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Absolute stability and stabilization of 2D Roesser Systems with Nonlinear Output Feedback

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Abstract— **This paper considers 2D systems described by the discrete Roesser model with linear dynamics in the forward path and a feedback path containing a memoryless, possibly time-varying, nonlinearity. Based on the extension of absolute stability theory to this class of systems, sufficient conditions for absolute p-stability are obtained and for the particular case of $p = 2$ linear matrix inequality based tests for this property are obtained, together with an algorithm to design a stabilizing nonlinear control law. The extension of these results to 2D discrete systems described by the Roesser model with Markovian jumps is also given. A numerical example to demonstrate the applicability and effectiveness of these new results concludes the paper.**

I. Introduction

Multidimensional systems propagate information in $n > 1$ independent directions but in this paper attention is restricted to the 2D case where the dynamics evolve over the right-upper quadrant of the associated plane. The study of 2D systems is motivated by many applications in, for example, image and signal processing and also by systems theoretic questions that cannot be solved by direct extension of standard, or 1D, theory. In terms of models for the dynamics, there is a much wider variety of signals possible in multidimensional systems where, for example, information propagation could be functions of discrete variables in both directions, of continuous variables in both directions, or a discrete variable in one direction and continuous in the other.

Consider the case when information propagation in both directions is a function of a discrete variable, for which there are two extensively studied state-space models. The Roesser model [1] defines a state vector for each direction of information propagation whereas the Fornasini-Marchesini model [2] uses a single state vector. Repetitive processes [3] also have a 2D systems structure but information propagation in one of the two directions only occurs over a finite duration. In control systems terms, repetitive processes do provide physical applications, such as iterative learning control, where a 2D systems approach can be applied, and this area has recently seen experimental verification studies [4].

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Stability analysis and control law design for 2D discrete linear systems has received considerable attention in the literature, including the case when there is uncertainty associated with the process model. A Linear matrix Inequality (LMI) approach to robust stabilization has also been extensively studied in, for example, [5]. The vast majority of the results currently available on control related analysis of 2D linear systems require the application of a linear state control law and hence, unless all state vector entries are available for measurement, an observer will be required for implementation.

A significant proportion of the literature on the control of multidimensional systems is based on a linear plant model and implementation. The first part of this paper deals with the case where the process model is linear and the feedback path contains a memoryless (in general time-varying) nonlinearity, which depends on the physical characteristics of the feedback channel. In 1D control systems theory this form of nonlinearity has been intensively studied in the framework of absolute stability theory [6] using, for example, the Popov criteria and the Kalman-Yacubovich-Popov (KYP) lemma. This paper first extends the absolute stability approach to 2D systems described by the Roesser model, where the resulting control design algorithms can in at least one case of interest be computed using LMIs.

The second area addressed in this paper is 2D discrete linear systems described by the Roesser model where failures in operation can occur which is modeled as random switching. In particular, the failures are modeled by a state-space models with jumps in the parameter values and/or structure governed by a Markov chain with a finite set of states, often termed Markovian jump systems or systems with random structure [7], [8]. Results on the development of control theory for such systems, which address issues such as stability, optimal and robust control problems in the 1D case can be found in, for example, [9]. In [10] results obtained for 1D Markovian jump systems are extended to investigate the problems of stabilization via state feedback and H_∞ control of 2D discrete-time Markovian jump systems described by the Roesser model. The new results in this part of the paper are for the case when the feedback control system of the first part also has failures modeled by Markovian jumps.

Throughout this paper the notation $M > 0$ (respectively) $M < 0$ is used to denote a symmetric positive-definite (respectively negative-definite) matrix. Also $M \geq 0$ (respectively $M \leq 0$) is used to denote a symmetric positive (respectively negative) semi-definite matrix. Let y be an $m \times 1$ vector with elements y_k, $1 \leq k \leq m$. Then $|y| =$

$$\left(\sum_{k=1}^{m}|y_k|^2\right)^{\frac{1}{2}} \text{ and } |y|^p = \left[\left(\sum_{k=1}^{m}|y_k|^2\right)^{\frac{1}{2}}\right]^p.$$

