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# Convergence and Stability Analysis for Iterative Dynamics with Application in Balanced Resource Allocation: A Trajectory Distance Based Lyapunov Approach

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*Abstract*— **This paper addresses the convergence and stability analysis for iterative processes such as numerical iterative algorithms by using a novel trajectory distance based approach. Iterative dynamics are widespread in distributed algorithms and numerical analysis. However, efficient analysis of convergence and sensitivity of iterative dynamics is quite challenging due to the lack of systematic tools. For instance, the trajectories of iterative dynamics are usually not continuous with respect to the initial condition. Hence, the classical dynamical systems theory cannot be applied directly. In this paper, a trajectory distance based Lyapunov approach is proposed as a means to tackling convergence and sensitivity to the initial condition of iterative processes. Technically the problem of convergence and sensitivity is converted into finiteness of trajectory distance and semistability analysis of discrete-time systems. A semidefinite Lyapunov function based trajectory distance approach is proposed to characterize convergence and semistability of iterative dynamics. Two examples are provided to elucidate the proposed method. Finally, the proposed framework is used to solve the convergence and stability of iterative algorithms developed for balanced resource allocation and damage mitigation problems under adversarial attacks.**

## I. Introduction

Suppose we want mobile sensors to detect the possible adversarial attacks and allocate the resources among different locations to counter losses; a primary question is: how should mobile sensors fulfill these global tasks by moving them based on merely neighboring information among the region? In operations research and management sciences, this problem is always recast as an optimization problem solved by one or several iterative algorithms. Now there are a lot of issues arising from performing this iterative process on the computer. The first question is: can you guarantee that the iterative process will stop eventually? This is related to the convergence problem for iterative dynamics. It seems that the classical dynamical systems theory provides vast tools on the convergence and stability analysis of discrete-time dynamical systems. Unfortunately, many iterative dynamics do not belong to dynamical systems due to the absence of the continuity property. Hence, it is quite hard to apply the existing results for dynamical systems to iterative processes. The second question is: how sensitive the iterative process will be with respect to the initial input value change or some parameter perturbation in the iterative process? This is related to the sensitivity analysis of iterative processes. In the dynamical systems theory, the relevant problems are continuous dependence of the solutions with respect to the initial condition and stability theory. This resemblance sheds light on our sensitivity research.

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To answer the first question, the first part of this paper develops a trajectory distance based Lyapunov approach to address convergence analysis of iterative dynamics. The trajectory distance method converts the convergence analysis into a finiteness test of an infinite, contingent trajectory distance series. The transformation helps us leverage many existing results and tools from dynamical systems theory to prove convergence of a non-dynamical system. By incorporating the Lyapunov-based approach, the proposed method creates a great flexibility of studying convergence for a quite broad spectrum of iterative dynamics.

To answer the second question, the second part of this paper focuses on the sensitivity analysis of iterative dynamics with respect to the initial condition. The motivation originates from the dynamical systems theory. Since the stability theory in dynamical systems deals with the sensitivity of the solutions with respect to the initial condition, we borrow this idea to extend it to the iterative dynamics case. More specifically, we use the recently developed notion of *semistability* [1]–[5] to address the sensitivity of the iterative process with respect to the initial condition change. We combine the trajectory distance method with the Lyapunov approach to derive a sufficient stability test for iterative dynamics. Finally, we apply these methods to address the convergence and stability of iterative algorithms depicting balanced resource allocation and damage mitigation problems in networks.

## II. A Trajectory Distance Based Lyapunov Analysis

### A. Convergence via Trajectory Distance

Consider the iterative dynamics given by

$$x(t+1)=F(t,x(t)),\quad x(t_0)=x_0,\\ t\in\overline{\mathbb{Z}}_+,\quad t\geq t_0,\tag{1}$$

where $t_0\in\overline{\mathbb{Z}}_+$, $x(t)\in\mathbb{R}^q$, and $F:\overline{\mathbb{Z}}_+\times\mathbb{R}^q\to\mathbb{R}^q$ is a mapping which is *not* necessarily continuous. A *solution* sequence or discrete *trajectory* to (1) with the initial condition $x(t_0)=x_0$, denoted by $s(\cdot,t_0,x_0)$, is defined as the map of the iterative dynamics given by $s:\overline{\mathbb{Z}}_+\times\overline{\mathbb{Z}}_+\times\mathbb{R}^q$ which satisfies $i)$ the *consistency* property $s(t_0,t_0,x_0)=x_0$

and $ii$) the *semigroup* property $s(k, t_0, s(\kappa, t_0, x_0)) = s(k + \kappa, t_0, x_0)$ for all $x_0 \in \mathbb{R}^q$, $t_0 \in \overline{\mathbb{Z}}_+$, and $k, \kappa \in \overline{\mathbb{Z}}_+$. We assume that a solution $s(t, t_0, x)$ to (1) exists for all $t \geq t_0$, $t \in \overline{\mathbb{Z}}_+$, and $x \in \mathbb{R}^q$, that is, we only consider (1) with *complete* solutions. But these solutions are *not* necessarily unique. A sufficient condition to guarantee the existence of complete solutions for (1) is to assume that $F(\cdot, \cdot)$ is *piecewise continuous*.

It is important to note that in general (1) is *not* a dynamical system due to the lack of continuity of the solution with respect to the initial condition. The continuous dependence on the initial condition is crucial in arriving at many conclusions in dynamical systems theory. Hence, one has to be very careful about applying the existing results for dynamical systems described by the similar form to the iterative dynamics (1).

