$. If there is a directed path from every node to every other node, the

This work was supported by the NSFC (60727002, 60774003, 60921001, 90916024), the MOE (20030006003), the COSTIND (A2120061303), the National 973 Program (2005CB321902).

Yan Cui is with the Seventh Research Division and the Department of Systems and Control, Beihang University, Beijing 100191, China cuiyan8013@163.com

Yingmin Jia is with the Seventh Research Division and the Department of Systems and Control, Beihang University, Beijing 100191, China. He is also with the Key Laboratory of MathematicsInformatics and Behavioral Semantics (LMIB), Ministry of Education, SMSS, Beihang University, Beijing 100191, China ymjia@buaa.edu.cn

Junping Du is with the Beijing Key Laboratory of Intelligent Telecommunications Software and Multimedia, School of Computer Science and Technology, Beijing University of Posts and Telecommunications, Beijing 100876, China junpingdu@126.com

Fashan Yu is with the School of Electrical Engineering and Automation, Henan Polytechnic University, Jiaozuo 454000, Henan, China yufs@hpu.edu.cn

graph is said to be strongly connected. Moreover, if there exists a node such that there is a directed path from every other node to this node, the directed graph is said to have spanning trees.

Lemma 1: [10] Let L be the Laplacian associated with the directed graph $\mathscr{G}$. Then L has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right-half plane. Furthermore, matrix L has exactly one zero eigenvalue if and only if the directed graph $\mathscr{G}$ has a spanning tree, and $\mathbf{1}_n$ is the corresponding eigenvector, i.e., $L\mathbf{1}_n = 0$.

Lemma 2: [13] Given the symmetry matrix $L_c = [L_{c_{ij}}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$,

$$L_{c_{ij}} = \begin{cases} \dfrac{n-1}{n}, & i = j, \\ -\dfrac{1}{n}, & i \neq j, \end{cases}$$

then the following statements hold:

1) The eigenvalues of L_c are 1 with multiplicity $n-1$ and 0 with multiplicity 1. The vectors $\mathbf{1}_n^T$ and $\mathbf{1}_n$ are the left and the right eigenvectors of L_c associated with the zero eigenvalue, respectively.

2) There must exist an orthogonal matrix U such that $U^T L_c U = \left[\begin{smallmatrix} I_{n-1} & 0 \\ * & 0 \end{smallmatrix}\right]$ and the last column is $\dfrac{\mathbf{1}_n}{\sqrt{n}}$. Furthermore, let $L \in \mathbb{R}^{n\times n}$ be the Laplacian of any directed graph, then $U^T L U = \left[\begin{smallmatrix} \bar{L} & 0 \\ U_2^T L U_1 & 0 \end{smallmatrix}\right]$, where $\bar{L} = U_1^T L U_1$ and $\bar{L} \in \mathbb{R}^{n\times(n-1)}$. For convenience, we denote $U = [U_1 \quad U_2]$, $U_1 \in \mathbb{R}^{n\times(n-1)}$ and $U_2 = \frac{\mathbf{1}_n}{\sqrt{n}} \in \mathbb{R}^{n\times 1}$.

III. CONTROL PROTOCOL AND SYSTEM DYNAMICS

A. $L_2 - L_\infty$ control and problem statement

We consider the multi-agent system consisting of n identical agents subject to external disturbances. Suppose the ith agent ($i \in \mathscr{I}$) has the dynamics

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t) + \omega_i(t), \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}$ and $v_i(t) \in \mathbb{R}$ are the position and velocity state of the ith agent, respectively. $\omega_i(t) \in \mathscr{L}_2[0,\infty)$ is the external disturbance and $\mathscr{L}_2[0,\infty)$ denotes the space of square integrable vector functions over $[0,\infty)$, and $u_i(t)$ is control input.

According to the special control objective, $z_i(t) = [z_{i1}(t)\ z_{i2}(t)]^T \in \mathbb{R}^2$ for $i \in \mathscr{I}$ is defined as controlled output functions, and it is appropriate to analyze the effect of external disturbances and parameter uncertainties.

$$\begin{aligned} z_{i1}(t) &= x_i(t) - \frac{1}{n}\sum_{j=1}^n x_j(t), \\ z_{i2}(t) &= v_i(t) - \frac{1}{n}\sum_{j=1}^n v_j(t), \end{aligned} \tag{2}$$

and $z(t) = [z_1^T(t), z_2^T(t), \cdots, z_n^T(t)]^T \in \mathbb{R}^{2n}$, $i = 1,2,\ldots,n$.

It is obvious that consensus of the second-order multi-agent system (1) can be achieved if and only if $\lim_{t\to+\infty} z(t) = 0$, i.e.

$$\lim_{t\to+\infty}[x_i(t) - x_j(t)] = 0, \quad \lim_{t\to+\infty}[v_i(t) - v_j(t)] = 0, \tag{3}$$

for all $i, j \in \mathscr{I}$.

And the attenuating ability of the multi-agent system against external disturbances can be quantitatively measured by the $L_2 - L_\infty$ performance index of the closed-loop transfer function matrix from the external disturbance input $\omega(t)$ to the controlled output $z(t)$ shown as

$$\|T_{z\omega}\|_{L_2-L_\infty} = \sup_{0\neq\omega(t)\in L_2[0,\infty)} \frac{\|z(t)\|_\infty}{\|\omega(t)\|_2}, \tag{4}$$

where, $\|z(t)\|_\infty^2 = \sup_t z^T(t)z(t)$, $\|\omega(t)\|_2^2 = \int_0^\infty \omega^T(t)\omega(t)dt$.