II. Problem formulation and preliminaries

The systems considered in this part of the paper are described by the 2D discrete linear systems Roesser state-space model

$$\begin{aligned}\begin{bmatrix} h(i+1,j) \\ v(i,j+1) \end{bmatrix} &= A\begin{bmatrix} h(i,j) \\ v(i,j) \end{bmatrix} + Bu(i,j),\\ z(i,j) &= C\begin{bmatrix} h(i,j) \\ v(i,j) \end{bmatrix},\end{aligned} \tag{1}$$

where $h \in \mathbb{R}^{n_h}$ and $v \in \mathbb{R}^{n_v}$ are the horizontal and vertical state vectors respectively and $u \in \mathbb{R}^{n_u}$ and $z \in \mathbb{R}^{n_z}$ are the input and output vectors, respectively. The boundary conditions are

$$h(0,j) = h_{0,j}, \;\; v(i,0) = v_{i,0}, \;\; i,j \in Z_+. \tag{2}$$

Also it is convenient to compatibly partition the matrices A, B and C in (1) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; C = [C_1 \; C_2].$$

Use will also be made of the local state vector for (1) defined as

$$x(i,j) = \begin{bmatrix} h^T(i,j) & v^T(i,j) \end{bmatrix}^T \tag{3}$$

In this paper, the subject of interest is the application of a nonlinear output feedback control law of the form

$$u = \varphi(z), \; \varphi(0) = 0, \tag{4}$$

to examples described by (1), where it is assumed that $\varphi(z)$ satisfies the inequality

$$z^TQz + 2z^TS\varphi(z) + \varphi^T(z)R\varphi(z) \geq 0, \; z \in \mathbb{R}^{n_z}, \tag{5}$$

where $Q = Q^T$, $R = R^T$ and S are matrices of compatible dimensions. This last inequality (5) is a standard constraint in absolute stability theory [6] for 1D systems, and absolute p-stability for the 2D system formed by applying (4) to (1) under (5) is defined as follows.

Definition 1: A 2D linear system (1) with control law (4) applied is said to be absolutely p-stable if for all boundary conditions (2) satisfying the inequality

$$\sum_{j=0}^{\infty}|h(0,j)|^p + \sum_{i=0}^{\infty}|v(i,0)|^p < \infty \tag{6}$$

and for all nonlinear functions $\varphi(z)$ satisfying (5)

$$|h(i,j)|^p + |v(i,j)|^p \to 0,$$

as $i+j \to \infty$.

The next step is to develop constructive conditions for absolute p -stability for the system described by (1), (4) and (5) in the form of computationally tractable stability test.

III. Stability Theory and Stabilization

Consider the following candidate Lyapunov function for the system considered in terms of the local state vector (3)

$$\begin{aligned} V(x(i,j)) &= V_1(h(i,j)) + V_2(v(i,j)), \; h \in \mathbb{R}^{n_h},\\ V_1(0) &= 0, \; V_2(0) = 0,\\ V_1(h(i,j)) &> 0, \; h \neq 0, \; V_2(v(i,j)) > 0, \; v \neq 0,\end{aligned} \tag{7}$$

with associated increment

$$\begin{aligned}\Delta V(x(i,j)) = V_1(h(i+1,j)) - V_1(h(i,j))\\ +V_2(v(i,j+1)) - V_2(v(i,j)).\end{aligned} \tag{8}$$

The following result is the 2D counterpart of a well known exponential stability theorem [6].

Theorem 1: Consider the 2D system described by (1) with control law (4) applied. Suppose also that, for all boundary conditions satisfying (6) and for all $\varphi(z)$ satisfying (5), there exist positive constants c_1, c_2, c_3 such that the Lyapunov function (7) and its associated increment (8) satisfy

$$\begin{aligned} c_1(|h(i,j)|^p + |v(i,j)|^p) \leq V(x(i,j))\\ \leq c_2(|h(i,j)|^p + |v(i,j)|^p),\end{aligned} \tag{9}$$

and

$$\Delta V(x(i,j)) \leq -c_3(|h(i,j)|^p + |v(i,j)|^p), \tag{10}$$

respectively. Then this system is absolutely p-stable.