On the other hand, many notions and results from dynamical systems theory can be borrowed to develop relevant results for iterative dynamics. In this paper, we follow this idea to develop a series of results on convergence and sensitivity to the initial condition for (1). Before we proceed, some notions enlightened by dynamical systems theory are given as follows. More specifically, we say a set $\mathcal{M} \subseteq \mathbb{R}^q$ to be *strongly positively invariant* (respectively, *weakly positively invariant*) with respect to (1) if for every solution (respectively, one solution) $s(t, t_0, x)$, $s(t, t_0, x) \in \mathcal{M}$ for all $x \in \mathcal{M}$ and $t \geq t_0$.

Let $\mathcal{G} \subseteq \mathbb{R}^q$ be strongly positively invariant with respect to (1). A function $V : \mathcal{G} \to \mathbb{R}$ is said to be *proper* relative to $\mathcal{G}$ if $V^{-1}(\mathcal{K})$ is a relatively compact subset of $\mathcal{G}$ for all compact subsets $\mathcal{K}$ of $\mathbb{R}$. The following result is a criterion for boundedness of the solutions to (1).

*Lemma 2.1:* Consider (1). If there exists a continuous function $U : \mathcal{G} \to \mathbb{R}$ that is proper relative to $\mathcal{G}$ and such that $U(x) \geq 0$ and $U(F(t, x)) \leq U(x)$ for all $x \in \mathcal{G}$ and $t \geq t_0$, then every solution to (1) in $\mathcal{G}$ is bounded relative to $\mathcal{G}$ for all $t \geq t_0$.

Next, we give the definition of convergence of a solution to (1). We say a solution $s(t, t_0, x)$ to (1) is *convergent* on $\mathcal{G}$ if $\lim_{t\to\infty} s(t, t_0, x)$ exists for all $x \in \mathcal{G}$ and $t \geq t_0$. A point $p \in \mathcal{G}$ is a *positive limit point* of the trajectory $s(t, t_0, x)$ if there exists a monotonic sequence $\{k_n\}_{n=0}^{\infty}$ of nonnegative numbers, with $\lim_{n\to\infty} k_n = \infty$, such that $\lim_{n\to\infty} x(k_n) = p$. The set of all positive limit points of all solutions of the form $s(t, t_0, x)$ is the *positive limit set* $\omega(t_0, x)$.

*Lemma 2.2:* Assume that every solution to (1) in $\mathcal{G}$ is bounded relative to $\mathcal{G}$. Then $\omega(t_0, x)$ is nonempty and closed. Furthermore, $s(t, t_0, x) \to \omega(t_0, x)$ as $t \to \infty$.

The following result provides a new Lyapunov-type test for the convergence of (1).

*Theorem 2.1:* Consider (1). Assume that every solution to (1) in $\mathcal{G}$ is bounded relative to $\mathcal{G}$. Furthermore, assume there exists a continuous function $V : \mathcal{G} \to \mathbb{R}$ such that $V(x) \geq 0$ and $V(F(t, x)) \leq V(x)$ for all $x \in \mathcal{G}$ and $t \geq t_0$. Let $\mathcal{R} \triangleq \{x \in \mathcal{G} : V(F(t, x)) = V(x), t \geq t_0\}$ and suppose that there exists a set $\mathcal{U}$ containing $\mathcal{R}$ that is relatively open in $\mathcal{G}$ and a lower semicontinuous, real-valued function $g$ defined on a closed interval containing $V(\mathcal{U})$ such that

$$\|F(t, x) - x\| \leq g(V(x)) - g(V(F(t, x))), \ \forall t \geq t^* \qquad (2)$$

on $\mathcal{U}$ for some $t^* \geq t_0$. Then every solution to (1) is convergent on $\mathcal{G}$. Furthermore, $\lim_{t\to\infty} x(t) \in \mathcal{R}$.

*Remark 2.1:* The function $U$ in Lemma 2.1 is *not* necessarily the same as the function $V$ in Theorem 2.1.

*Remark 2.2:* For the system (1), the infinite series $\sum_{t=0}^{\infty} \|x(t+1) - x(t)\|$ represents the normed distance of the solution to (1). Thus, the last part of the proof for Theorem 2.1 implies that if every solution to (1) has a finite distance, then this solution is convergent on $\mathcal{G}$. Indeed, the first part of the proof for Theorem 2.1 essentially shows that if the inequality (2) is satisfied, then every solution in $\mathcal{G}$ has a finite distance. Hence, we call this analysis a *trajectory distance based approach*.

### B. Sensitivity via Semistability

The previous subsection uses the trajectory distance approach to characterize the convergence of iterative dynamics (1). In this subsection, we further utilize this approach to exploit the sensitivity issue of (1) with respect to the initial condition. In dynamical systems theory, this issue is addressed by the continuous dependence of (1) with respect to the initial condition and stability theory. Since in general the solution to (1) loses the continuity property, it is more reasonable to borrow stability theory to develop relevant results for (1). More specifically, we use the recently developed notion of semistability [1]–[5] to characterize the sensitivity of (1) with respect to the initial condition. This is due to the reason that the convergence behavior of iterative dynamics is more aligned with the notion of semistability in dynamical systems.

To begin our discussion, we first introduce the definition of semistability for (1). This new concept is motivated from semistability theory of autonomous/switched systems [2], [5] and time-varying continuous-time/nonsmooth dynamical systems [6], [7]. To state this new concept, we define a *equilibrium point* of (1) to be a point $z \in \mathbb{R}^q$ satisfying $F(t, z) = z$ for all $t \geq t_0$. The set of all the equilibrium points of (1) is denoted by $\mathcal{E}$. We assume that $\mathcal{E}$ is nonempty.

