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Global stability of uncertain rational nonlinear systems with some positive states

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Abstract— **This paper presents LMI conditions for local and global asymptotic stability of rational uncertain nonlinear systems where some or all the state variables are constrained by the model to have definite signal. The uncertainties are modeled as real time varying parameters with magnitude and rate of variation bounded by given polytopes. The stability conditions are based on a rational Lyapunov function with respect to the states and uncertain parameters. A numerical example is used to illustrate the potential of the proposed results.**

I. INTRODUCTION

The use of the LMI framework to express stability conditions for nonlinear systems as a convex problem is recent. For instance, using quadratic Lyapunov functions and Linear Fractional Transformations ideas to represent a rational nonlinear system, convex conditions for stability analysis and state feedback design can be found in [1]. LMI methods to determine and maximize an estimate of the region of attraction may be found in [2], [3]. The stability conditions in both references are based on polynomial Lyapunov functions. The stability conditions are transformed into LMIs by using sum of squares relaxations based on the $S-Procedure$ in [2]. In [3] a particular system decomposition is adopted and the LMIs are obtained with the Finsler's Lemma. Analysis and design methods based on sum of squares polynomials has received a lot of attention this last decade [4]. A given polynomial is SOS if it can be represented as $\theta(x)'Q\theta(x)$ where Q is a constant positive definite matrix and $\theta(x)$ a vector of monomials. When the Lyapunov conditions for stability can be expressed as SOS polynomials, the stability problem can be solved with powerful LMI tools [5], [2]. It is shown in [6] that any locally exponentially stable system with a thrice differentiable vector field will have a polynomial Lyapunov function which decreases exponentially on that region. If in one hand, this shows that polynomial Lyapunov functions, not necessarily of SOS type, seems to be reach enough to tackle a wide class of stability problems, on the other hand it should be mentioned that the polynomial degree, although finite, has no a priori upper bound available to date. Thus stability conditions based on polynomial Lyapunov functions may lead to poor estimates of the region of attraction if the degree of the polynomial is not high enough. In view of this fact, it could be interesting to investigate the potential of rational Lyapunov functions in stability problems of nonlinear systems. This problem is studied in this paper and we present LMI conditions for the asymptotic stability of uncertain rational systems. The conservativeness of state dependent LMIs is reduced with the use of certain *scaling matrices*. The Lyapunov function used in this paper is more general than the polynomial based Lyapunov functions used in all previous mentioned references. Here the Lyapunov function is a general rational matrix function of the state and parameter vectors.

The paper is organized as follows. The next section is devoted to some preliminaries and definitions. Conditions for local stability problems are presented in the Section III. It is shown that if a certain LMI is satisfied at the vertices of a given bounded polytope then the origin is exponentially stable and the Lyapunov function is directly obtained from the LMI. Based on this Lyapunov function, the Section IV proposes additional LMI conditions to extend the results for global stability problems where some or all the state variables are constrained by the model to have definite signal. The results are illustrated through a numerical example and some concluding remarks end the paper.

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Notation

$\mathbb{R}^n$ denotes the n-dimensional Euclidian space. $\mathbb{R}^{p\times q}$ is the set of $p\times q$ real matrices. $\mathbb{I}_q$ denotes the set of integers $\{1,\ldots,q\}$. M' denotes the transpose of M. $\|.\|$ represents the 2-norm of vectors and its induced spectral norm of matrices. I_r denotes the $r\times r$ identity matrix. A $p\times q$ matrix of zeros is denoted by $0_{p\times q}$. The $i-th$ row and columns of a matrix M is represented by $row_i(M)$ and $col_i(M)$. The notation $[.]_{col}$, $[.]_{row}$, $[.]_{diag}$ denote matrices whose elements, indicated inside the brackets, are arranged as a column, row and diagonal arrays. $[M]^n_{col}$, $[M]^n_{row}$, $[M]^n_{diag}$ are matrices where the element M is repeated n times as a column, row and diagonal arrays. Similarly, $[M_i]^n_{col}$, $[M_i]^n_{row}$, $[M_i]^n_{diag}$ indicate the matrices are constructed accordingly from the set of elements $\{M_i,\ i\in\mathbb{I}_n\}$. $M>0$ (≥ 0) means that M is a symmetric positive definite (semi-definite) matrix. For two polytopes $\Pi_1\subset\mathbb{R}^{n_1}$ and $\Pi_2\subset\mathbb{R}^{n_2}$ the notation $\Pi_1\times\Pi_2$ represents a meta-polytope of dimension n_1+n_2 obtained by the cartesian product of Π_1,Π_2. $\vartheta(\Pi)$ represents the set of all vertices of the polytope Π. $\overline{\lambda}(.),\underline{\lambda}(.)$ denotes the maximum and minimum eigenvalues of $(.)$.

II. PRELIMINARIES

Let us start with some definitions. Consider the system

$$\dot{x}=f(x,\delta)\ ,\quad x_0\in\mathscr{X}\ ,\ \delta\in\mathscr{D}\ ,\ \dot{\delta}\in\check{\mathscr{D}}\tag{1}$$

where $x \in \mathbb{R}^n$ is the state with initial condition x_0, $\delta \in \mathbb{R}^d$ a vector of uncertain parameters, $f(x,\delta)$ is a continuous vector function with $f(0,\delta)=0$. $\mathscr{X}$ is a given polytope, not necessarily symmetric, that contains the origin and represents a desired set of initial states to be considered in the stability analysis. $\mathscr{D}, \check{\mathscr{D}} \subset \mathbb{R}^d$ are given polytopes representing the bounds on the magnitude of the parameter δ and its rate of variation $(\dot{\delta})$. The notation $(x,\delta,\dot{\delta}) \in \mathscr{X} \times \mathscr{D} \times \check{\mathscr{D}}$ means that $x \in \mathscr{X}$ $\delta \in \mathscr{D}$ and $\dot{\delta} \in \check{\mathscr{D}}$.

In this paper we are concerned with the Lyapunov stability of the system (1). More precisely, we are interested in determining a suitable Lyapunov function $v(x,\delta)$ that satisfies the following conditions $\forall (x,\delta,\dot{\delta}) \in \mathscr{X} \times \mathscr{D} \times \check{\mathscr{D}}$.

$$\begin{array}{l} \phi_3(\|x\|) \le v(x,\delta) := x'\mathbf{P}(x,\delta)x \le \phi_1(\|x\|) \\ \dot{v}(x,\delta) = x'\dot{\mathbf{P}}(x,\delta)x + 2x'\mathbf{P}(x,\delta)f(x,\delta) \le -\phi_2(\|x\|) \end{array} \quad (2)$$

where $\phi_1(.), \phi_2(.), \phi_3(.)$ are class $\mathscr{K}$ functions and $\mathbf{P}(.)$ is symmetric. The above conditions imply local uniform asymptotic stability of the origin [7].