Proof: It follows from (9) and (10) that

$$\begin{aligned} V_1(h(i+1,j)) &+ V_2(v(i,j+1))\\ -[V_1(h(i,j)) &+ V_2(v(i,j))]\\ \leq -c_3(|h(i,j)|^p &+ |v(i,j)|^p)\\ \leq -\frac{c_3}{c_2}[V_1(h(i,j)) &+ V_2(v(i,j))].\end{aligned} \tag{11}$$

Rearranging (11) yields

$$\begin{aligned} V_1(h(i+1,j)) &+ V_2(v(i,j+1))\\ \leq \lambda[V_1(h(i,j)) &+ V_2(v(i,j))],\end{aligned} \tag{12}$$

where $\lambda = \frac{c_2-c_3}{c_2}$. Also since $V_1(h)$ and $V_2(v)$ are positive definite, $0 < \lambda < 1$ (using (7), $c_2 > 0$ and (12)). Also for i from 0 to N and j from N to 0,

$$\begin{aligned} V_1(h(1,N)) &+ V_2(v(0,N+1))\\ \leq \lambda[V_1(h(0,N) &+ V_2(v(0,N)],\\ V_1(h(2,N-1)) &+ V_2(v(1,N))\\ \leq \lambda[V_1(h(1,N-1)) &+ V_2(v(1,N-1))],\\ V_1(h(3,N-2)) &+ V_2(v(2,N-1))\\ \leq \lambda[V_1(h(N,0)) &+ V_2(v(N,0))],\\ &\vdots\\ V_1(h(N+1,0)) &+ V_2(v(N,1))\\ \leq \lambda[V_1(h(N,0)) &+ V_2(v(N,0))].\end{aligned}$$

Adding both sides of these inequalities and using the trivial identity

$$\begin{aligned} V_1(h(0,N+1)) &+ V_2(v(N+1,0)) \\ = V_1(h(0,N+1)) &+ V_2(v(N+1,0)), \end{aligned}$$

gives

$$\begin{aligned} &\sum_{j=0}^{N+1} V_1(h(N+1-j,j)) + V_2(v(N+1-j,j)) \\ &\leq \lambda \left[\sum_{j=0}^{N} V_1(h(N-j,j)) + V_2(v(N-j,j)) \right] \\ &\qquad + V_1(h(0,N+1)) + V_2(v(N+1,0)) \end{aligned}$$

and hence

$$\begin{aligned} \sum_{j=0}^{N+1} V_1(h(N+1-j,j)) &+ V_2(v(N+1-j,j)) \\ \leq \sum_{j=0}^{N+1} \lambda^j (V_1(h(0,N+1-j)) &+ V_2(v(N+1-j,0)). \end{aligned}$$

Using (9) now gives

$$\begin{aligned} &\sum_{j=0}^{N} (|h(N-j,j)|^p + |v(N-j,j)|^p) \\ &\leq \alpha \left[\sum_{j=0}^{N} \lambda^j (|h(0,N-j)|^p + |v(N-j,0)|^p) \right], \end{aligned}$$

where $\alpha = c_2/c_1$, and hence

$$\begin{aligned} &\sum_{N=0}^{M} \sum_{j=0}^{N} (|h(N-j,j)|^p + |v(N-j,j)|^p) \\ &\leq \alpha[(1+\lambda+\ldots+\lambda^M)(|h(0,0)|^p + |v(0,0)|^p) \\ &\quad + (1+\lambda+\ldots+\lambda^{M-1})(|h(0,1)|^p + |v(1,0)|^p) \\ &\quad + (1+\lambda+\ldots+\lambda^{M-2})(|h(0,2)|^p + |v(2,0)|^p) \\ &\quad + \ldots + (|h(0,M)|^p + |v(M,0)|^p)] \\ &\leq \alpha[(1+\lambda+\ldots+\lambda^M) \sum_{N=0}^{M} (|h(0,N)|^p + |v(N,0)|^p)]. \end{aligned}$$

Since (6) holds and $0 < \lambda < 1$, it follows that the right-hand side of this last inequality is bounded as $M \to \infty$, hence the series on the left-hand side is convergent and

$$\sum_{j=0}^{N} (|h(N-j,j)|^p + |v(N-j,j)|^p) \to 0$$

as $N \to \infty$. Hence absolute p-stability holds. ■

Note that this last result is valid if the feedback path is time-varying, that is, $u = \varphi(i,j,z(i,j)),\ \varphi(i,j,0) = 0,\ i,j \in Z_+$.