Therefore we should design the protocol $u_i(t)$ meeting the following two conditions simultaneously:

1). The states of second-order multi-agent system is asymptotically stable, i.e. $\lim_{t\to+\infty} z(t) = 0$, without external disturbances;

2). Under the zero-valued initial state condition, $u_i(t)$ can make the closed-loop transfer function $T_{z\omega}$ satisfy the following condition

$$\|T_{z\omega}\|_{L_2-L_\infty} < \gamma, \tag{5}$$

here γ is a given positive scalar.

B. Control Protocol and System Dynamics

To solve consensus problem of the second-order multi-agent system, we design the following protocol

$$\begin{aligned} u_i(t) = &-v_i(t) + K_1 \sum_{j\in N_i}(a_{ij} + \Delta a_{ij}(t))[x_j(t-\tau) - x_i(t-\tau)] \\ &+ K_2 \sum_{j\in N_i}(a_{ij} + \Delta a_{ij}(t))[v_j(t-\tau) - v_i(t-\tau)] \end{aligned} \tag{6}$$

where $K_1 > 0$, $K_2 > 0$ are protocol parameters and τ is time-delay. a_{ij} is the weight of edge, $\Delta a_{ij}(t)$ denotes the uncertainty of a_{ij} with $|\Delta a_{ij}(t)| \leq \psi_{ij}$ for $i \neq j$ and $a_{ij} \neq 0$, and $|\Delta a_{ij}(t)| = 0$ otherwise, where ψ_{ij} is a constant for $i, j \in \mathscr{I}$.

In the multi-agent system (1), denote

$$\begin{aligned} &\eta(t) = [x_1(t), v_1(t), \ldots, x_n(t), v_n(t)]^T \in R^{2n}, \\ &A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \\ &B_1 = \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Then we can get the second-order multi-agent system dynamics of time-delay with protocol (6)

$$\begin{aligned} \dot{\eta}(t) &= (I_n \otimes A)\eta(t) - [(L+\Delta L)\otimes B_1]\eta(t-\tau) + (I_n \otimes B_2)\omega(t) \\ z(t) &= (L_c \otimes I_2)\eta(t) \end{aligned} \tag{7}$$

where $\omega(t) = [\omega_1(t)\ \omega_2(t) \ldots \omega_n(t)]^T \in \mathbb{R}^n$, the edge weights a_{ij} and the uncertainties $\Delta a_{ij}(t) (i, j \in \mathscr{I})$ are denoted by the corresponding Laplacian matrix L and ΔL, respectively. Then the uncertainty Laplacian matrix ΔL can be rewritten as $E_1\Sigma(t)E_2$, where $E_1 \in \mathbb{R}^{n\times|\varepsilon|}$, $E_2 \in \mathbb{R}^{|\varepsilon|\times n}$ are the determined constant matrices. $\Sigma(t) \in \mathbb{R}^{|\varepsilon|\times|\varepsilon|}$ reflects the uncertainties of the edges and satisfies $\Sigma^T(t)\Sigma(t) \le I$, see, [14], and the references therein.

IV. MAIN RESULTS

In the section, we will present consensus conditions in directed networks with time-delay on fixed and switching topologies.

Lemma 3: [16] For any real matrices $D \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{m\times n}$ with $F(t) \in \mathbb{R}^{m\times m}$ satisfying $\|F(t)\| \le 1$, and any scalar $\varepsilon > 0$, we have

$$DF(t)E + E^T F(t)^T D^T \le \varepsilon^{-1}DD^T + \varepsilon E^T E$$

Lemma 4: [17] (*Schur Complement Formula*) Given symmetric matrix $S \in \mathbb{R}^{n\times n}$ with the form $S = [S_{ij}], i, j \in \{1,2\}$, $S_{11} \in \mathbb{R}^{r\times r}$, $S_{12} \in \mathbb{R}^{r\times(n-r)}$, $S_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, then $S < 0$ if and only if $S_{11} < 0, \quad S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$. Or equivalently, $S_{22} < 0, \quad S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$.

A. Network with time-delay on fixed topology

Theorem 1: Consider a directed network with fixed topology and time-delay τ. For the multi-agent system(7), consensus can be achieved with $\|T_{z\omega}\|_{L_2-L_\infty} < \gamma$ (a given index $\gamma > 0$), if the symmetry matrix $P > 0$, $Q > 0$, $R > 0$ and P, Q, $R \in \mathbb{R}^{2(n-1)\times 2(n-1)}$ and positive scalars ε_1, ε_2, ε_3, ε_4, ε_5, ε_6 satisfy

$$\begin{aligned} X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} &< 0, \\ \begin{bmatrix} -P & (U_1 \otimes I_2)^T \\ * & -\gamma I \end{bmatrix} &< 0, \end{aligned} \tag{8}$$

where X_{11}, X_{12}, X_{22} respectively are (see(9))
here $\bar{L} = U_1^T L U_1$, $\bar{E}_2 = (U_1^T E_2^T E_2 U_1) \otimes I_2$, $\bar{B}_2 = U_1^T \otimes B_2$, $P_1 = P(I_{n-1} \otimes A - \bar{L} \otimes B_1) + (I_{n-1} \otimes A - \bar{L} \otimes B_1)^T P + Q + \tau(I_{n-1} \otimes A)^T R(I_{n-1} \otimes A)$, and $Q_1 = -Q + \varepsilon_1 \bar{E}_2 + \tau\varepsilon_2 \bar{E}_2 + \tau\varepsilon_3(\bar{L} \otimes B_1)^T(\bar{L} \otimes B_1) + \tau\varepsilon_4 \bar{E}_2 + \tau\varepsilon_5 \bar{E}_2 + \tau\varepsilon_6 \bar{E}_2$.