The complexity of the matrix function $\mathbf{P}(x,\delta)$ is an important aspect of the Lyapunov function. If in one hand the computational burden to numerically solve the stability problem grows with the complexity of the Lyapunov function, on the other hand less conservative results should be expected. In this paper we consider a Lyapunov function that is allowed to be a general rational function of the states and uncertain parameters. Several important class of functions, such as polynomials of any type and the well known quadratic function, are recovered as particular cases.

We end this preliminary section with a version of the Finsler's lemma and the definition of a Linear Annihilator.

Lemma 2.1 (Finsler's Lemma): Let $\mathscr{W} \subseteq \mathbb{R}^s$ be a polytopic set, $S(.) : \mathscr{W} \mapsto \mathbb{R}^{q\times q}$, $K(.) : \mathscr{W} \mapsto \mathbb{R}^{r\times q}$ be given matrix functions, with $S(.)$ symmetric. Let $Q(w)$ be a basis for the null space of $K(w)$. Then the following are equivalent:

(i) $\forall w \in \mathscr{W}$ the condition $z'S(w)z > 0$ is satisfied $\forall z \in \mathbb{R}^q :\ K(w)z = 0$.

(ii) $\forall w \in \mathscr{W}$ there exists a matrix function $L(w) : \mathscr{W} \mapsto \mathbb{R}^{q\times r}$ such that $S(w) + L(w)K(w) + K(w)'L(w)' > 0$.

(iii) $\forall w \in \mathscr{W}$ the condition $Q(w)'S(w)Q(w) > 0$ is satisfied. □

Two cases are of particular interest to this paper. The first is when $S(.), K(.)$ are affine functions and $L(.)$ is constrained to be constant. In this situation **(i)**,**(ii)** are no longer equivalent, but **(ii)** is clearly a sufficient polytopic LMI condition for **(i)**. The second case is when $S(.)$ is affine function and $K(.)$ is constrained to be constant, leading $Q(.)$ to be constant as well. In this case **(i)**,**(iii)** are yet equivalent and **(iii)** is a polytopic LMI with a smaller number of decision variable when compared to **(ii)**. The interest of these two polytopic LMI problems is that they are numerically efficient alternatives to the condition **(i)**, which is an infinite dimensional problem. See for instance [8] and Lemma 7.3 of [9] for more details on the Finsler's Lemma.

Another definition of interest is as follows.

Definition 2.1 (Linear Annihilator): Given a vector function $f(.) : \mathbb{R}^q \mapsto \mathbb{R}^s$, a matrix function $\aleph_f(.) : \mathbb{R}^q \mapsto \mathbb{R}^{r\times s}$ will be called a *Linear Annihilator of* $f(.)$ if it satisfies the following two requirements **(i)** $\aleph_f(.)$ is linear and **(ii)** $\aleph_f(z)f(z) = 0$, $\forall z \in \mathbb{R}^q$ of interest. □

Observe that the matrix representation of a Linear Annihilator is not unique in general. Suppose that $f(z) = z = \begin{bmatrix} z_1 & \dots & z_q \end{bmatrix}' \in \mathbb{R}^q$. Taking into account all possible pairs z_i, z_j for $i \neq j$ without repetition, i.e. for $\forall i,j \in \mathbb{I}_q$ with $j > i$, we get an annihilator where the number of rows are $r = \sum_{j=1}^{q-1} j$ and it is given by the formula

$$\aleph_z(z) = \begin{bmatrix} \psi_1(z) & Y_1(z) \\ \vdots & \vdots \\ \psi_{(q-1)}(z) & Y_{(q-1)}(z) \end{bmatrix} \quad (3)$$

$$Y_i(z) = -z_i I_{(q-i)} \ ,\ i \in \mathbb{I}_{q-1} \ ,\ \psi_1(z) = \begin{bmatrix} z_2 & \dots & z_q \end{bmatrix}'$$

$$\psi_i(z) = \begin{bmatrix} & z_{(i+1)} \\ 0_{(q-i)\times(i-1)} & \vdots \\ & z_q \end{bmatrix},\ i \in \{2,\dots,q-1\}$$

If $f(z) = \begin{bmatrix} z_1^2 & z_1 z_2 \end{bmatrix}'$ a linear annihilator is $\aleph_f(z) = \begin{bmatrix} z_2 & -z_1 \end{bmatrix}$. Observe that linear annihilators express a linear interdependence among the entries of $f(z)$. The conservativeness of state dependent LMIs can be reduced by combining the Linear Annihilator with the Finsler's Lemma.

III. Local Stability Results

Let us suppose that the system (1) can be represented as

$$\begin{array}{ll} \dot{x} = f(x,\delta) = A_0 x + A_1 \pi & x_0 \in \mathscr{X} \\ 0 = G(x,\delta)x + F(x,\delta)\pi & \delta \in \mathscr{D}\ ,\ \dot{\delta} \in \check{\mathscr{D}}. \\ x \in \mathbb{R}^n\ ,\ \pi \in \mathbb{R}^p\ ,\ \delta \in \mathbb{R}^d & \end{array} \quad (4)$$

Assumption (4a): $f(x,\delta)$ is a continuous function of $(x,\delta), \forall (x,\delta) \in \mathscr{X} \times \mathscr{D}$ with $f(0,\delta) = 0$ for all δ of interest. This assumption regards the existence and uniqueness of the solutions of the differential equation in a neighborhood $\mathscr{X}$ of the equilibrium point $x = 0$.

Assumption (4b): The matrices $F(x,\delta), G(x,\delta)$ are affine functions of (x,δ) and $F(x,\delta)$ is invertible for all values of $(x,\delta) \in \mathscr{X} \times \mathscr{D}$. Under this regularity assumption the decomposition (4) of $f(x,\delta)$ in terms of the nonlinear function π is well posed as $f(x,\delta) = (A_0 - A_1 F(x,\delta)^{-1} G(x,\delta))x$.

It is important to emphasize that the matrices of the system representation (4) can be easily obtained and there exist systematic ways for doing this. Due to space limitation these results are omitted here.

