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A Modified SDRE Design for the NonLinear Benchmark Problem with a Stability Proof

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Abstract— **An approximation to the SDRE control law for the non-linear benchmark problem is developed, which is a solution of an optimal control problem. Although the SDRE solution of the non-linear benchmark problem was derived using the tools of optimal control, the SDRE control law is not the solution of any optimal control problem. The new control law is obtained by approximating the existing SDRE control law in the least square sense from the class of control laws that are of the form required for them to be solutions of optimal control problems. Using the sum of squares tools the optimality criterion is calculated. As a byproduct, the stability of the approximating control law is established over a subset of the non-linear benchmark problem's state space.**

I. INTRODUCTION

The benchmark problem for nonlinear control design was introduced in [1]. A solution for the problem, using the state dependent Riccati equation (SDRE) method was presented in [2]. The paper showed by means of numerical simulations that the SDRE based control law stabilized the system, essentially over the entire state space of interest. The paper further showed by comparing the state space trajectories of the closed loop system with the numerically obtained open loop optimal trajectories that the use of the state dependent Riccati equation method results in a reasonable approximation to the optimal control law. The question may be asked: Is there an approximation to the SDRE control law which is the solution of an optimal control problem? The present paper derives such an approximation. If it can be shown that the criterion function, associated with the approximation of the SDRE controller is positive, then this implies the stability of closed loop system [3].

Nearly all known approximate methods for the control of nonlinear systems have been applied to the benchmark problem. A selection of the methods is: LPV based method, [4]; backstepping method [5], and [6]; LMI based [7]; a controller based on a series expansion of the HJBI equation, [8]; fuzzy control based approach [9]; and an experimental implementation of some control laws [10].

The paper relies on the well developed theory of the inverse optimal control problem, and the theory of positive polynomials. The inverse optimal control problem is concerned with the following: given a control law, find, if possible, a criterion function such that the given control law is a solution of the optimal control problem with the calculated criterion function. For the linear case, conditions for the existence of a quadratic criterion function may be found in [11], and [12] while a linear matrix inequality based calculation was presented in [13]. The inverse optimal control problem was extended to the nonlinear case in [14].

The use of positive polynomials in control was presented in [15], [16], and [17]. Many problems, including the construction of Lyupanov functions, in nonlinear control can be formulated as search problems for polynomials that are globally positive. Proving the global positivity of a polynomial is a difficult problem, but proving that a polynomial can be expressed as a sum of squares expression (SOS) is tractable. The Matlab add on software SOSTOOLS [18] (sum of squares optimization toolbox for Matlab) provides an easy to use interface. The program reformulates sum of squares programs as semidefinite programs, which can be solved very efficiently by several readily available programs, for example by SeDuMi [19].

The paper is organized as follows: In section II the benchmark problem for nonlinear control design is presented. In the following section III, the problem of finding an approximation for the SDRE based control of [2], which is also the solution of an optimal control problem is introduced. In the subsequent section IV the sum of squares formulation of the approximation problem is presented. The results of the approximation are compared to the SDRE solutions, by means of numerical simulations in section VI. The paper closes with summary and conclusion section.

II. THE NONLINEAR BENCHMARK PROBLEM AND ITS SDRE SOLUTION

Here the benchmark problem [1] is briefly reviewed. The system as shown in figure 1, consists of a cart of mass M constrained to move horizontally, and connected to a fixed wall by a spring of stiffness k. A proof mass, with mass m, and moment of inertia I about its center of mass is free to rotate about a point fixed to the cart. The distance from the proof mass pivot point to its center of mass is e. The control signal is the torque, N applied about the proof mass pivot point. There is in addition a horizontal disturbance force F acting on the cart. The purpose of the controller is to stabilize the system, so that it exhibits good settling behavior and to counteract the external disturbance. Let q and $\dot{q}$ represent the translational position and velocity of the cart, and let θ and $\dot{\theta}$ denote the angular position and velocity of the proof mass. When $\theta = 0$ the proof mass is perpendicular to the motion of the cart. The equations of motion of the system are

$$\begin{aligned}(M+m)\ddot{q} + kq &= -me(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + F \quad (1)\\ (I+me^2)\ddot{\theta} &= -me\ddot{q}\cos\theta + N \quad (2)\end{aligned}$$