Proof: Since the Laplacian matrix L has zero eigenvalues, it can be verified that the system (7) is unstable if A is an unstable matrix. So we need to conduct a model transformation.

Let

$$\begin{aligned} \hat{\eta}(t) =& \eta(t) - \mathbf{1}_n \otimes \Omega(t), \\ \Omega(t) =& \frac{1}{n}\sum_{i=1}^{n}\int_0^t e^{A(t-s)}B_2\omega_i(s)ds, \\ \hat{\eta}^1(t) =& (U_1 \otimes I_2)^T\hat{\eta}(t), \hat{\eta}^1(t-\tau) = (U_1 \otimes I_2)^T\hat{\eta}(t-\tau), \\ \hat{\eta}^2(t) =& (U_2 \otimes I_2)^T\hat{\eta}(t), \hat{\eta}^2(t-\tau) = (U_2 \otimes I_2)^T\hat{\eta}(t-\tau), \end{aligned} \tag{10}$$

where $\Omega(t)$ denotes the average of external disturbances; $\hat{\eta}(t)$ is the states of all agents, which takes out the average of external disturbances; $\hat{\eta}^1(t)$ and $\hat{\eta}^1(t-\tau)$ describes the disagreement states of all agents; and $\hat{\eta}^2(t)$ and $\hat{\eta}^2(t-\tau)$ depicts the average states of all agents.

By Lemma 2, U is an orthogonal matrix, we have $U_1^T U_2 = 0$. Since $U_2 = \mathbf{1}_n/\sqrt{n}$, $L\mathbf{1}_n = 0$ and $\Delta L\mathbf{1}_n = 0$, we have

$$\begin{aligned} &(U \otimes I_2)^T\dot{\eta}(t) \\ =&(U \otimes I_2)^T(I_n \otimes A)(U \otimes I_2)\begin{bmatrix} \hat{\eta}^1(t) \\ \hat{\eta}^2(t) \end{bmatrix} \\ &-(U \otimes I_2)^T[(L+\Delta L)\otimes B_1](U \otimes I_2)\begin{bmatrix} \hat{\eta}^1(t-\tau) \\ \hat{\eta}^2(t-\tau) \end{bmatrix} \\ &+(U \otimes I_2)^T(I_n \otimes A)(\mathbf{1}_n \otimes \Omega(t)) + (U^T \otimes B_2)\omega(t) \\ =&\begin{bmatrix} (I_{n-1} \otimes A) & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} \hat{\eta}^1(t) \\ \hat{\eta}^2(t) \end{bmatrix} \\ &-\begin{bmatrix} (U_1^T L U_1 + U_1^T \Delta L U_1)\otimes B_1 & 0 \\ (U_2^T L U_1 + U_2^T \Delta L U_1)\otimes B_1 & 0 \end{bmatrix}\begin{bmatrix} \hat{\eta}^1(t-\tau) \\ \hat{\eta}^2(t-\tau) \end{bmatrix} \\ &+\begin{bmatrix} (U_1^T \otimes B_2)\omega(t) \\ \sqrt{n}A\Omega(t) + B_2\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\omega_i(t) \end{bmatrix} \end{aligned} \tag{11}$$

Premultiplying the left-hand side of system (7) with the matrix $(U \otimes I_2)^T$ yields

$$\begin{aligned} &(U \otimes I_2)^T\dot{\eta}(t) \\ =&\begin{bmatrix} \dot{\hat{\eta}}^1(t) \\ \dot{\hat{\eta}}^2(t) \end{bmatrix} + (U \otimes I_2)^T(\mathbf{1}_n \otimes \dot{\Omega}(t)) \\ =&\begin{bmatrix} \dot{\hat{\eta}}^1(t) \\ \dot{\hat{\eta}}^2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^t Ae^{A(t-s)}B_2\omega_i(s)ds \\ +\frac{1}{\sqrt{n}}\sum_{i=1}^{n}B_2\omega_i(t) \end{bmatrix} \\ =&\begin{bmatrix} \dot{\hat{\eta}}^1(t) \\ \dot{\hat{\eta}}^2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{n}A\Omega(t) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}B_2\omega_i(t) \end{bmatrix} \end{aligned} \tag{12}$$

From (11) and (12), we can get $\hat{\eta}^1(t)$ is independent of $\hat{\eta}^2(t)$, but $\hat{\eta}^2(t)$ is dependent on $\hat{\eta}^1(t)$. Denote $\overline{\Delta L} = U_1^T \Delta L U_1$. Then the original system can be reduced to the following system

$$\begin{aligned} \dot{\hat{\eta}}^1(t) =& (I_{n-1} \otimes A)\hat{\eta}^1(t) - [(\bar{L} + \overline{\Delta L})\otimes B_1]\hat{\eta}^1(t-\tau) \\ &+ (U_1^T \otimes B_2)\omega(t) \\ z(t) =& (L_c \otimes I_2)\eta(t) = (U_1 \otimes I_2)\hat{\eta}^1(t) \end{aligned} \tag{13}$$