In the sequel we present LMI conditions for local stability of the origin of the system (4). We suppose that the polytopes $\mathscr{X}, \mathscr{D}$ and $\check{\mathscr{D}}$ are given. They represent, respectively, the desired set of initial conditions to be considered in the stability analysis and the bounds on the magnitude and rate of variation of the uncertain parameters. The problem to be solved is to find, if possible, a Lyapunov function $v(x,\delta) = x'\mathbf{P}(x,\delta)x$ that satisfies the conditions (2) pointwise $\forall (x,\delta,\dot{\delta}) \in \mathscr{X} \times \mathscr{D} \times \check{\mathscr{D}}$.

Consider the system (4) under Assumptions (4a,b) and the Lyapunov function candidate

$$v(x,\delta) := \pi_b' P \pi_b = x'\mathbf{P}(x,\delta)x\ ,\ \pi_b := [x,\pi]_{col} \quad (5)$$

where $G(x,\delta)x+F(x,\delta)\pi=0$ as in (4), $P\in\mathbb{R}^{(n+p)\times(n+p)}$ is a symmetric matrix to be determined and

$$\mathbf{P}(x,\delta)=\begin{bmatrix} I_n \\ -F(x,\delta)^{-1}G(x,\delta)\end{bmatrix}' P \begin{bmatrix} I_n \\ -F(x,\delta)^{-1}G(x,\delta)\end{bmatrix} \quad (6)$$

Without loss of generality, we consider both the system and the Lyapunov function are represented from the same π vector of nonlinear functions. It is important to emphasize that, with a proper choice of π we can consider any type of rational systems and Lyapunov functions. For instance, the Krasovskii stability method [7] for the system $\dot{x}=f(x)$, which is based on a Lyapunov function of the type $v(x)=f(x)'P_k f(x)$ may be obtained as particular case from (5) by choosing $P=\begin{bmatrix} A_0 & A_1\end{bmatrix}'P_k\begin{bmatrix} A_0 & A_1\end{bmatrix}$. The well known quadratic Lyapunov function $v(x)=x'P_0x$, with $P_0>0$, corresponds to the choice

$$P=\begin{bmatrix} I_n & 0_{n\times p}\end{bmatrix}'P_0\begin{bmatrix} I_n & 0_{n\times p}\end{bmatrix} \quad (7)$$

Rational Lyapunov functions of the type SOS are obtained with the constraint $P>0$.

Let us define the matrix C_b as

$$C_b(x,\delta)=\begin{bmatrix} G(x,\delta) & F(x,\delta)\end{bmatrix} \quad (8)$$

and observe from (4) that $C_b(x,\delta)$ is affine with respect to (x,δ) and satisfies $C_b(x,\delta)\pi_b=0$ with π_b given in (5).

Observe from Assumption (4b), the vector π is uniquely determined from (x,δ) through the relation $\pi=-F(x,\delta)^{-1}G(x,\delta)x$. However, as previously mentioned there are, in general, several pair of matrices $F(x,\delta),G(x,\delta)$ that could be used to characterize the same π function. In this context, the use of linear annihilators jointly with the Finsler's Lemma is an important feature of the method. In order to exploit the degrees of freedom associated with $\aleph_{\pi_b}(x,\delta)$, let us define

$$\begin{array}{c}\Gamma_b(x,\delta):=M_b\aleph_{\pi_b}(x,\delta)+\aleph_{\pi_b}(x,\delta)'M_b' \\ \aleph_{\pi_b}(x,\delta):=[\aleph_x(x),\aleph_\pi(x,\delta)]_{diag}\in\mathbb{R}^{(s_b)\times(n+p)}\end{array} \quad (9)$$

where $\aleph_{\pi_b}(x,\delta)\in\mathbb{R}^{(s_b)\times(n+p)}$ is a linear annihilator of π_b, $M_b\in\mathbb{R}^{(n+p)\times(s_b)}$ is a free scaling matrix to be determined and note that $\pi_b'\Gamma_b(x,\delta)\pi_b=0$. Observe that both $C_b(x,\delta),\aleph_{\pi_b}(x,\delta)$ can be viewed as linear annihilators of π_b as $C_b(x,\delta)\pi_b=0$ and $\aleph_{\pi_b}(x,\delta)\pi_b=0$. However, these matrices have different roles. While the first is associated with the particular pair of matrices $F(x,\delta),G(x,\delta)$ used to characterize the π function, the second introduces additional degrees of freedom, represented by the scaling matrix M_b, to alleviate the impact of the non-uniqueness of $F(x,\delta),G(x,\delta)$.

From standard convexity properties of LMIs, it is not difficult to check that the existence of $P=P'\in\mathbb{R}^{(n+p)\times(n+p)}$, $L_b\in\mathbb{R}^{(n+p)\times(p)}$, $M_b\in\mathbb{R}^{(n+p)\times(s_b)}$ satisfying

$$\begin{array}{c}P+L_bC_b(x,\delta)+C_b(x,\delta)'L_b'+\Gamma_b(x,\delta)>0 \\ \forall(x,\delta)\in\vartheta(\mathscr{X}\times\mathscr{D})\end{array} \quad (10)$$

is a sufficient LMI condition for the positiveness of $v(x,\delta)$ from (5) $\forall(x,\delta)\in\mathscr{X}\times\mathscr{D}$.

Next, the time derivative of $v(x,\delta)$ leads to

$$\dot{v}(x,\delta)=2\begin{bmatrix} x \\ \pi\end{bmatrix}'P\begin{bmatrix} A_0x+A_1\pi \\ \dot{\pi}\end{bmatrix} \quad (11)$$

As $G(x,\delta)x+F(x,\delta)\pi=0$, the vector $\dot{\pi}$ satisfies the relation

$$\dot{F}(x,\delta)\pi+F(x,\delta)\dot{\pi}+\dot{G}(x,\delta)x+G(x,\delta)(A_0x+A_1\pi)=0 \quad (12)$$