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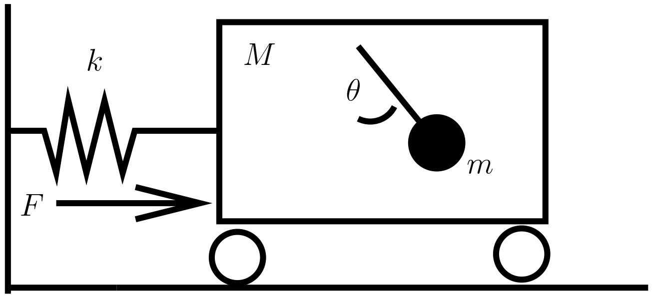

Fig. 1. The nonlinear benchmark problem

With the normalizations,

$$\xi = \sqrt{\frac{M+m}{I+me^2}}q \quad (3)$$

$$\tau = \sqrt{\frac{k}{M+m}}t \quad (4)$$

$$u_N = \frac{M+m}{k(I+me^2)}N \quad (5)$$

$$w = \frac{1}{k}\sqrt{\frac{M+m}{I+me^2}}F \quad (6)$$

the equations of motion become,

$$\ddot{\xi}+\xi = \epsilon\left(\dot{\theta}^2\sin\theta-\ddot{\theta}\cos\theta\right)+w \quad (7)$$

$$\ddot{\theta} = -\epsilon\ddot{\xi}\cos\theta+u_N \quad (8)$$

where all the differentiations are with respect to the normalized time, τ. The parameter ϵ,

$$\epsilon = \frac{me}{\sqrt{(I+me^2)(M+m)}} \quad (9)$$

represents the coupling between the rotational and translational degrees of freedom.

The first step in the SDRE solution of the benchmark problem [2] is to define new variables, $[x_1,x_2,x_3,x_4]' = [\xi,\dot{\xi},\sin\theta,\dot{\theta}\cos\theta]$. Once this is accomplished the non-dimensional equations written in first order form are,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})u+\boldsymbol{d}(\boldsymbol{x})w \quad (10)$$

where

$$u = u_N\cos\theta \quad (11)$$

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_2 \\ \frac{-x_1(1-x_3^2)+\epsilon x_3x_4^2}{(1-x_3^2)\Delta} \\ x_4 \\ \frac{\epsilon(1-x_3^2)^2x_1-x_3x_4^2}{(1-x_3^2)\Delta} \end{bmatrix} \quad (12)$$

$$\boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ -\frac{\epsilon}{\Delta} \\ 0 \\ \frac{1}{\Delta} \end{bmatrix} \quad (13)$$

TABLE I

BENCHMARK PROBLEM PARAMETER VALUES

Description	Parameter	Value	Units
Cart mass	M	1.3608	Kg
Arm mass	m	0.096	Kg
Arm eccentricity	e	0.0592	m
Arm inertia	I	0.0002175	Kg-m^2
Spring stiffness	k	186.3	N/m
Coupling parameter	ϵ	0.200	—-

$$\boldsymbol{d}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ \frac{1}{\Delta} \\ 0 \\ \frac{-\epsilon(1-x_3^2)}{\Delta} \end{bmatrix} \quad (14)$$

$$\Delta = 1-\epsilon^2+\epsilon^2x_3^2 \quad (15)$$

The values of the parameters of the problem are shown in table I.

A. The SDRE Solution

Next, the SDRE solution from [2] is reviewed. The solution of a nonlinear control problem using the SDRE method consists of the following steps:

1) Define a suitable criterion function
2) Express the dynamics in state dependent coefficient form.
3) Solve the resulting LQ type control problem point wise for each value of $\boldsymbol{x}$.
4) The controller is obtained by the usual LQ type formula for the optimal controller, but now all the coefficients depend on the state $\boldsymbol{x}$.