IV. Absolute quadratic stability and stabilization

A. LMI based stability test

In the quadratic case when $p = 2$, the following result is obtained.

Theorem 2: Consider a system described by (1) with a control law (4) satisfying (5) applied. Then the resulting controlled system is absolutely quadratically stable if the following LMIs are feasible

$$\begin{bmatrix} A^TPA - P + C^TQC + \varepsilon I & A^TPB + C^TS \\ B^TPA + S^TC & B^TPB + R \end{bmatrix} \leq 0, \quad (13)$$
$$P = \mathrm{diag}[P_1\ P_2] > 0.$$

Proof: Choose the candidate Lyapunov function as the quadratic form

$$\begin{aligned} V(x(i,j)) = h^T(i,j)P_1h(i,j) + v^T(i,j)P_2v(i,j), \\ P_1 > 0,\ P_2 > 0. \quad (14) \end{aligned}$$

To guarantee absolute quadratic stability of the system formed by applying (4) to (1), the increment of (14) should be negative for all $\varphi(z)$, satisfying (5). Applying the S procedure [11], this condition holds if for some $\varepsilon > 0$

$$\begin{aligned} &\Delta V(x(i,j)) + z^T(i,j)Qz(i,j) + 2z^T(i,j)S\varphi(z(i,j)) \\ &+\varphi(z^T(i,j))R\varphi(z(i,j)) \leq -\varepsilon(|h(i,j)|^2 + |v(i,j)|^2). \quad (15) \end{aligned}$$

Calculating increment and completing the square gives that if (14) and (15) are valid then Theorem 1 in this case holds with $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P)$, $c_3 = \varepsilon$, where $\lambda_{\min}(P)$, $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of P, respectively. ■

B. Stabilization via nonlinear feedback

Consider first the application of the following linear feedback control law defined in terms of the local state vector (3) as

$$u(i,j) = -Kx(i,j) \quad (16)$$

to (1). Then the controlled linear system is internally stable [12] if there exists a matrix $H = \mathrm{diag}[H_1\ H_2] > 0$ such that

$$(A-BK)^T H(A-BK) - H < 0, \quad (17)$$

or, the LMIs with variables X and Y

$$\begin{aligned} \begin{bmatrix} X & (AX-BY)^T \\ AX-BY & X \end{bmatrix} &> 0, \quad (18) \\ X = \mathrm{diag}[X_1\ X_2] &> 0, \end{aligned}$$

are feasible. In which case a stabilizing control law matrix is given by $K = YX^{-1}$.

Suppose that the system (1) has been stabilized by design of (16) and consider the modified nonlinear control law

$$u = \varphi(z) - Kx(i,j),\ \varphi(0) = 0. \quad (19)$$

Then we have the following result.

Theorem 3: Suppose that the system (1) has been stabilized by application of the linear state feedback (16) and the following LMI is feasible

$$\left[\begin{array}{cc} A_c^T P A_c - P + \bar{Q} + \varepsilon I & A_c^T P B + \bar{S} \\ B^T P A_c + \bar{S}^T & B^T P B + R \end{array}\right] \leq 0, \quad (20)$$
$$P = \mathrm{diag}[P_1 \; P_2] > 0,$$

where $A_c = A - BK,\ \bar{Q} = C^T Q C - C^T S K - K^T S^T C + K^T R K,\ \bar{S} = C^T S - K^T R$ and ε is positive scalar. Then systems described by (1) are absolutely quadratically stable under the nonlinear control law (19).

Proof: The result follows immediately on forming the controlled system state-space model and application of Theorem 2. ■

Consider the case when the control law is linear and actuated only by the system output, that is,

$$u(i,j) = -F z(i,j) \quad (21)$$

and suppose that F satisfies LMI

$$\left[\begin{array}{cc} Q - SF - (SF)^T & F^T \\ F & -R^{-1} \end{array}\right] \geq 0. \quad (22)$$

Then the following corollary to Theorem 3 holds.