Keeping in mind the matrices $F(x,\delta),G(x,\delta)$ are affine functions of (x,δ), consider the notation

$$\begin{array}{c}G(x,\delta):=G_0+\bar{G}(x)+\tilde{G}(\delta)\ ,\ \bar{G}(x)=\sum_{i=1}^n\bar{G}_ix_i \\ F(x,\delta):=F_0+\bar{F}(x)+\tilde{F}(\delta)\ ,\ \bar{F}(x)=\sum_{i=1}^n\bar{F}_ix_i \\ \tilde{G}(\delta)=\sum_{i=1}^d\tilde{G}_i\delta_i\ ,\ \tilde{F}(\delta)=\sum_{i=1}^d\tilde{F}_i\delta_i\end{array} \quad (13)$$

where x_i,δ_i are the entries of x,δ and $\bar{F}_i,\tilde{F}_i,\bar{G}_i,\tilde{G}_i,G_0,F_0$ are constant matrices of structure that can be easily obtained from $G(x,\delta),F(x,\delta)$. Then it follows that $\dot{F}(x,\delta),\dot{G}(x,\delta)$ are linear functions of $(\dot{x},\dot{\delta})$ given respectively by $\bar{F}(\dot{x})+\tilde{F}(\dot{\delta})$ and $\bar{G}(\dot{x})+\tilde{G}(\dot{\delta})$. Now observe the term $\bar{F}(\dot{x})\pi$ that appears in (12), may be rewritten as

$$\begin{aligned}\bar{F}(\dot{x})\pi=\sum_{j=1}^n\bar{F}_j\dot{x}_j\pi &= -\sum_{j=1}^n\bar{F}_j\dot{x}_jF(x,\delta)^{-1}G(x,\delta)x \\ &= \sum_{j=1}^n\bar{F}_jF(x,\delta)^{-1}G(x,\delta)\mu_j \end{aligned} \quad (14)$$

$$\mu_j=-x\dot{x}_j=-xE_j\dot{x}=-xE_j(A_0x+A_1\pi) \quad (15)$$

where $E_j:=row_j(I_n)$. Introducing another change of variable

$$\eta_j:=F(x,\delta)^{-1}G(x,\delta)\mu_j \quad (16)$$

we get $\bar{F}(\dot{x})\pi=\sum_{j=1}^n\bar{F}_j\eta_j=\bar{F}_a\eta\ ,\ \eta=[\eta_i]_{col}^n\ ,\ \bar{F}_a=[\bar{F}_i]_{row}^n$. The term $\bar{G}(\dot{x})x$ that appears in (12), may be rewritten as

$$\begin{array}{c}\bar{G}(\dot{x})x=\sum_{j=1}^n\bar{G}_j\dot{x}_jx=\sum_{j=1}^n\bar{G}_jxE_j\dot{x}=\bar{G}_a(x)\dot{x} \\ =\bar{G}_a(x)(A_0x+A_1\pi)\ ,\ \bar{G}_a(x):=\sum_{j=1}^n\bar{G}_jxE_j\end{array} \quad (17)$$

With the above expressions we can rewrite (12) as

$$\begin{aligned}&\bar{F}_a\eta+\tilde{F}(\dot{\delta})\pi+F(x,\delta)\dot{\pi} \\ &\quad +(\bar{G}_a(x)+G(x,\delta))(A_0x+A_1\pi)+\tilde{G}(\dot{\delta})x=0\end{aligned} \quad (18)$$

where $F(x,\delta),G(x,\delta)$ have the structure (13) and

$$\begin{array}{l}\mu=[\mu_i]_{col}^n=-E_b(x)(A_0x+A_1\pi) \\ E_b(x):=[xE_i]_{col}^n\ ,\ E_i:=row_i(I_n)\ ,\ F_b(x,\delta)\eta=G_b(x,\delta)\mu \\ F_b(x,\delta):=[F(x,\delta)]_{diag}^n\ ,\ G_b(x,\delta):=[G(x,\delta)]_{diag}^n \\ \bar{G}_a(x):=\sum_{i=1}^n\bar{G}_ixE_i\ ,\ \bar{F}_a:=[\bar{F}_i]_{row}^n\end{array} \quad (19)$$

The time derivative of $v(x,\delta)$ indicated in (11) can be rewritten as

$$\dot{v}(x,\delta)=\pi_a'(P_a+P_a')\pi_a\quad ,\quad \pi_a=[x,\pi,\dot{\pi},\mu,\eta]_{col} \quad (20)$$

$$\begin{array}{c}P_a:=\begin{bmatrix} PA_a \\ 0_{(p+n^2+np)\times(n+2p+n^2+np)}\end{bmatrix} \\ A_a:=\begin{bmatrix} A_0 & A_1 & 0_{n\times p} & 0_{n\times(n^2+np)} \\ 0_{p\times n} & 0_{p\times p} & I_p & 0_{p\times(n^2+np)}\end{bmatrix}\end{array} \quad (21)$$

Observe that n,p are the dimensions of x,π respectively. To represent the system and Lyapunov function we have used the vectors x,π. However, the time derivative of $v(x,\delta)$ has increased complexity and we need extra change of variables, namely $\dot{\pi}, \mu, \eta$, with dimensions p, n^2, np respectively, to render the expressions affine in $x, \delta, \dot{\delta}$.

By arranging in a single expression all the relations among the vectors $x, \pi, \dot{\pi}, \mu, \eta$ we get $C_a(x,\delta,\dot{\delta})\pi_a = 0$ where

$$C_a(x,\delta,\dot{\delta}) := \qquad (22)$$

$$\begin{bmatrix} G(x,\delta) & F(x,\delta) & 0_{p\times p} & 0_{p\times n^2} & 0_{p\times np} \\ W_1(x,\delta,\dot{\delta}) & W_2(x,\delta,\dot{\delta}) & F(x,\delta) & 0_{p\times n^2} & \bar{F}_a \\ E_b(x)A_0 & E_b(x)A_1 & 0_{n^2\times p} & I_{n^2} & 0_{n^2\times np} \\ 0_{np\times n} & 0_{np\times p} & 0_{np\times p} & -G_b(x,\delta) & F_b(x,\delta) \end{bmatrix}$$

$$W_1(x,\delta,\dot{\delta}) = \bar{G}_a(x)A_0 + \tilde{G}(\dot{\delta}) + G(x,\delta)A_0$$
$$W_2(x,\delta,\dot{\delta}) = \bar{G}_a(x)A_1 + \tilde{F}(\dot{\delta}) + G(x,\delta)A_1$$

Similarly to (9), we define $\Gamma_a(x,\delta,\dot{\delta})$ as

$$\Gamma_a(x,\delta,\dot{\delta}) := M_a \aleph_{\pi_a}(x,\delta,\dot{\delta}) + \aleph_{\pi_a}(x,\delta,\dot{\delta})' M_a' \qquad (23)$$

where $\aleph_{\pi_a}(x,\delta,\dot{\delta}) \in \mathbb{R}^{s_a\times(n+2p+n^2+np)}$ is a linear annihilator of π_a, $M_a \in \mathbb{R}^{(n+2p+n^2+np)\times s_a}$ is a free scaling matrix to be determined and note that $\pi_a'\Gamma_a(x,\delta,\dot{\delta})\pi_a = 0$.