The criterion function used in the solution of the benchmark problem was

$$J = \frac{1}{2}\int_0^\infty \left[\boldsymbol{x}'\boldsymbol{Q}(\boldsymbol{x})\boldsymbol{x}+u'R(\boldsymbol{x})u\right]\,dt \quad (16)$$

where

$$\boldsymbol{Q}(\boldsymbol{x}) = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1/(1-x_3^2) \end{bmatrix} \quad (17)$$

and

$$R(\boldsymbol{x}) = \frac{1}{1-x_3^2} \quad (18)$$

To express the dynamics in state dependent form, (10) is written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x}+\boldsymbol{B}(\boldsymbol{x})u \quad (19)$$

with

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{\Delta} & 0 & \frac{\epsilon x_4^2}{(1-x_3^2)\Delta} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon-\epsilon x_3^2}{\Delta} & 0 & 0 & -\frac{x_3x_4}{(1-x_3^2)\Delta} \end{bmatrix} \quad (20)$$

and

$$\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) \quad (21)$$

For the purpose of obtaining the SDRE solution the $\boldsymbol{x}$ dependencies in the matrices $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{B}(\boldsymbol{x})$, $\boldsymbol{Q}(\boldsymbol{x})$, and $R(\boldsymbol{x})$ are ignored, and the resulting optimal control problem is solved for each value of the state variable $\boldsymbol{x}$. The solution is obtained by solving the Riccati equation for each value of $\boldsymbol{x}$

$$\boldsymbol{A}'(\boldsymbol{x})\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{P}\boldsymbol{B}(\boldsymbol{x})R^{-1}(\boldsymbol{x})\boldsymbol{B}'(\boldsymbol{x})\boldsymbol{P} + \boldsymbol{Q}(\boldsymbol{x}) = 0 \tag{22}$$

The resulting Riccati equation solution is a function of the state $\boldsymbol{x}$. The feedback controller is

$$u = -R^{-1}(\boldsymbol{x})\boldsymbol{B}'(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\boldsymbol{x} \tag{23}$$

The resulting controller has the same structure as the LQ controller, except that all the coefficients are state dependent. The SDRE controller, by its construction, ensures that there is a neighborhood near the equilibrium value of $\boldsymbol{x} = 0$ such that the closed loop system is asymptotically stable, provided certain conditions are met, but there is no a priory theory promising global stability. In practice, in many cases the resulting controller is stable for large deviations from the equilibrium value.

III. THE APPROXIMATION

An approximation $\tilde{u}(\boldsymbol{x})$ to the SDRE controller (23) is sought such that $\tilde{u}(\boldsymbol{x})$ is a solution of a HJBI. For this purpose a positive definite cost-to-go function $V(\boldsymbol{x})$ is required such that it satisfies the equation

$$\min_{u}\left\{u'\frac{R}{2}(\boldsymbol{x})u + \nabla V'(\boldsymbol{x})\left[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u\right]\right\} + \tilde{Q}(\boldsymbol{x}) = 0 \tag{24}$$

for some positive $\tilde{Q}(\boldsymbol{x})$, which may be differ from $\boldsymbol{x}'\boldsymbol{Q}(\boldsymbol{x})\boldsymbol{x}/2$ in (16). The indicated minimization can be readily performed to obtain

$$\tilde{u} = -R^{-1}(\boldsymbol{x})\boldsymbol{g}'(\boldsymbol{x})\nabla V(\boldsymbol{x}) \tag{25}$$

which when substituted into the HJBI (24) yields

$$\tilde{Q}(\boldsymbol{x}) - \nabla V'(\boldsymbol{x})\boldsymbol{g}\frac{R^{-1}}{2}\boldsymbol{g}'\nabla V + \nabla V'(\boldsymbol{x})\boldsymbol{f} = 0 \tag{26}$$

The function $\tilde{Q}(\boldsymbol{x})$ is considered an unknown, but is required to be positive for the problem to make sense. The function $R(\boldsymbol{x})$ is taken as known from (18).