*Definition 2.1:* $i$) An equilibrium point $x_{\rm e} \in \mathcal{E} \cap \mathcal{G}$ of (1) is *Lyapunov stable* relative to $\mathcal{G}$ if, for every $\varepsilon > 0$ and $t_0 \in \overline{\mathbb{Z}}_+$, there exists $\delta = \delta(\varepsilon, x_{\rm e}, t_0) > 0$ such that $x(t_0) \in \mathcal{B}_\delta(x_{\rm e}) \cap \mathcal{G}$ implies that $x(t) \in \mathcal{B}_\varepsilon(x_{\rm e}) \cap \mathcal{G}$ for all $t \geq t_0$ and all the solutions $x(t)$, where $\mathcal{B}_\varepsilon(x)$ denotes the open ball centered at $x$ with radius $\varepsilon$. The iterative dynamics (1) is *Lyapunov stable* relative to $\mathcal{G}$ if every equilibrium point in $\mathcal{E} \cap \mathcal{G}$ is Lyapunov stable relative to $\mathcal{G}$.

$ii$) An equilibrium point $x_{\rm e} \in \mathcal{E} \cap \mathcal{G}$ of (1) is *semistable* relative to $\mathcal{G}$ if it is Lyapunov stable relative to $\mathcal{G}$ and, for every $t_0 \in \overline{\mathbb{Z}}_+$, there exists $\delta = \delta(x_{\rm e}, t_0) > 0$ such that $x(t_0) \in \mathcal{B}_\delta(x_{\rm e}) \cap \mathcal{G}$ implies that all the solution sequences of the form $\{x(t)\}_{t=t_0}^{\infty}$ converge to Lyapunov equilibria in $\mathcal{E} \cap \mathcal{G}$. The iterative dynamics (1) is *semistable* relative to $\mathcal{G}$ if every equilibrium point in $\mathcal{E} \cap \mathcal{G}$ is semistable relative to $\mathcal{G}$.

*Lemma 2.3:* Let $x \in \mathcal{G}$. If a point $z \in \omega(t_0, x) \cap \mathcal{G}$ is a Lyapunov stable equilibrium point relative to $\mathcal{G}$, then $\lim_{t\to\infty} s(t, t_0, x) = z$.

*Lemma 2.4:* Consider (1) and $x_{\rm e} \in \mathcal{E} \cap \mathcal{G}$. If there exists a lower semicontinuous function $U : \mathcal{G} \to \mathbb{R}$ such that $U(\cdot)$ is continuous at $x_{\rm e}$, $U(x_{\rm e}) = 0$, $U(x) > 0$ for all $x \in \mathcal{G}\backslash\{x_{\rm e}\}$, and $U(F(t, x)) \le U(x)$ for all $x \in \mathcal{G}$ and $t \ge t_0$, then $x_{\rm e}$ is Lyapunov stable relative to $\mathcal{G}$.

*Theorem 2.2:* Assume that all the conditions in Theorem 2.1 hold. Furthermore, assume that every point in $\mathcal{R}$ is a Lyapunov stable equilibrium point relative to $\mathcal{G}$. Then the iterative dynamics (1) is semistable relative to $\mathcal{G}$.

As we conclude this subsection, we present two examples to illustrate the proposed method. The first example is a discrete-time switched system.

*Example 2.1:* Consider the iterative dynamics given by

$$x(t+1) = f_{\sigma(t)}(x(t)), \quad x(0) = x_0, \quad t \ge 0, \tag{3}$$

where $x(t) \in \mathbb{R}^3$, $\sigma(t) = 0$ if $t$ is even and $\sigma(t) = 1$ if $t$ is odd,

$$f_0(x) = \begin{bmatrix} (1-\mu(t))|x_2| + \mu(t)|x_3| \\ (1-\mu(t))|x_3| + \mu(t)|x_1| \\ (1-\mu(t))|x_1| + \mu(t)|x_2| \end{bmatrix}, \tag{4}$$

$$f_1(x) = \begin{bmatrix} (1-\mu(t))|x_1| + \frac{\mu(t)}{2}|x_2| + \frac{\mu(t)}{2}|x_3| \\ (1-\mu(t))|x_2| + \frac{\mu(t)}{2}|x_3| + \frac{\mu(t)}{2}|x_1| \\ (1-\mu(t))|x_3| + \frac{\mu(t)}{2}|x_1| + \frac{\mu(t)}{2}|x_2| \end{bmatrix}, \tag{5}$$

$\mu(\cdot)$ is piecewise continuous, and $0 < \inf_{t\in[0,\infty)} \mu(t) \le \sup_{t\in[0,\infty)} \mu(t) < 1$. First, let $\mathcal{G} = \mathbb{R}^3$ and consider the function $U(x) = \|x\|_1$, where $\|\cdot\|_1$ denotes the 1-norm on $\mathbb{R}^3$. Clearly, $U(x(t+1)) - U(x(t)) = 0$, $t \in \overline{\mathbb{Z}}_+$. Furthermore, note that $U(x)$ is positive definite, and hence, is proper with respect to $\mathbb{R}^3$. Thus, it follows from Lemma 2.1 that all the solutions to (3) are bounded.