Corollary 1: Let $K = YX^{-1}$, where the pair (X, Y) is solution to (18) and LMI's (20) and (22) are feasible. Then system formed by applying the control law (21) to (1) is internally stable.

V. Roesser Models with Markovian Jumps and Nonlinear Feedback

The 2D systems considered in this section are described by the Roesser state-space model

$$\left[\begin{array}{c} h(i+1,j) \\ v(i,j+1) \end{array}\right] = A(\rho(i,j)) \left[\begin{array}{c} h(i,j) \\ v(i,j) \end{array}\right] + B(\rho(i,j)) u(i,j),$$
$$z(i,j) = C(\rho(i,j)) \left[\begin{array}{c} h(i,j) \\ v(i,j) \end{array}\right], \quad (23)$$

where $\rho(i,j)$ is an homogeneous Markov process with a finite set of states $\mathbb{N} = \{1, \ldots, \nu\}$ and transition probabilities given by

$$\begin{array}{rcl} \mathrm{P}[\rho(i,j+1) = l | \rho(i,j) = k] & = & \pi_{kl}, \\ \mathrm{P}[\rho(i+1,j) = l | \rho(i,j) = k] & = & \omega_{kl}. \end{array} \quad (24)$$

The remainder of the notation is the same as that for (1).

Consider a nonlinear output feedback control law of the form

$$u(i,j) = \varphi(z(i,j), \rho(i,j)), \; \varphi(0,r) = 0 \quad (25)$$

where it is assumed that $\varphi(z,r)$ satisfies the inequality

$$z^T Q(r) z + 2 z^T S(r) \varphi(z,r) + \varphi^T(z,r) R(r) \varphi(z,r) \geq 0, \; z \in \mathbb{R}^{n_z}, \text{ if } \rho(i,j) = r \quad (26)$$

and $Q(r) = Q^T(r),\ R(r) = R^T(r)$ and $S(r)$ are matrices of compatible dimensions.

Definition 2: A 2D system (23) with control law (25) applied is said to be stochastically absolutely p-stable if for all boundary conditions (2) satisfying the inequality (6) and for all nonlinear functions $\varphi(z)$ satisfying (26)

$$\mathrm{E}[|h(i,j)|^p + |v(i,j)|^p] \to 0,$$

as $i + j \to \infty$.

In the remainder of this section, general conditions for stochastic absolute p-stability of the system described by (23), (4) and (26) are developed together with a computationally tractable stability test for the quadratic case when $p = 2$.

Consider the candidate stochastic Lyapunov function

$$V(x,r) = V_1(h,r) + V_2(v,r), \; h \in \mathbb{R}^{n_h},$$
$$v \in \mathbb{R}^{n_v}, \; r \in \mathbb{N}; \; V_1(0,r) = 0, \; V_2(0,r) = 0, \; r \in \mathbb{N},$$
$$V_1(h,r) > 0, \; h \neq 0, \; V_2(v,r) > 0, v \neq 0, \; r \in \mathbb{N}, \quad (27)$$

with associated increment

$$\Delta V(x,r) = \mathrm{E}[V_1(h(i+1,j), \rho(i+1,j)) - V_1(h(i,j), \rho(i,j)) + V_2(v(i,j+1), \rho(i,j+1)) - V_2(v(i,j), \rho(i,j)) | x(i,j) = x, r(i,j) = r]. \quad (28)$$

Theorem 4: Consider the 2D system described by (23) with the control law (25) applied. Suppose also that for all boundary conditions satisfying

$$\mathrm{E}\left[\sum_{j=0}^{\infty} |h(0,j)|^p + \sum_{i=0}^{\infty} |v(i,0)|^p\right] < \infty \quad (29)$$

and for all $\varphi(z)$ satisfying (5), there exist positive constants $c_1, \; c_2, \; c_3$ such that the Lyapunov function (27) and its associated increment (28) satisfy

$$c_1(|h(i,j)|^p + |v(i,j)|^p) \leq V(x(i,j), \rho(i,j)) \leq c_2(|h(i,j)|^p + |v(i,j)|^p), \quad (30)$$

and

$$\Delta V(x(i,j), \rho(i,j)) \leq -c_3(|h(i,j)|^p + |v(i,j)|^p). \quad (31)$$

Then this controlled system is stochastically absolutely p-stable.