In general there does not exist a function $V(\boldsymbol{x})$ such that the SDRE controller (23) is expressible as

$$u(\boldsymbol{x}) = -R^{-1}(\boldsymbol{x})\boldsymbol{g}'(\boldsymbol{x})\nabla V(\boldsymbol{x}) \tag{27}$$

as this would require that

$$\frac{\partial\left(\boldsymbol{P}(\boldsymbol{x})\boldsymbol{x}\right)_i}{\partial x_j} = \frac{\partial\left(\boldsymbol{P}(\boldsymbol{x})\boldsymbol{x}\right)_j}{\partial x_i} \tag{28}$$

which is a generalization to n dimensions of the well known condition that a vector function is expressible as a gradient of a scalar function if and only if its curl is equal to zero. (If $\boldsymbol{P}(\boldsymbol{x})$ is symmetric and has no $\boldsymbol{x}$ dependence, then the conditions *are* satisfied.)

Ideally the approximation problem should be combined with the calculation of $V(\boldsymbol{x})$ and $\tilde{Q}(\boldsymbol{x})$ in a single step. This however leads to a difficult problem. A necessary condition that $\tilde{u}(\boldsymbol{x})$ should satisfy is that it should be expressible in the form (25). In this paper the approach taken is first to find a polynomial function $V_0(\boldsymbol{x})$, which when substituted for $V(\boldsymbol{x})$ in (25) results in a controller $\tilde{u}(\boldsymbol{x})$ which approximates the SDRE controller $u(\boldsymbol{x})$ in the least squares sense. The approximation in the least squares sense was chosen in the interest of simplicity and tractability. The approximation requirement defines the cost to go function $V(\boldsymbol{x})$ only partially so that some freedom remains to satisfy the positivity constraints. In the second step the cost to go function $V(\boldsymbol{x})$ is calculated, such that $\tilde{u}(\boldsymbol{x})$ is unchanged, but the positivity requirements are satisfied. Although there is no a priory reason to assume that a suitable cost to go function which satisfies the HJBI equation exists, the calculations show that specifically for the nonlinear benchmark problem a suitable cost to go function $V(\boldsymbol{x})$ can be found.

There is a certain similarity here to the well known inverse optimal control problem. The solution used here may be viewed as an extension of the LMI based conditions in [13], p 147 to polynomial systems.

Here $V_0(\boldsymbol{x})$ is restricted to be a polynomial, and in fact because $\boldsymbol{f}$ is an odd function, and $\boldsymbol{g}$ is an even function of $\boldsymbol{x}$, $V_0(\boldsymbol{x})$ may be restricted to contain only terms whose sum of exponents are even. The constraints $V(\boldsymbol{x}) > 0$ and $\tilde{Q}(\boldsymbol{x}) > 0$ in this paper are replaced by an easily tested sufficient condition for the positivity of a polynomial is used, namely that the polynomial is expressible as a sum of squares (SOS). This sufficient condition is presented in the next section.

IV. THE SOS FORMULATION OF THE APPROXIMATION

At this point the approximating $\tilde{u}(\boldsymbol{x})$ controller has been obtained. The next problem addressed here is:

- Find polynomial functions, $V(\boldsymbol{x}) > 0$ and $\tilde{Q}(\boldsymbol{x}) > 0$ such that the functions satisfy (26), and
- constrain $V(\boldsymbol{x})$ so that the controller $\tilde{u}(\boldsymbol{x})$ in (25) remains unchanged by requiring

$$R^{-1}(\boldsymbol{x})\boldsymbol{g}'(\boldsymbol{x})\nabla V(\boldsymbol{x}) = R^{-1}(\boldsymbol{x})\boldsymbol{g}'(\boldsymbol{x})\nabla V_0(\boldsymbol{x}) \tag{29}$$

A. SOS Methods

The purpose of this section is to provide a very brief introduction to the main ideas of SOS methods. This is done via an example from [15].