Next, consider the function $V(x) = \sum_{i=1}^3 \sum_{j=1}^3 ||x_{i+j}| - x_i|$, where $x_4 = x_1$ and $x_5 = x_2$. Note that

$$\begin{aligned} &|x_{i+1}(t+1) - x_i(t)| \\ &\quad \le (1-\mu(t))||x_{i+1}(t)| - x_i(t)| \\ &\quad + \frac{\mu(t)[3+(-1)^{t+1}]}{4}||x_{i+2}(t)| - x_i(t)| \\ &\quad + \frac{\mu(t)[1+(-1)^{t+1}]}{4}||x_{i+3}(t)| - x_i(t)| \\ &\quad \le \rho \sum_{j=1}^3 ||x_{i+j}(t)| - x_i(t)|, \end{aligned} \tag{6}$$

where $\rho = \sup_{t\in[0,\infty)}\{1 - \mu(t), \frac{1}{4}\mu(t)[3 + (-1)^{t+1}], \frac{1}{4}\mu(t)[1 + (-1)^{t+1}]\}$ which is a constant between 0 and 1, for every $i = 1, 2, 3$ and every $t \ge 0$. Thus,

$$\begin{aligned} \|x(t+1) - x(t)\|_1 &\le \rho \sum_{i=1}^3 \sum_{j=1}^3 ||x_{i+j}(t)| - x_i(t)| \\ &= \rho V(x(t)), \quad t \in \overline{\mathbb{Z}}_+. \end{aligned} \tag{7}$$

On the other hand, since $V(x(t+1)) \le \xi \sum_{i=1}^3 \sum_{j=1}^3 ||x_{i+j}(t)| - |x_i(t)|| \le \xi V(x(t))$, $t \in \overline{\mathbb{Z}}_+$, where $\xi = \sup_{t\in[0,\infty)}\{|1-2\mu(t)|, \mu(t)\}$ which is a constant between 0 and 1, it follows that $V(x(t+1)) \le V(x(t))$ and $V(x(t)) \le (1/(1-\xi))[V(x(t)) - V(x(t+1))]$, $t \in \overline{\mathbb{Z}}_+$, and hence, for every $t \ge 0$, $\|x(t+1) - x(t)\|_1 \le (\rho/(1-\xi))[V(x(t)) - V(x(t+1))]$. Thus, with $g(x) = (\rho/(1-\xi))x$, it follows from Theorem 2.1 that all the solutions are convergent.

Finally, to show Lyapunov stability of (3) relative to $\mathcal{G} = \{x \in \mathbb{R}^3 : x_i \ge 0, i = 1, 2, 3\}$, note that (3) can be rewritten as $x(t+2) = A_2(t+1)A_1(t)x(t)$ for even $t \ge 2$ and $x(t+2) = A_1(t+1)A_2(t)x(t)$ for odd $t \ge 1$, where

$$A_1(t) = \begin{bmatrix} 0 & 1-\mu(t) & \mu(t) \\ \mu(t) & 0 & 1-\mu(t) \\ 1-\mu(t) & \mu(t) & 0 \end{bmatrix}, \tag{8}$$

$$A_2(t) = \begin{bmatrix} 1-\mu(t) & \frac{\mu(t)}{2} & \frac{\mu(t)}{2} \\ \frac{\mu(t)}{2} & 1-\mu(t) & \frac{\mu(t)}{2} \\ \frac{\mu(t)}{2} & \frac{\mu(t)}{2} & 1-\mu(t) \end{bmatrix}. \tag{9}$$

Note that $A_1(t)$ and $A_2(t)$ are nonnegative matrices for all $t \ge 0$. Furthermore, the row sums or the column sums for both $A_1(t)$ and $A_2(t)$ are 1 for all $t \ge 0$. Then it follows from the Perron-Frobenius theorem [8] that $\|A_1(t)\| \le 1$ and $\|A_2(t)\| \le 1$ for all $t \ge 0$. Next, note that $\mathcal{R} = \{x \in \mathcal{G} : x_1 = x_2 = x_3 = a \ge 0\}$ and consider $W(x) = \|x - a[1,1,1]^{\rm T}\|_1 = \sum_{i=1}^3 |x_i - a|$. Then $W(x(t+2)) \le \|A_1(t+1)\|\|A_2(t)\|W(x(t)) \le W(x(t))$ or $W(x(t+2)) \le \|A_2(t+1)\|\|A_1(t)\|W(x(t)) \le W(x(t))$, $t \ge 1$. Hence, for every $\varepsilon > 0$, one can always find $\delta = \varepsilon > 0$ such that $\sum_{i=1}^3 ||x_i(0)| - a| = \sum_{i=1}^3 |x_i(0) - a| < \delta$ implies that $\|x(t) - a[1,1,1]^{\rm T}\| < \varepsilon$ for all $t \ge 0$. Now by definition, (3) is Lyapunov stable relative to $\mathcal{G}$. Therefore, it follows from Theorem 2.2 that the iterative dynamics (3) is semistable relative to $\mathcal{G}$.

The second example is a discrete-time quantized system.

*Example 2.2:* Consider the iterative dynamics given by

$$x(t+1) = f(x(t)), \quad x(0) = x_0, \quad t \ge 0, \tag{10}$$

where $x(t) \in \mathbb{R}^2$, $f$ is given by

$$f(x) = \begin{bmatrix} (1-\mu(t))\lfloor x_1 \rfloor + \mu(t)\lfloor x_2 \rfloor \\ \mu(t)\lfloor x_1 \rfloor + (1-\mu(t))\lfloor x_2 \rfloor \end{bmatrix}, \tag{11}$$