We can use the reduced-order system (13) instead of the multi-agent system (7) to investigate the $L_2 - L_\infty$ performance. It follows that if $\lim_{t\to+\infty}\hat{\eta}^1(t) = 0$, then $\lim_{t\to+\infty}z(t) = 0$. So, whether the multi-agent system (7) can reach consensus is only related to the component $\hat{\eta}^1(t)$. Then we analysis the robust $L_2 - L_\infty$ performance of the multi-agent system.

Define a Lyapunov-Krasovskii function

$$\begin{aligned} V(t) =& \hat{\eta}^{1T}(t)P\hat{\eta}^1(t) + \int_{t-\tau}^{t}\hat{\eta}^{1T}(s)Q\hat{\eta}^1(s)ds \\ &+ \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\hat{\eta}}^{1T}(s)R\dot{\hat{\eta}}^1(s)dsd\theta \end{aligned} \tag{14}$$

$$X_{11} = \begin{bmatrix} P(I_{n-1}\otimes A-\bar{L}\otimes B_1)+(I_{n-1}\otimes A-\bar{L}\otimes B_1)^T P+Q+\tau(I_{n-1}\otimes A)^T R(I_{n-1}\otimes A) & 0 & P\bar{B}_2+\tau(I_{n-1}\otimes A)^T R\bar{B}_2 \\ * & Q_1 & -\tau(\bar{L}\otimes B_1)^T R\bar{B}_2 \\ * & * & -\gamma I+\tau\bar{B}_2^T R\bar{B}_2 \end{bmatrix},$$

$$X_{12} = \left[\begin{matrix} P(\bar{L}\otimes B_1) & P[(U_1^T E_1)\otimes B_1] & \tau(I_{n-1}\otimes \bar{A})^T R & \tau(I_{n-1}\otimes \bar{A})^T R[(U_1^T E_1)\otimes \bar{B}_1] & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau(\bar{L}\otimes B_1)^T R & \tau(\bar{L}\otimes B_1)^T R[(U_1^T E_1)\otimes B_1] \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right.$$
$$\left.\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{B}_2^T R[(U_1^T E_1)\otimes B_1] \end{matrix}\right], \tag{9}$$

$$X_{22} = diag\left\{-\frac{R}{\tau}, -\varepsilon_1 I, -\tau\varepsilon_3 R, -\tau\varepsilon_4 I, -\tau R, -\tau\varepsilon_2 I, \begin{bmatrix} -\tau R & \tau R(U_1^T E_1)\otimes B_1 \\ * & -\tau\varepsilon_5 I \end{bmatrix}, -\tau\varepsilon_6 I\right\},$$

Firstly, we study the system without the external disturbance $\omega(t)$. The time derivative of $V(t)$ along the solution of the system (13) is

$$\begin{aligned}\dot{V}(t) =& 2\hat{\eta}^{1T}(t)P\dot{\hat{\eta}}^1(t)+\hat{\eta}^{1T}(t)Q\hat{\eta}^1(t) \\ & -\hat{\eta}^{1T}(t-\tau)Q\hat{\eta}^1(t-\tau)+\tau\dot{\hat{\eta}}^{1T}(t)R\dot{\hat{\eta}}^1(t) \\ & -\int_{t-\tau}^{t}\dot{\hat{\eta}}^{1T}(s)R\dot{\hat{\eta}}^1(s)ds \\ =& 2\hat{\eta}^{1T}(t)P(I_{n-1}\otimes A)\hat{\eta}^1(t)-2\hat{\eta}^{1T}(t)P(\bar{L}+\overline{\Delta L})\otimes B_1 \\ & \hat{\eta}^1(t-\tau)+\hat{\eta}^{1T}(t)Q\hat{\eta}^1(t)-\hat{\eta}^{1T}(t-\tau)Q\hat{\eta}^1(t-\tau) \\ & +\tau\dot{\hat{\eta}}^{1T}(t)R\dot{\hat{\eta}}^1(t)-\int_{t-\tau}^{t}\dot{\hat{\eta}}^{1T}(s)R\dot{\hat{\eta}}^1(s)ds \end{aligned} \tag{15}$$

From the Newton-Leibniz formula $\delta_1(t-\tau)=\delta_1(t)-\int_{t-\tau}^{t}\dot{\delta}_1(s)ds$. By Lemma 3, we have

$$\dot{V}(t)\le\hat{\eta}^{1T}(t)\bar{N}_1\hat{\eta}^1(t)+\hat{\eta}^{1T}(t-\tau)\bar{N}_2\hat{\eta}^1(t-\tau), \tag{16}$$