In order to find an expression for $\aleph_{\pi_a}(x,\delta,\dot{\delta})$ observe that $\aleph_{\pi_b}(x,\delta)\pi_b = 0$. Then we get $\dot{\aleph}_{\pi_b}(x,\delta)\pi_b + \aleph_{\pi_b}(x,\delta)\dot{\pi}_b = 0$. As $\aleph_{\pi_b}(x,\delta)$ is affine with respect to x,δ it follows that $\dot{\aleph}_{\pi_b}(x,\delta)$ is linear with respect to $\dot{x},\dot{\delta}$ and we consider the notation

$$\begin{array}{ll} \aleph_{\pi_b}(x,\delta) = H_0 + \bar{H}(x) + \tilde{H}(\delta), & \dot{\aleph}_{\pi_b}(x,\delta) = \bar{H}(\dot{x}) + \tilde{H}(\dot{\delta}) \\ \bar{H}(x) = \sum_{i=1}^n \bar{H}_i x_i, & \bar{H}(\dot{x}) = \sum_{i=1}^n \bar{H}_i \dot{x}_i \\ \tilde{H}(\delta) = \sum_{i=1}^d \tilde{H}_i \delta_i, & \tilde{H}(\dot{\delta}) = \sum_{i=1}^d \tilde{H}_i \dot{\delta}_i \end{array} \qquad (24)$$

where $H_0, \bar{H}_i, \tilde{H}_i \in \mathbb{R}^{s_b\times(n+p)}$ are fixed matrices of structure that can be easily obtained from $\aleph_{\pi_b}(x,\delta)$.

Considering the additional notation

$$\pi_b = J_0 x + J_1 \pi \quad , \quad J_0 = \begin{bmatrix} I_n \\ 0_{p\times n} \end{bmatrix} \quad , \quad J_1 = \begin{bmatrix} 0_{n\times p} \\ I_p \end{bmatrix} \qquad (25)$$

and proceeding as in (14),(17) we get

$$\begin{array}{c} \bar{H}(\dot{x})\pi_b = \sum_{i=1}^n \bar{H}_i J_1 \eta_i - \sum_{i=1}^n \bar{H}_i J_0 \mu_i = \bar{H}_a \eta + \bar{H}_b \mu \\ \bar{H}_a := [\bar{H}_i J_1]_{row}^n \quad , \quad \bar{H}_b := -[\bar{H}_i J_0]_{row}^n \end{array} \qquad (26)$$

Finally, the expression $\dot{\aleph}_{\pi_b}(x,\delta)\pi_b + \aleph_{\pi_b}(x,\delta)\dot{\pi}_b = 0$ can be rewritten as $\bar{H}_a\eta + \bar{H}_b\mu + \tilde{H}(\dot{\delta})\pi_b + \aleph_{\pi_b}(x,\delta)\dot{\pi}_b = 0$ and with (25) we get $\bar{H}_a\eta + \bar{H}_b\mu + \tilde{H}(\dot{\delta})(J_0 x + J_1\pi) + \aleph_{\pi_b}(x,\delta)(J_0 A_0 x + J_0 A_1 \pi + J_1\dot{\pi}) = 0$, that in turn yields

$$\aleph_{\pi_a}(x,\delta,\dot{\delta}) = \begin{bmatrix} \aleph_{\pi_b}(x,\delta) & 0_{s_b\times p} & 0_{s_b\times n^2} & 0_{s_b\times np} \\ 0_{s_b\times(n+p)} & 0_{s_b\times p} & \aleph_{\mu}(x) & 0_{s_b\times np} \\ 0_{s_b\times(n+p)} & 0_{s_b\times p} & 0_{s_b\times n^2} & \aleph_{\eta}(x,\delta) \\ W_a & W_b & \aleph_{\pi_b}(x,\delta)J_1 & \bar{H}_b \quad \bar{H}_a \end{bmatrix}$$

$$W_a = \tilde{H}(\dot{\delta})J_0 + \aleph_{\pi_b}(x,\delta)J_0 A_0 \qquad (27)$$
$$W_b = \tilde{H}(\dot{\delta})J_1 + \aleph_{\pi_b}(x,\delta)J_0 A_1$$

where $\aleph_{\pi_b}$ is a linear annihilator of π_b, $\tilde{H}(\dot{\delta}), \bar{H}_a, \bar{H}_b$ are defined from (24),(26) and the annihilators $\aleph_{\mu}(x)$ and $\aleph_{\eta}(x)$ are obtained as follows. Observe from (15) that $\mu_i = -x\dot{x}_i$ and as $\dot{x}_i$ is a scalar we conclude μ_i and x have the same linear annihilators, i.e. $\aleph_{\mu_i}(x) = \aleph_x(x)$. As $\mu(x) = [\mu_i]_{col}^n$ we get $\aleph_{\mu}(x) := [\aleph_x(x)]_{diag}^n$.

Moreover, from (16) it follows that $\eta_i = F(x,\delta)^{-1}G(x,\delta)\mu_i = \pi\dot{x}_i$ and thus η_i and π have the same linear annihilator, i.e. $\aleph_{\eta_i}(x,\delta) = \aleph_{\pi}(x,\delta)$. As $\eta(x) = [\eta_i]_{col}^n$ we get $\aleph_{\eta}(x,\delta) := [\aleph_{\pi}(x,\delta)]_{diag}^n$.

The matrices $C_a(x,\delta,\dot{\delta}), \aleph_{\pi_a}(x,\delta,\dot{\delta})$ may be viewed as annihilators of π_a but, similarly to (9), they have different roles in the problem.

As $C_a(x,\delta,\dot{\delta})\pi_a = 0$ and $\pi_a'\Gamma_a(x,\delta,\dot{\delta})\pi_a = 0$, from (20) and the Finsler's Lemma we get the following sufficient LMI condition for the negativeness of $\dot{v}(x,\delta)$.

$$P_a + P_a' + C_a(x,\delta,\dot{\delta})'L_a' + L_a C_a(x,\delta,\dot{\delta}) + \Gamma_a(x,\delta,\dot{\delta}) < 0$$
$$\forall (x,\delta,\dot{\delta}) \in \vartheta(\mathscr{X}\times\mathscr{D}\times\check{\mathscr{D}}) \qquad (28)$$

where L_a, M_a are free scaling matrices having the same dimensions of C_a', $\aleph_{\pi_a}'$ and must be determined jointly with P that characterizes P_a from (21). If the LMI (28) is satisfied for some L_a, M_a, P, then the function $\dot{v}(x,\delta)$ is negative definite.