$\lfloor\cdot\rfloor$ denotes the floor function, $\mu(\cdot)$ is piecewise continuous, and $0 < \inf_{t\in[0,\infty)} \mu(t) \le \sup_{t\in[0,\infty)} \mu(t) < 1$. Let $\mathcal{G} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \ge 0, x_2 \ge 0\}$, which is a nonnegative orthant. Clearly $\mathcal{G}$ is strongly positively invariant. Next, consider $U(x) = \|x\|_2^2 = x_1^2 + x_2^2$. Then it follows that $U(x)$ satisfies $U(f(x)) - U(x) \le -\varrho(\lfloor x_1 \rfloor - \lfloor x_2 \rfloor)^2$ for all $x \in \mathcal{G}$, where $\varrho = \sup_{t\in[0,\infty)} 2\mu(t)(1-\mu(t))$, and hence, it follows from it follows from Lemma 2.1 that every solution to (10) is bounded relative to $\mathcal{G}$. Furthermore, note that $\lfloor x_1 \rfloor - \lfloor x_2 \rfloor$ is an integer, which implies that $(\lfloor x_1 \rfloor - \lfloor x_2 \rfloor)^2 \ge |\lfloor x_1 \rfloor - \lfloor x_2 \rfloor|$

for every $(x_1, x_2) \in \mathcal{G}$. Thus,

$$\begin{aligned} \|f(x)-x\|_1 &= \epsilon|\lfloor x_1\rfloor - \lfloor x_2\rfloor| \\ &\leq \epsilon(\lfloor x_1\rfloor - \lfloor x_2\rfloor)^2 \\ &\leq \frac{\epsilon}{\varrho}[U(x)-U(f(x))], \quad x\in\mathcal{G}, \quad (12) \end{aligned}$$

where $\epsilon = \sup_{t\in[0,\infty)}\{1-\mu(t), \mu(t)\}$. Now it follows from Theorem 2.1 that every solution to (10) is convergent.

Next, we claim that $\mathcal{R} = \{(x_1, x_2) \in \mathcal{G} : x_1 = x_2 = k \in \overline{\mathbb{Z}}_+\}$. Indeed, this assertion follows from the similar arguments as in the proof of Lemma 3.5 of [9]. It follows from (11) that $(1-\mu(t))x_1(t)+\mu(t)x_2(t)-1 < x_1(t+1) \leq (1-\mu(t))x_1(t)+\mu(t)x_2(t)$ and $\mu(t)x_1(t)+(1-\mu(t))x_2(t)-1 < x_2(t+1) \leq \mu(t)x_1(t) + (1-\mu(t))x_2(t)$ for all $t \geq 0$ and $x(0) \in \mathcal{G}$. Hence, (10) can be rewritten as

$$x_1(t+1) = \lfloor(1-\mu(t))x_1(t)+\mu(t)x_2(t)\rfloor, \quad (13)$$
$$x_2(t+1) = \lfloor\mu(t)x_1(t)+(1-\mu(t))x_2(t)\rfloor, \quad (14)$$

for every $t \geq 0$ and $x(0) \in \mathcal{G}$, which implies that $x_i(t) \in \overline{\mathbb{Z}}_+$ for all $t \geq 1$ and $i = 1, 2$.

Suppose $x_1(0) \geq x_2(0)$. If $x_1(t) \geq x_2(t)$ for some $t \geq 0$, then it follows from (11), (13), and (14) that $\lfloor x_2(t)\rfloor \leq x_2(t+1) \leq x_1(t+1) \leq \lfloor x_1(t)\rfloor$. Similarly, assume $x_1(0) \leq x_2(0)$. If $x_1(t) \leq x_2(t)$ for some $t \geq 0$, then we have $\lfloor x_1(t)\rfloor \leq x_1(t+1) \leq x_2(t+1) \leq \lfloor x_2(t)\rfloor$. Hence, by induction, $x_1(t) \preceq x_2(t)$ and $\lfloor x_1(t)\rfloor \preceq x_1(t+1) \preceq x_2(t+1) \preceq \lfloor x_2(t)\rfloor$ for all $t \geq 0$, where $\preceq$ means that all the terms have the same inequality order, that is, either $\leq$ or $\geq$ altogether. Since $x_i(t) \in \overline{\mathbb{Z}}_+$ for all $t \geq 1$ and $i = 1, 2$, it follows that one of the two sequences $\{x_i(t)\}_{t=1}^{\infty}$, $i = 1, 2$, is nonincreasing while the other one is nondecreasing. Moreover, since $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = k^* \in \overline{\mathbb{Z}}_+$, it follows that $x_1(t) \preceq k^* \preceq x_2(t)$ for all $t \geq 1$.

To show that $k^*[1,1]^{\mathrm{T}}$ is Lyapunov stable, we take $V(x) = \|x - k^*[1,1]^{\mathrm{T}}\|_1 = |x_1 - k^*| + |x_2 - k^*|$. Since $x_1(t) \preceq k^* \preceq x_2(t)$ for all $t \geq 1$, it follows that $V(x(t)) = |x_1(t) - x_2(t)|$ for all $t \geq 1$. In this case, note that $x_i(t) \in \overline{\mathbb{Z}}_+$ for all $t \geq 1$ and $i = 1, 2$, $V(x(t+1)) = |x_1(t+1) - x_2(t+1)| = |1-2\mu(t)||x_1(t) - x_2(t)| \leq (\sup_{t\in[0,\infty)} |1-2\mu(t)|)V(x(t))$ for all $t \geq 1$. This implies that $V(x(t)) \leq (\sup_{t\in[0,\infty)} |1-2\mu(t)|)^{t-1}V(x(1)) = (\sup_{t\in[0,\infty)} |1-2\mu(t)|)^{t-1}|x_1(1)-x_2(1)| \leq (\sup_{t\in[0,\infty)} |1-2\mu(t)|)^{t}|\lfloor x_1(0)\rfloor - \lfloor x_2(0)\rfloor|$ for every $t \geq 1$. Hence, the sequence $\{V(x(t))\}_{t=1}^{\infty}$ is monotonically decreasing to zero regardless of the value $|\lfloor x_1(0)\rfloor - \lfloor x_2(0)\rfloor|$ as long as $(x_1(0), x_2(0)) \in \mathcal{G}$. Thus, for every $\varepsilon > 0$, one can always find $\delta = \varepsilon > 0$, such that for every $|x_1(0) - k^*| + |x_2(0) - k^*| < \delta$, $(x_1(0), x_2(0)) \in \mathcal{G}$, $|x_1(t) - k^*| + |x_2(t) - k^*| < \varepsilon$ for all $t \geq 0$. By definition of Lyapunov stability, $k^*[1,1]^{\mathrm{T}}$ is Lyapunov stable relative to $\mathcal{G}$. Finally, note that $k^*$ is the limit of both $x_1(t)$ and $x_2(t)$, by definition of semistability, (10) is semistable relative to $\mathcal{G}$.