where

$$\begin{aligned}\bar{N}_1 =& P(I_{n-1}\otimes A-\bar{L}\otimes B_1)+(I_{n-1}\otimes A-\bar{L}\otimes B_1)^T P+Q \\ & +\tau(I_{n-1}\otimes A)^T R(I_{n-1}\otimes A)+\frac{1}{\varepsilon_1}P\bar{E}_1 P+\tau P(\bar{L}\otimes B_1)R^{-1} \\ & (\bar{L}\otimes B_1)^T P+\frac{\tau}{\varepsilon_3}(I_{n-1}\otimes A)^T R(I_{n-1}\otimes A)+\frac{\tau}{\varepsilon_4}(I_{n-1}\otimes A)^T \\ & R\bar{E}_1 R(I_{n-1}\otimes A), \\ \bar{N}_2 =& -Q+\varepsilon_1\bar{E}_2+\tau(\bar{L}\otimes B_1)^T R(\bar{L}\otimes B_1)+\tau(\overline{\Delta L}\otimes B_1)^T R(\overline{\Delta L} \\ & \otimes B_1)+\frac{\tau}{\varepsilon_2}(\bar{L}\otimes B_1)^T R\bar{E}_1 R(\bar{L}\otimes B_1)+\tau\varepsilon_2\bar{E}_2+\tau\varepsilon_3(\bar{L} \\ & \otimes B_1)^T(\bar{L}\otimes B_1)+\tau\varepsilon_4\bar{E}_2. \end{aligned}$$

where $\bar{E}_1=[(U_1^T E_1)\otimes B_1][(U_1^T E_1)\otimes B_1]^T$ and $\bar{E}_2=(U_1^T E_2^T E_2 U_1)\otimes I_2$, and ε_1, ε_2, ε_3, ε_4 are positive scalars.

A sufficient condition for $\dot{V}(t)<0$ is $\bar{N}=\left[\begin{smallmatrix}\bar{N}_1 & 0\\ 0 & \bar{N}_2\end{smallmatrix}\right]<0$. The multi-agent system (13) is asymptotically stable. Therefore, all the agents can reach consensus.

By Lemma 4, we have (see(17) and (18)), here

$$\begin{aligned}\theta_1 =& -Q+\varepsilon_1\bar{E}_2+\tau(\bar{L}\otimes B_1)^T R(\bar{L}\otimes B_1)+\tau/\varepsilon_2(\bar{L}\otimes B_1)^T R \\ & \bar{E}_1 R(\bar{L}\otimes B_1)+\tau\varepsilon_2\bar{E}_2+\tau\varepsilon_3(\bar{L}\otimes B_1)^T(\bar{L}\otimes B_1)+\tau\varepsilon_4\bar{E}_2. \end{aligned}$$

By Lemma 3,

$$\theta\le\begin{bmatrix}\theta_1+\tau\varepsilon_5\bar{E}_2 & 0 \\ 0 & -\tau R+\dfrac{\tau}{\varepsilon_5}R\bar{E}_1 R\end{bmatrix}<0,$$

for any $\varepsilon_5>0$.

Then by Lemma 4 again, we get the equation (see(19))

We discuss the performance with $\omega(t)$ in the following. Denote $\bar{B}_2=U_1^T\otimes B_2$, we have

$$\begin{aligned}\dot{V}(t)\le& \hat{\eta}^{1T}(t)\bar{N}_1\hat{\eta}^1(t)+\hat{\eta}^{1T}(t-\tau)\bar{N}_2\hat{\eta}^1(t-\tau)+2\hat{\eta}^{1T}(t)P\bar{B}_2 \\ & \omega(t)+\tau\omega^T(t)\bar{B}_2^T R\bar{B}_2\omega(t)-2\tau\hat{\eta}^{1T}(t)(I_{n-1}\otimes A)^T R\bar{B}_2 \\ & \omega(t)-2\tau\hat{\eta}^{1T}(t)(\bar{L}+\overline{\Delta L})\otimes B_1]^T R\bar{B}_2\omega(t) \\ \le& \hat{\eta}^{1T}(t)\bar{N}_1\hat{\eta}^1(t)+\hat{\eta}^{1T}(t-\tau)\bar{N}_2\hat{\eta}^1(t-\tau)+2\hat{\eta}^{1T}(t)P\bar{B}_2 \\ & \omega(t)+\tau\omega^T(t)\bar{B}_2^T R\bar{B}_2\omega(t)+2\tau\hat{\eta}^{1T}(t)(I_{n-1}\otimes A)^T R\bar{B}_2 \\ & \omega(t)-2\tau\hat{\eta}^{1T}(t)(\bar{L}\otimes B_1)^T R\bar{B}_2\omega(t)+\varepsilon_6\tau\hat{\eta}^{1T}(t-\tau)\bar{E}_2 \\ & \hat{\eta}^1(t-\tau)+\frac{\tau}{\varepsilon_6}\omega(t)^T\bar{B}_2^T R\bar{E}_1 R\bar{B}_2\omega(t). \end{aligned}$$

here $\varepsilon_6>0$.

Under the zero-valued initial state condition $V(0)=0$, we define the cost performance for any $T>0$,

$$\begin{aligned}J_T =& V(t)-\gamma\int_0^T\omega(t)^T\omega(t)dt=\int_0^T[\dot{V}(t)-\gamma\omega(t)^T\omega(t)]dt \\ \le& \int_0^T\zeta^T(t)Y\zeta(t)dt, \end{aligned} \tag{20}$$

where

$$Y=\begin{bmatrix}\bar{N}_1 & 0 & P\bar{B}_2+\tau(I_{n-1}\otimes A)^T RB_2 \\ * & \bar{N}_2+\varepsilon_6\tau\bar{E}_2 & -\tau(\bar{L}\otimes B_1)^T R\bar{B}_2 \\ * & * & -\gamma I+\dfrac{\tau}{\varepsilon_6}\bar{B}_2^T R\bar{E}_1 R\bar{B}_2+\tau\bar{B}_2^T R\bar{B}_2\end{bmatrix},$$

and $\zeta^T(t)=[\hat{\eta}^{1T}(t)\quad \hat{\eta}^{1T}(t-\tau)\quad \omega^T(t)]$.