With these results we can prove the following theorem.

Theorem 3.1: Consider the uncertain nonlinear system (4) with Assumptions (4a,b) and suppose the polytopes $\mathscr{X}, \mathscr{D}, \check{\mathscr{D}}$ are given.
Suppose there exist matrices P, L_a, M_a, L_b, M_b satisfying the LMIs (10),(28) for $(x,\delta,\dot{\delta})$ at the vertices of the polytope $\mathscr{X}\times\mathscr{D}\times\check{\mathscr{D}}$. Then the origin of the system (4) is locally asymptotically stable and $v(x,\delta) = x'\mathbf{P}(x,\delta)x$ with the structure (6) is a Lyapunov function that satisfies (2).

Proof: Suppose now the conditions of the Theorem 3.1 are satisfied. Then by convexity they are also satisfied $\forall (x,\delta,\dot{\delta}) \in \mathscr{X}\times\mathscr{D}\times\check{\mathscr{D}}$. Define the positive constants

$$\varepsilon_1 = \max_{x\in\mathscr{X},\delta\in\mathscr{D}} \overline{\lambda}(P + L_b C_b(x,\delta) + C_b(x,\delta)'L_b' + \Gamma_b(x,\delta))$$
$$\varepsilon_3 = \min_{x\in\mathscr{X},\delta\in\mathscr{D}} \underline{\lambda}(P + L_b C_b(x,\delta) + C_b(x,\delta)'L_b' + \Gamma_b(x,\delta))$$
$$\varepsilon_2 = \max_{x\in\mathscr{X},\delta\in\mathscr{D}} \overline{\lambda}(M'M),\ M := F(x,\delta)^{-1}G(x,\delta) \qquad (29)$$

As $P + L_b C_b + C_b' L_b' + \Gamma_b > 0$, let us multiply this inequality by $\pi_b := [x,\pi]_{col}$ to the right and by its transpose to the left. Keeping in mind that $C_b\pi_b = 0$, $\pi_b'\Gamma_b\pi_b = 0$ and $\pi_b' P \pi_b = x'\mathbf{P}(x,\delta)x$ as $\pi = -F(x,\delta)^{-1}G(x,\delta)x$, we get

$$\varepsilon_3\|\pi_b\|^2 \le v(x,\delta) = x'\mathbf{P}(x,\delta)x \le \varepsilon_1\|\pi_b\|^2\ ,\ \forall (x,\delta)\in\mathscr{X}\times\mathscr{D}$$

On the other hand $\|x\|^2 \le \|\pi_b\|^2 \le (\varepsilon_2+1)\|x\|^2$, $\forall (x,\delta) \in \mathscr{X}\times\mathscr{D}$. Thus $v(x,\delta) = x'\mathbf{P}(x,\delta)x$ satisfies the bounds in (2) $\forall (x,\delta) \in \mathscr{X}\times\mathscr{D}$ with

$$\phi_3 = \varepsilon_3\|x\|^2\ ,\ \phi_1 = \varepsilon_1(\varepsilon_2+1)\|x\|^2 \qquad (30)$$

Similar arguments are used to show the bounds on $\dot{v}(x,\delta)$. Define the positive constant ε_4 as

$$\varepsilon_4 = \min_{x\in\mathscr{X},\delta\in\mathscr{D},\dot{\delta}\in\check{\mathscr{D}}} \underline{\lambda}(-N) \qquad (31)$$
$$N := P_a + P_a' + C_a(x,\delta,\dot{\delta})'L_a' + L_a C_a(x,\delta,\dot{\delta}) + \Gamma_a(x,\delta,\dot{\delta})$$

Recall from (20),(22),(23) that $C_a\pi_a = 0$, $\pi_a'\Gamma_a\pi_a = 0$. Thus from (20),(28),(31) we get

$$\dot{v}(x,\delta) = \pi_a' N \pi_a \le -\varepsilon_4\|\pi_a\|^2 \qquad (32)$$

As $\|\pi_a\|^2 = \|x\|^2 + \|\pi\|^2 + \|\dot{\pi}\|^2 + \|\mu\|^2 + \|\eta\|^2$ we conclude $\|\pi_a\|^2 > \|x\|^2$ whenever $\|x\| \neq 0$ which in turn implies $\dot{v}(x,\delta) < -\varepsilon_4\|x\|^2$ and we conclude $\dot{v}(x,\delta)$ satisfies the bounds in (2) $\forall (x,\delta,\dot{\delta}) \in \mathscr{X} \times \mathscr{D} \times \check{\mathscr{D}}$ with $\phi_2 = \varepsilon_4 \|x\|^2$, which completes the proof. □

Observe the uncertain nonlinear system (4) is locally exponentially stable whenever the LMI conditions of the Theorem 3.1 are strictly satisfied and the sets $\mathscr{X}$ of initial conditions and $\mathscr{D}$ of uncertain parameters are bounded. This follows direct from (30),(29) and the boundedness of $\mathscr{X}, \mathscr{D}$.

IV. GLOBAL STABILITY

This section is devoted to the study of global stability problems and the idea is to find additional conditions under which the results for local stability of the previous section can be extended for global stability.

It turns out from the Theorem 3.1 that the global stability of the system (4) is guaranteed if the LMIs (10),(28) are satisfied with $\mathscr{X} = \mathbb{R}^n$. The problem of concern is then to find under what additional conditions these state-dependent LMIs may be satisfied $\forall x \in \mathbb{R}^n$.

Let us start with the following result.

Lemma 4.1: Consider the LMI problem $\Psi(W,z) > 0$ where $z \in \mathbb{R}^{n_z}$ is a vector of parameters and W the decision variable to be tuned. Suppose that $\Psi(.)$ can be decomposed as

$$\Psi(W,z) = \Psi_0(W) + \sum_{i=1}^{n_z} \Psi_i(W) z_i \tag{33}$$

where z_i are the entries of z and $\Psi_i(W)$ are affine functions of W.