## III. Balanced Resource Allocation and Damage Mitigation Algorithms

The balanced iterative algorithm design for damage mitigation and resource allocation in this paper is based on the iterative dynamics given by the form of (1). To explicitly state the connection between (1) and our proposed balanced iterative algorithm, we consider a network characterized by a strongly connected directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ consisting of the set of nodes $\mathcal{V} = \{1, \ldots, q\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where each edge $(i, j) \in \mathcal{E}$ is an ordered pair of distinct nodes. The set of neighbors of node $i$ is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Finally, we denote the value of the node $i \in \{1, \ldots, q\}$ at time $t$ by $x_i(t) \in \mathbb{R}$.

Each node $i$ holds an initial value on the network $x_i(0) \in \mathbb{R}$. The network permits the allowed communication between two nodes if and only if they are neighbors. We are interested in rearranging the scalar values at the nodes of a network to some equilibrium distribution, via a distributed algorithm in which the nodes only communicate with their neighbors, so that all the scalar values in those nodes are eventually distributed in some pattern. This final distribution pattern is not necessarily a fixed equilibrium pattern, but could be a continuum of a set of patterns. Furthermore, when we model resource allocation problems realistically, the limited capacity constraints such as quantization and the total resource need to be imposed for the balanced iterative algorithm design.

Surprisingly enough, the distributed algorithm design is of form (1). To elucidate this, we consider distributed nonlinear iterations over the network $\mathbb{G}$ given by the form

$$\begin{aligned} x_i(t+1) &= W_{(i,i)}(t, x_i(t), x_i(t))x_i(t) \\ &\quad + \sum_{j\in\mathcal{N}_i} W_{(i,j)}(t, x_i(t), x_j(t))x_j(t), \\ &\quad i = 1, \ldots, q, \quad t = 0, 1, 2, \ldots, \quad (15) \end{aligned}$$

where $W_{(i,j)} : \overline{\mathbb{Z}}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denotes the weight on $x_j$ at node $i$, which in general, is a function of time $t$ and the states variables $x_i$ and $x_j$ [10], [11]. Another physical interpretation of this model is given by the next section and [10]. Letting $W_{(i,j)}(\cdot,\cdot,\cdot) = 0$ for $j \neq \mathcal{N}_i$, this iteration can be rewritten as a compact form

$$x(t+1) = W(t, x(t))x(t), \quad t = 0, 1, 2, \ldots, \quad (16)$$

where $x(t) = [x_1(t), \ldots, x_q(t)]^{\mathrm{T}} \in \mathbb{R}^q$. The constraint on the matrix function $W(\cdot,\cdot,\cdot)$ can be expressed as $W(\cdot,\cdot,\cdot) \in \mathcal{W}$, where $\mathcal{W} = \{W \in \mathbb{R}^{q\times q} : W_{(i,j)} = 0 \text{ if } (i,j) \notin \mathcal{E} \text{ and } i \neq j\}$.

The damage mitigation problem in this paper is defined as the reduction of adversarial attack effectiveness. To effectively model the damage mitigation problem under adversarial attacks, we use a *nonnegative system* [10] technique with network routing in peer-to-peer networks to build up a non-fixed structure *mobile sensor network* [12] for detecting damages caused by adversarial threats. Specifically, we model a road network in an urban area as a strongly connected, non-fixed structure directed graph in peer-to-peer networks. The nodes in the graph represent some critical sites in the city that may be attacked by adversarial attacks. By a non-fixed structure graph, we mean that if there is no real road between two nodes, the mobile sensors in these two nodes can set up a wireless communication link between them so that the graph structure is not necessarily fixed to the actual road network of a city itself. The dynamics of damage mitigation evolves as follows: We let $x_i(t)$ denote the number of mobile sensors that node $i$ has at time $t$. Those mobile sensors are used to detect possible adversarial attacks at node $i$ and travel along the graph network to mitigate damages based on some algorithms. At the initial time, the number of mobile sensors at node $i$ is given by $x_i(0)$. In the $t$th time slot, let node $i$ contact some neighboring node $j$ to see how many mobile sensors both nodes have. Then at this time, to reduce possible dangers posed by adversarial attacks, both nodes will relocate their mobile sensors so that eventually the number of mobile sensors at each node is proportional to the severity of risks and damages by adversarial attacks. Of course, the neighboring nodes for node $i$ may not be just one node for a connected graph.