By Lemma 4 (Schur Complement Formula), we have $Y<0$ if $X<0$. Further, if $Y<0$ holds, $J_T<0$, i.e. $V(t)<\gamma\int_0^T\omega^T(t)\omega(t)dt$.

For $\begin{bmatrix}-P & (U_1\otimes I_2)^T \\ * & -\gamma I\end{bmatrix}<0$, by Lemma 4 (Schur Complement Formula) we have $(U_1\otimes I_2)^T(U_1\otimes I_2)-\gamma P<0$.

It is easy to get $\|z(t)\|_\infty^2<\gamma V(t)<\gamma^2\|\omega(t)\|_2^2$.

For any $0\ne\omega(t)\in L_2[0,\infty)$, $\|T_{zw}\|_{L_2-L_\infty}=\frac{\|z(t)\|_\infty}{\|\omega(t)\|_2}<\gamma$.

By Lemma 3 and Lemma 4 , we have $Y<0$ if $X<0$. Therefore, all agents can reach consensus with $\|T_{z\omega}(s)\|_{L_2-L_\infty}<\gamma$ under the condition (8). □

Note that the Laplacian matrix of switching topology $\mathscr{G}_{\sigma(t)}$ is denoted as $L_{\sigma(t)}$, where $\sigma(t)$ is the switching signal at t moment that determines the topology.

$$\bar{N}_1 < 0 \Leftrightarrow N_1 = \begin{bmatrix} P_1 & P(\bar{L}\otimes B_1) & P[(U_1^T E_1)\otimes B_1] & \tau(I_{n-1}\otimes A)^T R[I+(U_1^T E_1)\otimes B_1] \\ * & -\frac{R}{\tau} & 0 & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\tau(\varepsilon_3 R+\varepsilon_4 I) \end{bmatrix} < 0, \tag{17}$$

$$\bar{N}_2 < 0 \Leftrightarrow$$
$$\theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & -\tau R \end{bmatrix} + \tau \begin{bmatrix} [(E_2U_1)\otimes I_2]^T \\ 0 \end{bmatrix} \Sigma^T \begin{bmatrix} 0 & [(U_1^T E_1)\otimes B_1]^T R \end{bmatrix} + \tau \begin{bmatrix} 0 \\ R[(U_1^T E_1)\otimes B_1] \end{bmatrix} \Sigma \begin{bmatrix} (E_2U_1)\otimes I_2 & 0 \end{bmatrix} < 0, \tag{18}$$

$$N_2 = \begin{bmatrix} Q_1+\tau\varepsilon_4\bar{E}_2+\tau\varepsilon_5\bar{E}_2 & \tau(\bar{L}\otimes B_1)^T R & \tau(\bar{L}\otimes B_1)^T R[(U_1^T E_1)\otimes B_1] & 0 & 0 \\ * & -\tau R & 0 & 0 & 0 \\ * & * & -\tau\varepsilon_2 I & 0 & 0 \\ * & * & * & -\tau R & \tau R[(U_1^T E_1)\otimes B_1] \\ * & * & * & * & -\tau\varepsilon_5 I \end{bmatrix} < 0. \tag{19}$$

B. Network with time-delay on switching topology

Theorem 2: Consider a directed network with switching topologies and time-delay τ. For the multi-agent system(7), consensus can be achieved with $\|T_{z\omega}\|_{L_2-L_\infty} < \gamma$ (a given index $\gamma > 0$), if the symmetry matrix $P > 0$, $Q > 0$, $R > 0$ and P, Q, $R \in \mathbb{R}^{2(n-1)\times 2(n-1)}$ and positive scalars ε_1, ε_2, ε_3, ε_4, ε_5, ε_6 satisfy

$$X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} < 0, \qquad \begin{bmatrix} -P & (U_1\otimes I_2)^T \\ * & -\gamma I \end{bmatrix} < 0, \tag{21}$$

where X_{11}, X_{12}, X_{22} respectively are (see(22)) here $\bar{L}_{\sigma(t)} = U_1^T L_{\sigma(t)} U_1$, $\bar{E}_{2\sigma(t)} = (U_1^T E_{2\sigma(t)}^T E_{2\sigma(t)} U_1)\otimes I_2$, $\bar{B}_2 = U_1^T \otimes B_2$, $Q_{1\sigma(t)} = -Q + \varepsilon_1 \bar{E}_{2\sigma(t)} + \tau\varepsilon_2 \bar{E}_{2\sigma(t)} + \tau\varepsilon_3 (\bar{L}_{\sigma(t)} \otimes B_1)^T (\bar{L}_{\sigma(t)} \otimes B_1) + \tau\varepsilon_4 \bar{E}_{2\sigma(t)} + \tau\varepsilon_6 \bar{E}_{2\sigma(t)}$, $R_1 = \tau(I_{n-1}\otimes \bar{A})^T R[(U_1^T E_{1\sigma(t)})\otimes \bar{B}_1]$ and $R_2 = \tau(\bar{L}_{\sigma(t)}\otimes B_1)^T R[(U_1^T E_{1\sigma(t)})\otimes B_1]$.