Suppose that it is required to establish that the polynomial $P(\boldsymbol{x})$ is positive semidefinite

$$P(\boldsymbol{x}) = 4x_1^2 + 4x_1^3x_2 - 7x_1^2x_2^2 - 2x_1x_2^3 + 10x_2^4 \tag{30}$$

If there exists a matrix $\boldsymbol{S} > 0$ such that

$$P(\boldsymbol{x}) = \boldsymbol{z}'\boldsymbol{S}\boldsymbol{z} \tag{31}$$

where

$$\boldsymbol{z}' = \begin{bmatrix} x_1^2 & x_2^2 & x_1x_2 \end{bmatrix} \tag{32}$$

TABLE II

ELLIPSOID AXES LENGTHS

Axis	length
l_1	1.5
l_2	1.4
l_3	1
l_4	1.5

then obviously $P(\boldsymbol{x}) \geq 0$. A necessary condition for the equality of (30) and (31) may be derived by equating coefficients. The derived condition for the equality results in a matrix

$$\boldsymbol{S} = \begin{bmatrix} 4 & -\lambda & 2 \\ -\lambda & 10 & -1 \\ 2 & -1 & -7+\lambda \end{bmatrix} \tag{33}$$

for all values of λ. Finding a λ such that $\boldsymbol{S} \geq 0$ is a semidefinite program, which is solvable using one of the very efficient codes available. In this case $\lambda = 6$ results in $\boldsymbol{S} \geq 0$, proving that (30) is positive semidefinite.

B. The Details of the Approximation

On examining the SDRE solution in [2], the region of interest for the problem may be confined to the ellipsoid

$$\mathcal{R}(\boldsymbol{x}, s) = \frac{x_1^2}{l_1^2} + \frac{x_2^2}{l_2^2} + \frac{x_3^2}{l_3^2} + \frac{x_4^2}{l_4^2} - s^2 \leq 0 \tag{34}$$

The lengths of the ellipsoid axes are shown in table II. When $s = 1$ the entire region of interest is covered; smaller values of s represent subsets of the region of interest. In this paper $s < 1$ for all cases studied.

In general, the least squares approximation only partially defines $V(\boldsymbol{x})$. If $V(\boldsymbol{x})$ is a second order polynomial, there are 10 coefficients to determine, whereas there are only 4 coefficients in $\tilde{u}(\boldsymbol{x})$; if a fourth order approximation is employed, 45 coefficients are needed to determine in $V(\boldsymbol{x})$, but there are only 24 coefficients in a third order polynomial $\tilde{u}(\boldsymbol{x})$.

Since $R^{-1}(\boldsymbol{x}) \neq 0$, (29) may be simplified to

$$\boldsymbol{g}'(\boldsymbol{x})\nabla V(\boldsymbol{x}) = \boldsymbol{g}'(\boldsymbol{x})\nabla V_0(\boldsymbol{x}) \tag{35}$$

Substituting (35) in (26),

$$\tilde{Q}(\boldsymbol{x}) - \nabla V_0'(\boldsymbol{x})\boldsymbol{g}\frac{R^{-1}}{2}\boldsymbol{g}'\nabla V_0 + \nabla V'(\boldsymbol{x})\boldsymbol{f} = 0 \tag{36}$$

which is now linear in $\nabla V(\boldsymbol{x})$. At this point, the problem is to find positive $V(\boldsymbol{x})$ and $\tilde{Q}(\boldsymbol{x})$ in accordance with (36) and (35). The equality (36) contains quotients of polynomials. To make use of SOS methods, it needs to be reformulated as a polynomial. To accomplish this, the denominator of each term in the dynamics of the benchmark problem (10) is expressed explicitly,

$$\dot{\boldsymbol{x}} = \frac{1}{(1-x_3^2)\Delta}\boldsymbol{F}(\boldsymbol{x}) + \frac{1}{\Delta}\boldsymbol{G}(\boldsymbol{x})u + \frac{1}{\Delta}\boldsymbol{D}(\boldsymbol{x})w \tag{37}$$

where

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{F}(\boldsymbol{x})}{(1-x_3^2)\Delta} \tag{38}$$

$$\boldsymbol{g}(\boldsymbol{x}) = \frac{\boldsymbol{G}(\boldsymbol{x})}{\Delta} \tag{39}$$

$$\boldsymbol{d}(\boldsymbol{x}) = \frac{\boldsymbol{D}(\boldsymbol{x})}{\Delta} \tag{40}$$