Proof: It follows from taking the expectation of both sides in (30) and (31) that

$$\mathrm{E}[V_1(h(i+1,j), \rho(i+1,j)) + V_2(v(i,j+1), \rho(i,j+1))] \leq \lambda \mathrm{E}[V_1(h(i,j), \rho(i,j)) + V_2(v(i,j), \rho(i,j))],$$

where $\lambda = \frac{c_2 - c_3}{c_2}$. By the same arguments as in the proof of Theorem 1, $0 < \lambda < 1$. Writing out the inequalities for i from 0 to N and j from N to 0, adding both sides and using the trivial identity

$$\mathrm{E}[V_1(h(0,N+1), \rho(0,N+1)) + V_2(v(N+1,0), \rho(N+1,0))] = \mathrm{E}[V_1(h(0,N+1), \rho(0,N+1)) + V_2(v(N+1,0), \rho(N+1,0))],$$

gives

$$\sum_{j=0}^{N+1}\mathrm{E}[V_1(h(N+1-j,j),\rho(N+1-j,j)) \\ +V_2(v(N+1-j,j),\rho(N+1-j,j))] \\ \le \lambda\sum_{j=0}^{N}\mathrm{E}[V_1(h(N-j,j),\rho(N-j,j)) \\ +V_2(v(N-j,j),\rho(N-j,j))] \\ +\mathrm{E}[V_1(h(0,N+1),\rho(0,N+1)) \\ +V_2(v(N+1,0),\rho(N+1,0))].$$

Hence

$$\sum_{j=0}^{N+1}\mathrm{E}[V_1(h(N+1-j,j),\rho(N+1-j,j)) \\ +V_2(v(N+1-j,j),\rho(N+1-j,j))] \\ \le \sum_{j=0}^{N+1}\lambda^j\mathrm{E}[V_1(h(0,N+1-j),\rho(0,N+1-j)) \\ +V_2(v(N+1-j,0),\rho(N+1-j,0))].$$

Using (30) now gives

$$\mathrm{E}\left[\sum_{j=0}^{N}(|h(N-j,j)|^p+|v(N-j,j)|^p)\right] \\ \le \alpha\mathrm{E}\left[\sum_{j=0}^{N}\lambda^j(|h(0,N-j)|^p+|v(N-j,0)|^p)\right],$$

where $\alpha = c_2/c_1$, and hence

$$\sum_{N=0}^{M}\mathrm{E}\left[\sum_{j=0}^{N}(|h(N-j,j)|^p+|v(N-j,j)|^p)\right] \\ \le \alpha[(1+\lambda+\ldots+\lambda^M)\mathrm{E}(|h(0,0)|^p+|v(0,0)|^p) \\ +(1+\lambda+\ldots+\lambda^{M-1})\mathrm{E}(|h(0,1)|^p+|v(1,0)|^p) \\ +(1+\lambda+\ldots+\lambda^{M-2})\mathrm{E}(|h(0,2)|^p+|v(2,0)|^p) \\ +\ldots+\mathrm{E}(|h(0,M)|^p+|v(M,0)|^p)] \\ \le \alpha[(1+\lambda+\ldots+\lambda^M)\mathrm{E}\left[\sum_{N=0}^{M}(|h(0,N)|^p+|v(N,0)|^p)\right].$$

Since (29) holds and $0<\lambda<1$, it follows that the right-hand side of this inequality is bounded as $M\to\infty$, hence the series on the left-hand side is convergent and

$$\mathrm{E}\left[\sum_{j=0}^{N}(|h(N-j,j)|^p+|v(N-j,j)|^p)\right]\to 0$$

as $N\to\infty$, and the stability property is established. ■

In the case of $p=2$, and choose the candidate stochastic Lyapunov function as the quadratic form

$$V(x,r)) = h^TP_1(r)h+v^TP_2(r)v, \\ P_1(r)>0,\ P_2(r)>0,\ r\in\mathbb{N}. \quad (32)$$

To guarantee stochastic absolute quadratic stability of the controlled system, the increment of this function must be negative for all $\varphi(z,r)$, satisfying (26). Applying the S procedure [11], this condition holds provided

$$\Delta V(x,r)+z^TQ(r)z+2z^TS(r)\varphi(z,r) \\ +\varphi^T(z,r)R(r)\varphi(z,r)\le -\varepsilon(|h|^2+|v|^2), \quad (33) \\ x\in\mathbb{R}^{n_x},\ r\in\mathbb{N}.$$

Remark 1: For ease of presentation, the notation $A_r = A(r)$, $B_r = B(r)$ etc is used from this point onwards.