Proof: The proof of Theorem 2 can be straightly derived from Theorem 1. Meanwhile, it should be emphasized that all possible $I_{n-1}\otimes A - \bar{L}_{\sigma(t)}\otimes B_1$ should share a common Lypunov function $V(t)$. $\square$

Remark 1: For constant time-delay τ, the matrix equation (8) and (21) are both BMIs.

V. Simulation Results

In this section, we present some numerical simulations to illustrate the theoretical results obtained in the previous sections. These simulations are performed with four agents, whose initial conditions are all zeros. Fig.1 depicts four different network $\{G_a, G_b, G_c, G_d\}$. The switching mode starts at G_a and the order is $G_a \to G_b \to G_c \to G_d \to G_a$. Moreover, the topology of the multi-agent system switches every 0.01 s to the next states. It is assumed that the weights a_{ij} are all 1 and the uncertainty of each edge satisfies $|\Delta a_{ij}| \le 0.01$. Let the performance index $\gamma = 1$. In practical situation, external disturbances usually are unpredictable. Especially, let it be white noise $w(t)$. Then the disturbance $\omega(t) = [1\ -1\ 2\ 3]^T w(t)$.

We present the simulation results for the network with time-delay and fixed topology Fig.1 G_a. Let $K_1 = 5$ and $K_2 = 3$. Applying Theorem 1, we can obtain time-delay $\tau = 0.227$ s by solve the BMIs (8). Fig. 2 shows the position trajectories and velocity trajectories with the disturbance $\omega(t) = [1\ -1\ 2\ 3]^T w(t)$ and Fig. 4 shows the corresponding peak value trajectory of the controlled output $|z(t)|$ and energy trajectory disturbance signal $\omega(t)$. From these figures, consensus is asymptotically achieved when satisfying the performance $\|T_{zw}\|_{L_2-L_\infty} < \gamma$.

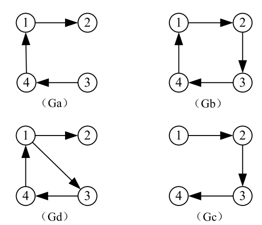

Fig. 1. Four directed graphs

Then the simulation results are given for the network with switching topology and time-delay. Applying Theorem 2, we can get time-delay $\tau = 0.203$ s. Fig. 3 describes the position trajectories and velocity trajectories. Fig. 5 depicts the corresponding peak value trajectory of the controlled output $|z(t)|$ and energy trajectory disturbance signal $\omega(t)$. Consensus is asymptotically achieved with performance $\|T_{zw}\|_{L_2-L_\infty} < \gamma$, too.

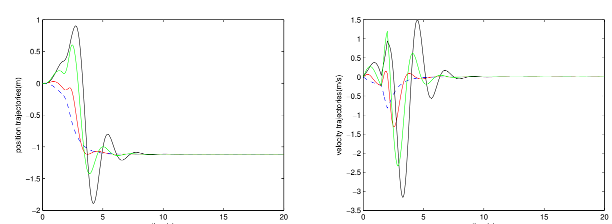

Fig. 2. Position trajectories and velocity trajectories of network with fixed topology and time-delay

VI. Conclusions

In this paper, we have employed the $L_2 - L_\infty$ control method to solve the consensus problem of multi-agent system subjected to external disturbances and parameter uncertainties with fixed and switching topologies. Neighbor-based

$$
\begin{aligned}
X_{11} &= \begin{bmatrix} P(I_{n-1}\otimes A-\bar{L}_{\sigma(t)}\otimes B_1)+(I_{n-1}\otimes A-\bar{L}_{\sigma(t)}\otimes B_1)^TP+Q+\tau(I_{n-1}\otimes A)^TR(I_{n-1}\otimes A) & 0 & P\bar{B}_2+\tau(I_{n-1}\otimes A)^TR\bar{B}_2 \\ * & Q_{1\sigma(t)} & -\tau(\bar{L}_{\sigma(t)}\otimes B_1)^TR\bar{B}_2 \\ * & * & -\gamma I+\tau\bar{B}_2^TR\bar{B}_2 \end{bmatrix}, \\
X_{12} &= \begin{bmatrix} P(\bar{L}_{\sigma(t)}\otimes B_1) & P[(U_1^TE_{1\sigma(t)})\otimes B_1] & \tau(I_{n-1}\otimes\bar{A})^TR & R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau(\bar{L}_{\sigma(t)}\otimes B_1)^TR & R_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{B}_2^TR[(U_1^TE_{1\sigma(t)})\otimes B_1] \end{bmatrix}, \\
X_{22} &= diag\left\{-\frac{R}{\tau},-\varepsilon_1 I,-\tau\varepsilon_3 R,-\tau\varepsilon_4 I,-\tau R,-\tau\varepsilon_2 I,\begin{bmatrix} -\tau R & \tau R(U_1^TE_{1\sigma(t)})\otimes B_1 \\ * & -\tau\varepsilon_5 I \end{bmatrix},-\tau\varepsilon_6 I\right\},
\end{aligned} \tag{22}
$$