The mathematical expression of the damage mitigation model described above can be written as (15) for each node, or equivalently, (16) for a vector form, where every element of $W$ represents the weight of the number of mobile sensors available for every node and used for designing algorithms. The structure of $W$ is not fixed except for connectivity. Furthermore, since $x_i(t)$ represents the number of mobile sensors, $x_i(t)$ is always a nonnegative integer, that is, $x_i(t) \in \overline{\mathbb{Z}}_+$ for all $i$ and all $t$. In this case, (16) should be interpreted as the following form

$$\begin{aligned} x_i(t+1) &= W_{(i,i)}(t, x_i(t), x_i(t))\lfloor x_i(t)\rfloor \\ &\quad + \sum_{j\in\mathcal{N}_i} W_{(i,j)}(t, x_i(t), x_j(t))\lfloor x_j(t)\rfloor, \\ &\quad i = 1, \ldots, q, \quad t = 0, 1, 2, \ldots, \end{aligned} \tag{17}$$

and hence, it becomes a nonnegative system [10] over $\overline{\mathbb{Z}}_+^q$ whose states are always in the nonnegative orthant in the state space. Now the balanced coordination algorithm design problem associated with this model becomes the following: How many mobile sensors should be allocated for each node at every instant of time by guaranteeing that all the nodes eventually achieve a steady state? To answer this question, in the next section we will first use the compartmental model [10] integrated with quantization effects [9] to design a particular form for $W_{(i,j)}$ in (17). Then we will analyze the convergence of (17) by using the trajectory distance based method developed in Section II. The merit of the trajectory distance based method is that it works perfectly for the general nonlinear form of iterations like (17), which cannot be solved by use of many existing methods in the literature [11], [13]–[18] due to the requirement of either possessing the linear form of iterations or satisfying the continuity property for the vector field. In our setting, neither of these conditions are needed for (17). Hence, this proposed method seems to be a very promising one for studying the convergence of nonlinear iterations of the form (17). Finally, the limit value of $x(t)$ represents the proportion of mobile sensors should be distributed among the nodes to counter adversarial attacks (for high-risk areas, a large proportion will be given).

## IV. Compartmental Model Characterization and Analysis

In this section, we apply our previously developed results to design balanced coordinated resource allocation and damage mitigation algorithms by considering a class of switched/time-varying, networked iterative dynamics with nonnegativity and quantization constraints. Specifically, consider the networked iteration given by

$$\begin{aligned} x_i&(t+1) \\ &= \lfloor x_i(t)\rfloor - \sum_{j=1, j\neq i}^{n} \mathcal{C}_{(j,i)} a_{ji}(t, x_j(t), x_i(t)) \lfloor x_i(t)\rfloor \\ &\quad + \sum_{j=1, j\neq i}^{n} \mathcal{C}_{(i,j)} a_{ij}(t, x_i(t), x_j(t)) \lfloor x_j(t)\rfloor, \\ &\quad i = 1, \ldots, n, \quad t \in \overline{\mathbb{Z}}_+, \quad x_i(0) \geq 0, \end{aligned} \tag{18}$$

or in the vector form, $x(t+1) = A(t, x(t))\lfloor x(t)\rfloor$, where $\lfloor x(t)\rfloor$ should be understood elementwise, for $i, j = 1, \ldots, n$, $\mathcal{C}_{(i,j)}$ represents the $(i,j)$th element of the *connectivity matrix* $\mathcal{C}$ [2] defined by $\mathcal{C}_{(i,j)} = 1$ if $j \in \mathcal{N}_i$, $\mathcal{C}_{(i,j)} = 0$ if $j \notin \mathcal{N}_i$ and $i \neq j$, and $\mathcal{C}_{(i,i)} = -\sum_{j=1}^{n} \mathcal{C}_{(i,j)}$,

$$A_{(i,j)}(t, x(t)) = \begin{cases} 1 - \sum_{l=1}^{n} \mathcal{C}_{(l,i)} a_{li}(t, x_l(t), x_i(t)), & i = j, \\ \mathcal{C}_{(i,j)} a_{ij}(t, x_i(t), x_j(t)), & i \neq j, \end{cases} \tag{19}$$

$a_{ij}(\cdot,\cdot,\cdot)$ is such that a complete solution to (18) exists, $a_{ij}(t, x_i(t), x_j(t)) > 0$, $a_{ii}(t, x_i(t), x_i(t)) \equiv 0$, and $\sum_{l=1}^{n} a_{li}(t, x_l(t), x_i(t)) \leq 1$, for all $x \in \overline{\mathbb{R}}_+^n$ and $t \in \overline{\mathbb{Z}}_+$. This network system is a compartmental model [10] representing a mass balance equation physically in which $x_i$ denotes the mass (and hence a nonnegative quantity) of the $i$th subsystem of the compartmental system. Note that since at any given instant of time mass can only be transported, stored, or discharged but not created and the maximum amount of mass that can be transported and/or discharged cannot exceed the mass in a compartment, it follows that $1 \geq \sum_{l=1}^{n} a_{li}(t, x_l(t), x_i(t))$, which interprets this constraint in physics.

Next, we present a step-by-step convergence analysis for (18). First, it follows from [10] that the nonnegative quadrant $\mathcal{G} = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \ldots, n\}$ is positively invariant under (18).

*Lemma 4.1:* Consider (18). Then $x_i(t+1) \in \overline{\mathbb{Z}}_+$ for every $x_i(0) \geq 0$, $i = 1, \ldots, n$, and $t \geq 0$. Furthermore, $x(t+1) = A(t, x(t))x(t)$ for all $t \geq 1$.

*Lemma 4.2:* Every solution in $\mathcal{G}$ is bounded relative to $\mathcal{G}$ for all $t \geq 0$.

### A. Consensus Based Convergence Analysis

The convergence analysis in this subsection is based on the consensus analysis technique developed in [19] whose method is originated from [20]. In this subsection, the graph topology $\mathbb{G}$ associated with (18) is focused on a *connected proximity graph* which is undirected.

*Assumption 4.1:* $\mathbb{G}$ is a connected proximity graph.

Note that $\mathbb{G}$ is a connected proximity graph if and only if $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ and rank $\mathcal{C} = n - 1$.