Theorem 5: Consider a system described by (23) with a control law (25) satisfying (26) applied. Then the resulting controlled system is stochastically absolutely quadratically stable if the following LMI is feasible

$$L_r\le 0,\ P_r=\mathrm{diag}[P_{1r}\ P_{2r}]>0,\ r\in\mathbb{N},$$

where

$$L_r=\begin{bmatrix} A_r^T\bar{P}_rA_r-P_r+\bar{Q}_r & A_r^T\bar{P}_rB_r+C_r^TS_r \\ B_r^T\bar{P}_rA_r+S_r^TC_r & B_r^T\bar{P}_rB_r+R_r\end{bmatrix},$$

and

$$\bar{P}_r=\mathrm{diag}\left[\sum_{l=1}^{\nu}P_{1l}\pi_{rl}\ \sum_{l=1}^{\nu}P_{2l}\omega_{rl}\right],\ \bar{Q}_r=C_r^TQ_rC_r+\varepsilon I.$$

Proof: Follows from the constructing increment of the Lyapunov function, completing the square in (33), and applying Theorem 4. ■

Suppose that there exists linear state feedback control law

$$u(i,j)=-K_rx(i,j),\ \text{if}\ \rho(i,j)=r \quad (34)$$

such that the controlled system formed by applying (34) to (23) is stochastically quadratically stable. Then it follows from Theorem 4 that a sufficient condition for stochastic quadratic stability of this system is existence of a matrix

$$H_r=\mathrm{diag}[H_{1r}\ H_{2r}]>0,$$

such that

$$(A_r-B_rK_r)^T\bar{H}_r(A_r-B_rK_r)-H_r<0, \quad (35)$$

where $H_r = \mathrm{diag}\left[\sum_{l=1}^{\nu}H_{1l}\pi_{rl}\ \sum_{l=1}^{\nu}H_{2l}\omega_{rl}\right]$. Also (35) is solvable with respect to the stabilizing pair (H_r,K_r) provided the following LMIs with variables X_r, Y_r

$$\begin{bmatrix} M_{11r} & M_{12r} \\ M_{12r}^T & M_{22r}\end{bmatrix}>0,\ X = \mathrm{diag}[X_{1r}\ X_{2r}]>0, \quad (36)$$

$r\in\mathbb{N}$ are feasible, where

$$M_{11r}=X_r,\ M_{22r}=\mathrm{diag}[X_1\ldots X_\nu],\ M_{12r} \\ =\begin{bmatrix}(A_{11r}X_{1r}-B_{1r}Y_{1r})\pi_{r1}^{\frac{1}{2}} & (A_{12r}X_{2r}-B_{1r}Y_{2r})\omega_{r1}^{\frac{1}{2}} \\ (A_{21r}X_{1r}-B_{2r}Y_{1r})\pi_{r1}^{\frac{1}{2}} & (A_{22r}X_{2r}-B_{2r}Y_{2r})\omega_{r1}^{\frac{1}{2}}\end{bmatrix} \\ \ldots\begin{bmatrix}(A_{11r}X_{1r}-B_{1r}Y_{1r})\pi_{r\nu}^{\frac{1}{2}} & (A_{12r}X_{2r}-B_{1r}Y_{2r})\omega_{r\nu}^{\frac{1}{2}} \\ (A_{21r}X_{1r}-B_{2r}Y_{1r})\pi_{r\nu}^{\frac{1}{2}} & (A_{22r}X_{2r}-B_{2r}Y_{2r})\omega_{r\nu}^{\frac{1}{2}}\end{bmatrix}.$$

If these LMIs are feasible, a stabilizing control law matrix is given by $K_r = Y_rX_r^{-1}$.