Substituting (38), and (39) into (36) yields,

$$\tilde{Q}(\boldsymbol{x}) - \nabla V_0'(\boldsymbol{x})\frac{\boldsymbol{G}}{\Delta}\frac{R^{-1}}{2}\frac{\boldsymbol{G'}}{\Delta}\nabla V_0(\boldsymbol{x}) + \frac{\nabla V'(\boldsymbol{x})\boldsymbol{F}}{(1-x_3^2)\Delta} = 0 \tag{41}$$

Since in the largest region of interest, in this paper, $0 < s < 1$, the inequality $(1 - x_3^2)\Delta^2 > 0$ holds, multiply by this factor to obtain

$$\begin{aligned} \mathbb{Q}(\boldsymbol{x}) = {} & \frac{1}{2}\nabla V_0'(\boldsymbol{x})\boldsymbol{G}\boldsymbol{G'}\nabla V_0(\boldsymbol{x})(1-x_3^2)^2 \\ & -\frac{\Delta}{2}\left[\nabla V'(\boldsymbol{x})\boldsymbol{F} + \boldsymbol{F}'\nabla V(\boldsymbol{x})\right] \end{aligned} \tag{42}$$

where

$$\mathbb{Q}(\boldsymbol{x}) = \tilde{Q}(\boldsymbol{x})(1-x_3^2)\Delta^2 \tag{43}$$

and the explicit form of $R(\boldsymbol{x})$ from (18) was used.

The positivity requirement $\tilde{Q}(\boldsymbol{x})$ is now replaced by one for $\mathbb{Q}(\boldsymbol{x})$. This positivity needs to be enforced only over the region $\mathcal{R}(\boldsymbol{x}, s_t) \leq 0$ (34). A sufficient condition for this is the existence of a positive polynomial $\lambda(\boldsymbol{x})$, such that

$$\mathbb{Q}(\boldsymbol{x}) + \lambda(\boldsymbol{x})\mathcal{R}(\boldsymbol{x}, s_t) \geq 0 \tag{44}$$

The SOS problem consists of

1) The polynomials
   a) $V(\boldsymbol{x}) - \epsilon_V||\boldsymbol{x}||^2$,
   b) $\lambda(\boldsymbol{x})$,
   c) $\mathbb{Q}(\boldsymbol{x}) - \epsilon_Q||\boldsymbol{x}||^2 + \lambda(\boldsymbol{x})\mathcal{R}(\boldsymbol{x}, s_t)$

   are SOS polynomials,
2) the equality constraint (35),

which is now replaced by

$$\boldsymbol{G}'(\boldsymbol{x})\nabla V(\boldsymbol{x}) = \boldsymbol{G}'(\boldsymbol{x})\nabla V_0(\boldsymbol{x}) \tag{45}$$

and $\epsilon_V = 0.05$ and $\epsilon_Q = 0.01$ are small positive numbers to ensure the positive definiteness of $V(\boldsymbol{x})$ and $\mathbb{Q}(\boldsymbol{x})$. These requirements translate directly to an SOS program.

V. RESULTS

Three cases were studied. In the first case the SDRE solution (23) was used but with $\boldsymbol{x} = 0$ in all the coefficient matrices,

$$u = -R^{-1}(\boldsymbol{0})\boldsymbol{B}'(\boldsymbol{0})\boldsymbol{P}(\boldsymbol{0})\boldsymbol{x} \tag{46}$$

This is equivalent to linearizing the benchmark problem about $\boldsymbol{x} = 0$ and using the LQ solution of the linearized problem. The second case a quadratic approximation is used for $V(\boldsymbol{x})$ in generating $\tilde{u}(\boldsymbol{x})$; while in the third case a quartic approximation was used. The results are summarized in table III. The least squares approximation relied on the use of random values for $\boldsymbol{x}$ uniformly distributed in the ellipsoid $\mathcal{R}(\boldsymbol{x}, s_a)$ for the second and third cases. The s_a values used for the second and third cases are shown in column 3 in table III. The last column of table III shows s_t, which defines the size of the region where there was a feasible solution for the SOS program. This parameter also defines the size of

TABLE III

RESULTS

Case	Approximation order	s_a	s_t
1	1	—-	0.3
2	1	0.5	0.5
3	3	0.75	0.625

the invariant region for the control $\tilde{u}(\boldsymbol{x})$. For cases 1 and 2 $\lambda(\boldsymbol{x})$ was a 6th order polynomial, and for the third case it was an 8th order polynomial.