Then $\Psi(W,z) > 0$ is satisfied for some W and $\forall z \in \mathbb{R}^{n_z}$ if and only if $\Psi_0(W) > 0$ and $\Psi_i(W) = 0$, $\forall i \in \mathbb{I}_{n_z}$. □

Proof: The sufficiency is trivial and the necessity follows by contradiction. Suppose $\Psi(W,z) > 0$ is satisfied $\forall z \in \mathbb{R}^{n_z}$ and $\Psi_i(W) \neq 0$ for some $i \in \mathbb{I}_{n_z}$. As z_i, $i \in \mathbb{I}_{n_z}$ are independent parameters consider the situation where $z_j = 0$, $\forall j \neq i$ and $z_i \in \{-\infty, \infty\}$. Then if $\Psi(W,z) = \Psi_0(W) + z_i\Psi_i(W) > 0$ is satisfied for $z_i \to \infty$ we conclude this condition cannot be satisfied for $z_i \to -\infty$. This shows $\Psi_i(W) = 0$ $\forall i \in \mathbb{I}_{n_z}$ are necessary conditions. Moreover $\Psi_0(W) > 0$ is necessary to satisfy $\Psi(W,z) > 0$ when $z = 0$, completing the proof. □

In order to generalize the local stability conditions of the Theorem 3.1 to global stability let us represent the decision variables of the LMIs (10),(28) through the notation $W := [P, M_a, L_a, M_b, L_b]_{diag}$ and consider the affine decomposition of the LMIs (10),(28) in terms of the state x as follows.

$$\begin{aligned}\Psi(W,x,\delta) &= P + L_b C_b(x,\delta) + C_b(x,\delta)' L_b' + \Gamma_b(x,\delta) \\ &= \Psi_0(W,\delta) + \sum_{i=1}^{n} \Psi_i(W) x_i \end{aligned}\tag{34}$$

$$\begin{aligned}\Phi(W,x,\delta,\dot{\delta}) &= P_a' + P_a + L_a C_a(x,\delta,\dot{\delta}) + C_a(x,\delta,\dot{\delta})' L_a' \\ &+ \Gamma_a(x,\delta,\dot{\delta}) = \Phi_0(W,\delta,\dot{\delta}) + \sum_{i=1}^{n} \Phi_i(W) x_i \end{aligned}\tag{35}$$

Corollary 4.1: Consider the system (4) with Assumptions (4a,b), let $\mathscr{D}, \check{\mathscr{D}}$ be given and $\mathscr{X} = \mathbb{R}^n$. Consider the affine decomposition (35),(34) . Suppose there exists W solving the following convex LMI problem

$$\begin{aligned}&\Psi_0(W,\delta) > 0,\ \Phi_0(W,\delta,\dot{\delta}) < 0,\ \forall(\delta,\dot{\delta}) \in \vartheta(\mathscr{D} \times \check{\mathscr{D}}) \\ &\Psi_i(W) = 0\ ,\ \Phi_i(W) = 0\ ,\ \forall i \in \mathbb{I}_n \end{aligned}\tag{36}$$

Then the origin of the system (4) is globally asymptotically stable $\forall(\delta,\dot{\delta}) \in \mathscr{D} \times \check{\mathscr{D}}$ and $v(x,\delta) = x'\mathbf{P}(x,\delta)x$ with the structure (6) is a Lyapunov function for the system. □

Proof. Observe (36) implies $\Psi(W,x,\delta) > 0$ from (34) and $\Phi(W,x,\delta,\dot{\delta}) < 0$ from (35) are satisfied $\forall(\delta,\dot{\delta}) \in \mathscr{D} \times \check{\mathscr{D}}$ and $\forall x \in \mathbb{R}^n$. Moreover, both $\Psi(W,x,\delta)$ and $\Phi(W,x,\delta,\dot{\delta})$ are independent of x, i.e. $\Psi(W,x,\delta) = \Psi_0(W,\delta)$ and $\Phi(W,x,\delta,\dot{\delta}) = \Phi_0(W,\delta,\dot{\delta})$. This implies the positive constants $\varepsilon_4, \varepsilon_3, \varepsilon_1$ from (31),(29) are bounded and given by

$$\begin{aligned}\varepsilon_4 &= \min_{\mathscr{X}\times\mathscr{D}\times\check{\mathscr{D}}} \underline{\lambda}(-\Phi(W,x,\delta,\dot{\delta})) = \min_{\mathscr{D}\times\check{\mathscr{D}}} \underline{\lambda}(-\Phi_0(W,\delta,\dot{\delta})) \\ \varepsilon_1 &= \max_{\mathscr{X}\times\mathscr{D}} \overline{\lambda}(\Psi(W,x,\delta)) = \max_{\mathscr{D}} \overline{\lambda}(\Psi_0(W,\delta)) \\ \varepsilon_3 &= \min_{\mathscr{X}\times\mathscr{D}} \underline{\lambda}(\Psi(W,x,\delta)) = \min_{\mathscr{D}} \underline{\lambda}(\Psi_0(W,\delta))\end{aligned}$$

However it is not possible to show, in general, the boundedness of ε_2 from (29) when $\mathscr{X} = \mathbb{R}^n$. In this case the conditions (2) are satisfied with

$$\begin{aligned}&\phi_3 = \varepsilon_3 \|x\|^2\ ,\ \phi_1 = \varepsilon_1 (\varepsilon_2(x) + 1)\|x\|^2\ ,\ \phi_2 = \varepsilon_4 \|x\|^2 \\ &\varepsilon_2(x) = \max_{\delta\in\mathscr{D}} \overline{\lambda}(M'M),\ M := F(x,\delta)^{-1} G(x,\delta)\end{aligned}$$

Observe ϕ_1 is positive definite and bounded for bounded $\|x\|$. The proof is completed from Theorem 3.1 and [7]. □

Remark 4.1: The Lemma 4.1 can also be used to solve stability problems where the uncertain parameters have arbitrary rates of variation, i.e. $\check{\mathscr{D}} = \mathbb{R}^d$. In this case it suffices to consider the affine decomposition of $\Phi(W,x,\delta,\dot{\delta})$ in terms of $\dot{\delta}_i$ and apply the Lemma 4.1. This situation will lead to a Lyapunov function that is not parameter dependent. □

Following the same lines of the above corollary it is possible to derive stability conditions for systems where some or all the state variables are constrained by the model to have definite signal.