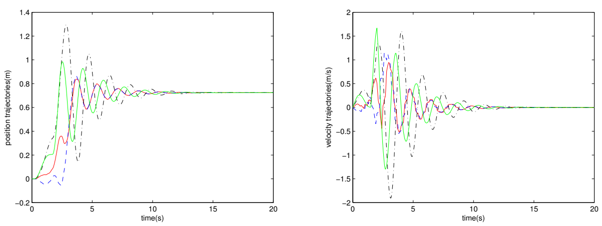

Fig. 3. Position trajectories and velocity trajectories of network with switching topology and time-delay

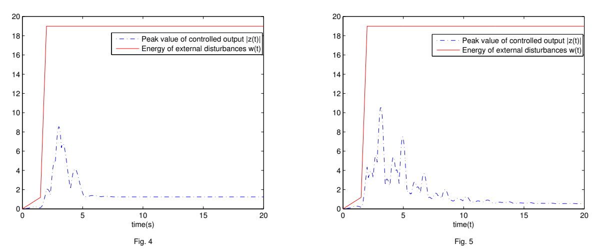

Fig. 4. Peak value trajectory of the controlled output $|z(t)|$ and energy trajectory disturbance signal $\omega(t)$ of network with fixed topology and time-delay

Fig. 5. Peak value trajectory of the controlled output $|z(t)|$ and energy trajectory disturbance signal $\omega(t)$ of network with switching topology and time-delay

control protocols with time-delay have been proposed for each agent. And some conditions are derived to ensure the consensus of multi-agent system with the desired L_2-L_∞ performance. Finally, numerical simulations are provided to show the effectiveness of our theoretical results. Further, the time-delay considered in this paper is constant and uniform, then time-varying delays or asymmetric time-delays can be also investigated in the future.

VII. ACKNOWLEDGMENTS

The authors gratefully acknowledge suggestions and comments from the reviewers.

REFERENCES

[1] R. W. Beard, J. Lawton and F. Y. Hadaegh, “A coordination architecture for spacecraft formation control,” *IEEE Trans. Contr. Sys. Tech.*, vol. 9, 2001, pp. 777-790.

[2] R. W. Beard, T. W. McLain, M. A. Goodrich *et al*., “Coordinated target assignment and intercept for unmanned air vehicles,” *IEEE Trans. on Robot. and Automat.*, vol. 18, 2002, pp. 911-922.

[3] G. Lafferriere, A.Williams, J. Caughman *et al*., “Decentralized control of vehicle formations,” *Sys. Contr. Lett.*, vol. 54, 2005, pp. 899-910.

[4] J. A. Stankovic, T. E. Abdelzaher *et al*., “Realtime communication and coordination in embedded sensor networks,” *Proceedings of the IEEE*, vol. 91, 2003, pp. 1002-1022.

[5] A. Tiwari and J. Fung, “Polyhedral Cone Invariance Applied to Rendezvous of Multiple Agents,” *in Proceedings of the 43rd IEEE Conference on Decision and Control*, 2004, Atlantis, Paradise Island, Bahamas, pp. 165-170.

[6] A. Jadbabaie, J. Lin and A.S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Trans. Automat. Control*, vol. 48, 2003, pp. 988-1001.

[7] L. Moreau, “Stability of multiagent systems with time-dependent communition links,” *IEEE Trans. Automat. Contr.*, vol. 50, 2005, pp. 169-182.

[8] D. B. Kingston and R. W. Beard, “Discrete-time average-consensus under switching network topologies,” *in Proceedings of the 2006 American Control Conference*, New York, IEEE, 2006, pp. 3551-3556.

[9] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Trans. Automatic Control*, vol. 49, 2004, pp. 1520-1533.

[10] W. Ren and R.W. Beard, “Consensus seeking in multi-agent systems under dynamically changing interaction topologies,” *IEEE Trans. Automat. Control*, vol. 50, 2005, 655-661.

[11] Y. Hong, L. Gao, D. Cheng *et al*., “Tracking control for multi-agent consensus with an active leader and variable topology,” *Automation*, vol. 42, 2006, pp. 1177-1182.

[12] W. Ren, “On consensus algorithms for double-integrator dynamics,” *IEEE Transactions on Automatic Control*, vol. 53, 2008, pp. 1503-1509.

[13] P. Lin, Y. Jia and L. Li, “Distributed robust H_∞ consensus control in directed networks of agents with time-delay, *Systems and control letters*,” vol. 57, 2008, pp. 643-653.

[14] H. Tanner, A. Jadbabaie and G. Pappas, “Stable flocking of mobile agents, part II: Dynamic topology,” *in Proceedings of IEEE Conference on Decision and Control*, Hawaii, 2003, pp. 2016-2021.

[15] Y. Liu, Y. Jia, “H_∞ approach to the consensus problem of multi-agent systems”, *Proceedings of the 29th Chinese Control Conference*, Beijing, China, 2010, pp. 4542-4547.

[16] C. E. De Souza and X. Li, “Delay-dependent robust H_∞ control of uncertain linear state-delayed systems,” *Automatica*, vol. 35, 1999, pp. 1313-1321.

[17] Y. M. Jia, *Robust H_∞ control*, Science Press, Beijing, 2007, pp. 144-145.