The control laws actually used in the three cases were

- case 1

$$\tilde{u}(\boldsymbol{x}) = 2.549x_1 + 1.3535x_2 - 0.30358x_3 - 0.99225x_4 \tag{47}$$

- case 2

$$\tilde{u}(\boldsymbol{x}) = 2.6478x_1 + 1.2388x_2 - 0.25616x_3 - 1.2605x_4 \tag{48}$$

- case 3

$$\begin{aligned}\tilde{u}(\boldsymbol{x}) = &-0.0066459x_1^3 - 0.0028213x_1^2x_2\\ &+0.013489x_1^2x_3 - 0.00058937x_1^2x_4 - 0.020082x_1x_2^2\\ &-0.0043404x_1x_2x_3 - 0.0066176x_1x_2x_4 + 1.4x_1x_3^2\\ &-0.68271x_1x_3x_4 + 0.48273x_1x_4^2 + 2.5847x_1\\ &-0.0013398x_2^3 - 0.0053901x_2^2x_3\\ &+0.0074509x_2^2x_4 + 1.1873x_2x_3^2\\ &+2.7761x_2x_3x_4 - 2.3855x_2x_4^2 + 1.3747x_2\\ &+0.078168x_3^3 - 2.0677x_3^2x_4 + 0.49352x_3x_4^2\\ &-0.28955x_3 - 1.1507x_4^3 - 0.95685x_4\end{aligned} \tag{49}$$

Although the control laws for cases one and two do not differ markedly, the use of the least squares approximation to calculate the controller based on the SDRE solution results in a substantially bigger invariant set for case two. On the other hand the use of a third order approximation in case three results in only a small increase in the invariant set. The resulting functions $V(\boldsymbol{x})$ and $\mathbb{Q}(\boldsymbol{x})$ are fourth order and tenth order polynomials in four variables and are not included in the paper.

The question may be asked: How tight are the bounds calculated for the invariant sets? This is answered in the next section, where the state trajectories of the system using the approximated control laws are compared with the state trajectories corresponding to the use of the full SDRE controller.

VI. SIMULATION RESULTS

The third order approximation to the SDRE derived control law was used to simulate the closed loop performance in the benchmark problem. The simulation results, together with the corresponding results from the SDRE based control law are shown in the figures 2–5 Figures 2 and 3 show the responses of the approximate and the SDRE controller when the initial state is $[1, 0.5, 0, 0]$, which corresponds to