Suppose that the system has been stabilized by the design of (34) and consider the modified nonlinear control law

$$u(i,j) = \varphi(z(i,j), r(i,j)) - Kx(i,j),\ \varphi(0,r) = 0. \quad (37)$$

Then we have the following result.

Theorem 6: Suppose that the system (23) is has been stabilized by application of the linear state feedback (34) and the following LMI is feasible

$$\begin{bmatrix} A_{cr}^T \bar{P}_r A_{cr} - P_r + \bar{Q}_r + \varepsilon I & A_{cr}^T \bar{P}_r B_r + \bar{S}_r \\ B_r^T \bar{P}_r A_{cr} + \bar{S}_r^T & B_r^T \bar{P}_r B_r + R_r \end{bmatrix} \leq 0, \quad (38)$$
$$P_r = \mathrm{diag}[P_{1r}\ P_{2r}] > 0,\ r \in \mathbb{N},$$

where $A_{cr} = A_r - B_r K_r,\ \bar{Q}_r = C_r^T Q_r C_r - C_r^T S_r K_r - K_r^T S_r^T C_r + K_r^T R_r K_r,\ \bar{S}_r = C_r^T S_r - K_r^T R_r$ and ε is positive scalar. Then systems described by (23) are stochastically absolutely quadratically stable under the nonlinear control law (37).

Proof: Follows from the same arguments used to prove Theorem 3 with appropriate modifications in the stochastic setting. ■

VI. NUMERICAL EXAMPLE

Consider the case of (23) with two modes

$$A_1 = \begin{bmatrix} 0.9992 & 0.0197 & 0.0008 \\ -0.0821 & 0.9696 & 0.0821 \\ 0.0153 & 0.0002 & 0.9847 \end{bmatrix}, B_1 = \begin{bmatrix} -0.0015 \\ -0.1458 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.9891 & 0.0198 & 0.0109 \\ -1.0797 & 0.9761 & 1.0797 \\ 0.0166 & 0.0002 & 0.9834 \end{bmatrix}, B_2 = \begin{bmatrix} -0.0045 \\ -0.4454 \\ 0 \end{bmatrix},$$

$n_h = 2,\ n_v = 1$ and with an unknown matrix of transition probabilities of the Markov chain. It is assumed that only horizontal variables are measurable and the control law is

$$z(i,j) = Fh(i,j), \quad (39)$$
$$u(i,j) = \varphi(z(i,j)). \quad (40)$$

where

$$\mu_1 \leq \varphi(z)/z \leq \mu_2,\ \mu_1 = 0.38,\ \mu_2 = 1.6,\ \varphi(0) = 0. \quad (41)$$

Algorithm 3 from [13] is used to compute linear stabilizing non-switching control law (39) that stabilizes the system with $u = z$ for arbitrary transition probabilities. The algorithm gives $F = [27.594\ \ 1.63792]$. Also it follows from (41) that (26) holds with $Q(r) = -\mu_1\mu_2,\ S(r) = (\mu_1 + \mu_2)/2$ and $R = -1$, and since $C(r) = F$ the LMI (34) is feasible with $\bar{P} = P = \mathrm{diag}[P_1\ P_2] > 0$. Hence by Theorem 5 the system is stochastically absolutely quadratically stable and the control law (39) has robustness property in the sense that it stabilizes the system for arbitrary transition probabilities between the modes and for all nonlinearities in the feedback channel (40) satisfying (41).

VII. CONCLUSIONS AND FURTHER WORK

This paper has addressed the development of absolute stability theory for analysis and stabilizing control law design for the class of 2D discrete linear systems described by the Roesser model, firstly for nonlinear output feedback and then with Markovian jumps in system parameters. The resulting design algorithms can be computed using LMIs for the quadratic case ($p = 2$) and an illustrative numerical example has been given.

Corollary 1 gives simple LMI based algorithm for computing of linear output stabilizing control law. This result appears to be conservative since if (20) holds then all control laws satisfying (5) are stabilizing and this aspect requires further research.. Also the problem of choosing the matrices Q, S and R in (5) is open. These questions and the extension of the given results to other classes of 2D systems are currently under investigation. The problem of robust stabilization in the presence of parameter uncertainty in the linear dynamics of such systems is also open.

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