*Theorem 4.1:* Assume that Assumption 4.1 holds. Furthermore, assume that there exist $p_i > 0$, $i = 1, \ldots, n$, such that $p_i/p_j$ is rational and $a_{ji}(t, x_j, x_i)p_j = p_i a_{ij}(t, x_i, x_j)$ for all $i, j = 1, \ldots, n$, $t \geq 1$, and $x \in \mathcal{G}$. Let $n_i \geq 1$ be the number of neighbors of the $i$th agent, that is, $|\mathcal{N}_i| = n_i$. If $p_i \geq \sup_{t\in[1,\infty),x_i,x_j\in\mathbb{R}}\{a_{ij}(t, x_i, x_j), a_{ji}(t, x_j, x_i)\}$ and $p_i < (1/n_i)$ for all $i, j = 1, \ldots, n$, $i \neq j$, then every solution to (18) is convergent relative to $\mathcal{G}$. Furthermore, $\lim_{t\to\infty} x(t) \in \mathcal{E}_{\mathrm{ss}} = \{x \in \mathcal{G} : x_i = (a/p_i) \in \overline{\mathbb{Z}}_+, i = 1, \ldots, n, a \geq 0\}$.

We note that so far no stability information is mentioned here. As a matter of fact, generally (18) is *not* Lyapunov stable relative to $\mathcal{G}$ for $n \geq 3$, needless to say semistability. This is due to the property of the floor function which creates a large jump between $x(1)$ and $x(0)$ (note that $x(t + 1) = A(t, x(t))x(t)$ for $t \geq 1$). On the other hand, if we take $\mathcal{G} = \overline{\mathbb{Z}}_+^n$, then one can show that every point in $\mathcal{E}_{\mathrm{ss}}$ is Lyapunov stable relative to $\mathcal{G}$, and hence, (18) is semistable relative to $\mathcal{G}$. Thus, the stability of (18) really depends on the choice of $\mathcal{G}$.

*Lemma 4.3:* Assume that Assumption 4.1 holds. Furthermore, assume that there exist $p_i > 0$, $i = 1, \ldots, n$, such that $a_{ji}(t, x_j, x_i)p_j = p_i a_{ij}(t, x_i, x_j)$ for all $i, j = 1, \ldots, n$, $t \geq 1$, $x \in \mathcal{G}$, and $\mathcal{G} = \overline{\mathbb{Z}}_+^n$. Then every point in $\mathcal{E}_{\mathrm{ss}}$ is Lyapunov stable relative to $\mathcal{G}$ for all $t \geq 0$.

*Theorem 4.2:* Assume that all the conditions of Theorem 4.1 are satisfied. Furthermore, assume $\mathcal{G} = \overline{\mathbb{Z}}_+^n$. Then (18) is semistable relative to $\mathcal{G}$ for all $t \geq 0$.

### B. Quadratic Form Based Convergence Analysis

The previous subsection assumes that $\mathbb{G}$ is an undirected graph. In this subsection, we remove this restriction by taking $\mathbb{G}$ to be directed. The following technical assumption guarantees the final distribution pattern of (18).

*Assumption 4.2:* There exist constants $\ell_{ij} \geq 0$, $i, j = 1, \ldots, n$, $i \neq j$, such that $\ell_{ij}/\ell_{ji}$ is rational and

$$\begin{bmatrix} \mathcal{C}_{(i,j)}a_{ij}^2 & -\mathcal{C}_{(i,j)}\mathcal{C}_{(j,i)}a_{ij}a_{ji} \\ -\mathcal{C}_{(j,i)}\mathcal{C}_{(i,j)}a_{ji}a_{ij} & \mathcal{C}_{(j,i)}a_{ji}^2 \end{bmatrix} \leq \begin{bmatrix} \mathcal{C}_{(i,j)}\ell_{ij}^2 & -\mathcal{C}_{(i,j)}\mathcal{C}_{(j,i)}\ell_{ij}\ell_{ji} \\ -\mathcal{C}_{(j,i)}\mathcal{C}_{(i,j)}\ell_{ji}\ell_{ij} & \mathcal{C}_{(j,i)}\ell_{ji}^2 \end{bmatrix} \quad (20)$$

for all $i, j = 1, \ldots, n$, $i \neq j$.

*Lemma 4.4:* Assume that Assumption 4.2 holds. Then $\|A(t, x)x - x\|_2 \leq \sqrt{n-1}\sqrt{x^{\mathrm{T}}\mathcal{L}x}$ for every $t \geq 1$ and $x \in \mathcal{G}$.

*Theorem 4.3:* Assume that Assumption 4.2 holds. Furthermore, assume that there exists $\alpha \in [0, 1)$ such that $A^{\mathrm{T}}(t, x)\mathcal{L}A(t, x) \leq \alpha\mathcal{L}$ for all $t \geq 1$ and $x \in \mathcal{G}$. Then every solution to (18) is convergent relative to $\mathcal{G}$. Furthermore, $\lim_{t\to\infty} x(t) \in \{x \in \mathcal{G} : \mathcal{L}x = 0\} \cap \overline{\mathbb{Z}}_+^n$.

Finally, for the stability of (18), we have similar results as before.

*Lemma 4.5:* Assume that Assumption 4.2 holds. Furthermore, assume that $\mathcal{G} = \overline{\mathbb{Z}}_+^n$. Then every point in $\{x \in \mathcal{G} : \mathcal{L}x = 0\}$ is Lyapunov stable relative to $\mathcal{G}$ for all $t \geq 0$.

*Theorem 4.4:* Assume that all the conditions of Theorem 4.3 are satisfied. Furthermore, assume that $\mathcal{G} = \overline{\mathbb{Z}}_+^n$. Then (18) is semistable relative to $\mathcal{G}$ for all $t \geq 0$.

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