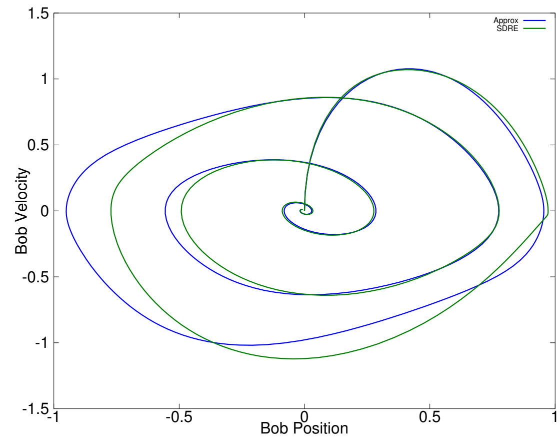

Fig. 2. Pendulum bob phase plane response, initial condition 1

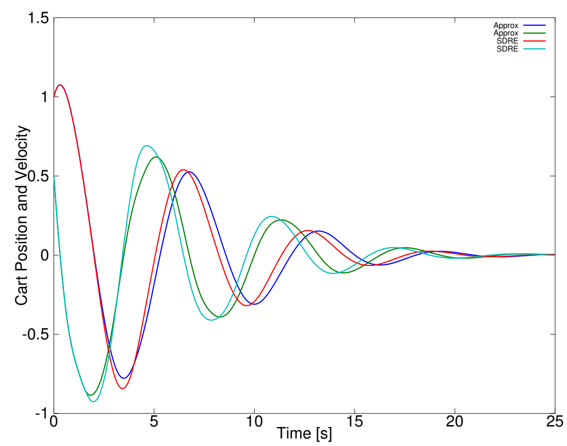

Fig. 3. Cart position and velocity, initial condition 1

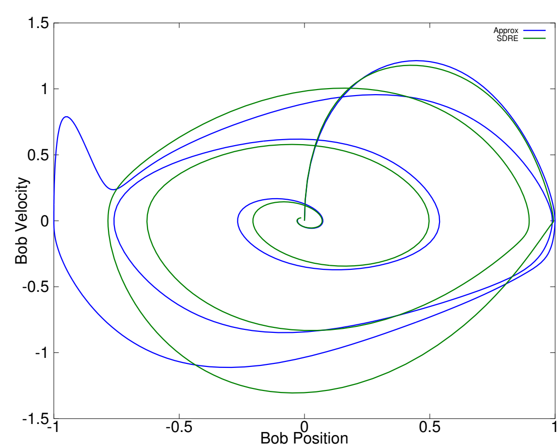

Fig. 4. Pendulum bob phase plane response, initial condition 2

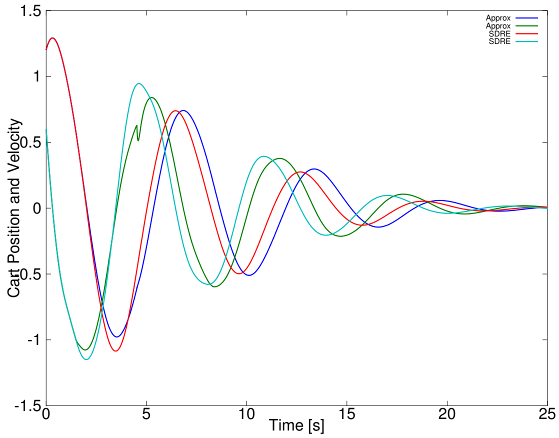

Fig. 5. Cart position and velocity, initial condition 2

$s = 0.75$. In this case the cart position and velocity follow closely the corresponding SDRE position and velocity, while the bob position and velocity approximate the corresponding SDRE trajectory a bit less well. According to the SOS based stability calculation this initial point is outside the invariant set so that the SOS based stability calculation *is* conservative. For the second set of simulations, the initial state was $[1.2, 0.6, 0, 0]$, for an $s = 0.9$. In this case there is still reasonable agreement in the cart position and velocity plot in figure 5, but the bob position and velocity phase plot does show some deviation from the corresponding paths in the SDRE case. In assessing the tightness of the stability bound it should be noted that the paper set out to calculate an approximation to the SDRE controller, which is a solution to a HJBI equation. The calculated running cost penalty function $\tilde{Q}(\boldsymbol{x})$ was evaluated outside the region indicated by the parameter s_t in table III, and it was found that $\tilde{Q}(\boldsymbol{x})$ takes on negative values for some values of $\boldsymbol{x}$ corresponding to $s_t = 0.7$, so as far as the problem that was set out in this paper there is only a small amount of conservatism.

VII. CONCLUSION AND SUMMARY

An approximation has been derived for the SDRE control law for the non-linear benchmark problem, that is also the solution of an optimal control problem. The approximated control law is obtained by approximating the SDRE control law in the least squares sense from among the class of control laws that are solutions of optimal control problems. For the optimal control problem to make sense the criterion function was required to be positive. The positivity was established using the sum of squares relaxation of the positivity of a polynomial. This then implies the asymptotic stability of the controlled system. The approximated controller was compared to the SDRE controller by means of a closed loop simulation, and it was shown that it is a reasonable approximation to the SDRE based controller. This then shows that at least inside a subset of the original benchmark problem an approximation of the SDRE controller *can* be found such that it is a solution of an optimal control problem, and it is possible to calculate a positive criterion function for this optimal control problem. The simulation results show that the calculated invariant set under approximates the actual invariant set.

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