Corollary 4.2: Consider the system (4) under Assumptions (4a,b), with

$$\mathscr{X} := \left\{ \begin{array}{l} x = [\ x_1 \ \ \dots \ \ x_n\ ]' \text{ such that} \\ x_i \quad \text{unconstrained for } i \in \mathfrak{I}_{unc} \\ x_i \geq 0 \quad \text{for } i \in \mathfrak{I}_{pos} \\ x_i \leq 0 \quad \text{for } i \in \mathfrak{I}_{neg} \\ \mathfrak{I}_{unc} + \mathfrak{I}_{pos} + \mathfrak{I}_{neg} = \mathbb{I}_n \end{array} \right\} \tag{37}$$

and let $\mathscr{D}, \check{\mathscr{D}}$ be given polytopes. Suppose $\mathscr{X}$ is positively invariant for the system trajectories, i.e. the state variables $x_i(t)$ for $i \in \mathfrak{I}_{pos} + \mathfrak{I}_{neg}$ do not change of signal for $t \geq t_0$ if $x(t_0) \in \mathscr{X}$. Consider the matrices $\Psi_i(.), \Phi_i(.)$ given by the affine decomposition in (35),(34). Suppose the following LMI problem is feasible for some $W := [P, M_a, L_a, M_b, L_b]_{diag}$.

$$\begin{aligned}&\Psi_0(W,\delta) > 0,\ \Phi_0(W,\delta,\dot{\delta}) < 0,\ \forall(\delta,\dot{\delta}) \in \vartheta(\mathscr{D} \times \check{\mathscr{D}}) \\ &\Psi_i(W) = 0\ ,\ \Phi_i(W) = 0\ ,\ \forall i \in \mathfrak{I}_{unc} \\ &\Psi_i(W) \geq 0\ ,\ \Phi_i(W) \leq 0\ ,\ \forall i \in \mathfrak{I}_{pos} \\ &\Psi_i(W) \leq 0\ ,\ \Phi_i(W) \geq 0\ ,\ \forall i \in \mathfrak{I}_{neg}\end{aligned}\tag{38}$$

Then the trajectories of the system (4) converge asymptotically to the origin $\forall(\delta,\dot{\delta}) \in \mathscr{D} \times \check{\mathscr{D}}$ and $\forall x(t_0) \in \mathscr{X}$ in (37). Moreover $v(x,\delta) = x'\mathbf{P}(x,\delta)x$ with the structure (6) is a Lyapunov function for the system. □

Remark 4.2: The proof follows the same lines of that of the Corollary 4.1. However, as the conditions (38) guarantee $v(x,\delta) > 0$ and $\dot{v}(x,\delta) < 0$ pointwise in $\mathscr{X}$ the assumption that $\mathscr{X}$ in (37) is positively invariant is essential for the stability condition. Observe that any trajectory in the interior of $\mathscr{X}$ does not cross the surface $\mathscr{H}_i = \{x: \ x_i = 0\}$, for some $i \in \mathfrak{I}_{neg}$, if $\dot{x}_i \leq 0 \ \forall x \in \mathscr{X} \bigcap \mathscr{H}_i$ and $\forall \delta \in \mathscr{D}$. Similarly, for a surface $\mathscr{F}_i = \{x: \ x_i = 0\}$, for some $i \in \mathfrak{I}_{pos}$, we must have $\dot{x}_i \geq 0 \ \forall x \in \mathscr{X} \bigcap \mathscr{F}_i$ and $\forall \delta \in \mathscr{D}$. See [10] for references on this point. If these conditions are not guaranteed to be satisfied the Corollary 4.2 cannot be used and the region of attraction must be estimated with the methods for local stability developed in the previous section. □

V. NUMERICAL EXAMPLE

Consider the problem of estimating the region of attraction of the origin of the following system, borrowed from [11]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_1^2 x_2 \\ -x_2 \end{bmatrix} \tag{39}$$

This system has nonlinearities of degree 3, and for this reason we adopt $\pi = \begin{bmatrix} x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{bmatrix}'$ and in this case the system representation is

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G(x) = \begin{bmatrix} -x_1 I_2 \\ -x_2 row_2(I_2) \\ 0_{4\times 2} \end{bmatrix}, F(x) = \begin{bmatrix} I_3 & 0_{3\times 4} \\ -x_1 I_3 & \\ -x_2 row_3(I_3) & I_4 \end{bmatrix}$$

The linear annihilators of x, π are $\aleph_x(x) = \begin{bmatrix} x_2 & -x_1 \end{bmatrix}$ and

$$\aleph_\pi(x) = \begin{bmatrix} x_2 & -x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 \end{bmatrix}$$

Observe $\aleph_{\pi_b} = [\aleph_x, \aleph_\pi]_{diag}$ and $\aleph_{\pi_a}$ is given from (27).

Now define the regions $\mathscr{X}_1 = \{x: \ x_1 \leq 0, \ x_2 \geq 0\}$ and $\mathscr{X}_2 = \{x: \ x_1 \geq 0, \ x_2 \leq 0\}$ and observe they correspond to the second and fourth quadrants of the state space. Observe in addition that both sets $\mathscr{X}_1, \mathscr{X}_2$ satisfy the conditions for positive invariance according to Remark 4.2. Then we may use the Corollary 4.2 to verify the stability of system trajectories in these sets. It turns out that the conditions of the Corollary 4.2 are satisfied for these two sets individually and therefore any trajectory in these sets are asymptotically stable. The LMIs were solved with SeDuMi and Yalmip [5] interface to Matlab. Even a quadratic Lyapunov function satisfies the conditions of the Corollary 4.2 for both $\mathscr{X}_1, \mathscr{X}_2$ individually. As $\mathscr{X}_1, \mathscr{X}_2$ are positively invariant sets the region $\mathscr{X}_1 \bigcup \mathscr{X}_2$ is a region of attraction of the system and it is very close to the true region of attraction obtained from [11] and represented by the dashed (red) curves.

Four state trajectories (blue) for some $x_0 \in \mathscr{X}_1 \bigcup \mathscr{X}_2$ are also shown in the figure.

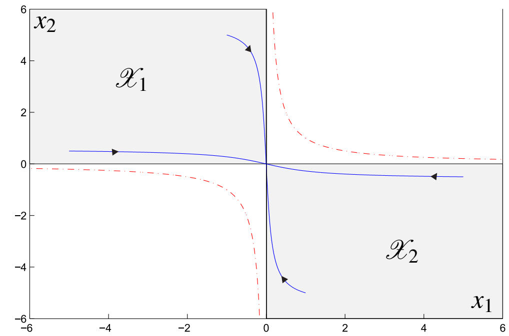

Fig. 1. Estimated regions of attraction for system (39).

VI. CONCLUDING REMARKS

The main contribution of this paper is to present an LMI technique for global asymptotic stability of rational systems where some or all the state variables are constrained by the model to have definite signal. The stability conditions are based on a rational Lyapunov function and the effectiveness of the approach is illustrated through a numerical example. Important results, comments and references were omitted here due to space limitation but they can be found in the full version of this paper (submitted for publication). The full version also treats the problems of regional stability, robust regions of attraction, presents LMI conditions to check the positive invariance of $\mathscr{X}$ in (37) , establishes connections with some results in the literature and presents a comparative study through several numerical examples.

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