2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

# Rendezvous in space with minimal sensing and coarse actuation

Soumya Ranjan Sahoo, Ravi N. Banavar and Arpita Sinha

*Abstract*— **In this paper we propose a control law for achieving rendezvous of autonomous vehicles capable of moving in three-dimensional (3D) space by using minimal data and quantized control. A pre-assigned graph uniquely assigns the pursuer-target pair in a cyclic manner. The measurement required for the proposed control law is the position from which the target vehicle moves out of the field-of-view of the pursuing vehicle. A Lyapunov function is chosen to find a range for the field-of-view which would guarantee rendezvous under the proposed control law.**

## I. INTRODUCTION

Autonomous vehicle systems have found potential applications in military operations, search and rescue, environment monitoring, commercial cleaning, material handling, and homeland security. While single vehicles performing solo missions have yielded some benefits, greater benefits will come from the cooperation of a team of vehicles. A multi-agent system is robust to failure than a single agent. It is more effcient than individual agents in certain cases. It is also possible to reduce the size of the agents and operational cost, increase system reliability in case of a multi-agent system. This has aroused interest of the control community in cooperative control and consensus algorithms. [1] mention various consensus algorithms in multi-agent coordination. Control strategies to achieve consensus like formation control or rendezvous under cyclic pursuit have been developed; see, e.g., [2], [3], [4], [5], [6], [7], [8].

Sometimes tasks have to be performed with minimal data available. Minimal data availability may be due to lower bandwidth available, security reasons, compact design of agents or availability of less sensors due to failure of other sensors of the agents. Focusing on minimal data helps increase the robustness of the system, results in simpler and compact design of agents, and lowers the production cost. Both quantised control and reduced sensing focus on minimal data. In [9] it has been shown that using minimal data, rendezvous of agents on a plane can be achieved. The agent just detects whether its target(agent assigned to it) moves out of its left or right side of the *windshield* and the turns accordingly to bring the latter into its view(or *windshield*).

However, autonomous agents like unmanned aerial vehicles (UAVs), autonomous underwater vehicles (AUVs) always move in a 3D space. Further, these agents may not be on a plane initially. So the control laws in a planar case will not hold and these need to be modified for agents capable of moving in 3D space. In this paper we present the rendezvous of agents in 3D space. The agent in our case is considered to be an aerial vehicle.

Though the work presented here is on the lines of [9], the 3D problem has significant differences. A few points are:

- The geometry in 3D is more involved than the 2D case.
- The extension to the 3D case of the notion of minimal sensing and coarse actuation is non-trivial.
- We present a lower bound on the angular speed (maneuverability) of each vehicle.
- The model considered and the assumptions made on the maneuvering capabilities (yaw and pitch maneuvers) of vehicles in 3D are realistic.

Lastly, we provide more insight into the problem through simulations. Some observations have also been made regarding the change in angular speed.

The paper is organized as follows: section 2 presents about the vehicle model, the sensors and the control law used to achieve rendezvous. Section 3 presents a few preliminaries from graph theory, the formation of a Lyapunov function, and a condition on the angle of the *windshield* and the angular speed. In section 4 simulation results for the problem have been presented and discussed.

## II. PROBLEM FORMULATION

We assume $n$ agents in the system, each having the control characteristic like an UAV. The schematic of the agent is shown in Fig. 1. The agent can yaw and pitch with constant angular speed, and move straight with constant speed. Each agent has a conical field-of-view with infinite range within which it trys to maintain its target. The target for the $i^{th}$ agent is $(i+1) mod\ n$. The control is applied only when the target moves out of the windshield. We also assume that all the agents have their target within their windshield initially. The vehicle model, sensors and control have been discussed in the following subsections.

### A. Vehicle Model

Let $p_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ be the position of the $i^{th}$ agent in an earth-fixed frame and $(\alpha_i,\ \beta_i,\ \gamma_i)$ be the Euler angles corresponding to the Z-Y-Z Euler angle convention for transformation from the body-fixed frame to the earth-fixed frame. See Fig. 1.

- The forward (linear) velocity of the vehicle is only along its body $X_b$ axis and its magnitude $(v_i)$ remains constant.
- The vehicle can rotate about its body $Y_b$-axis (pitch) and about its body $Z_b$-axis (yaw).

Soumya Ranjan Sahoo is with Systems and Control Engineering, IIT Bombay, Mumbai, India soumya@sc.iitb.ac.in

Ravi N. Banavar, Professor, Systems and Control Engineering, IIT Bombay, Mumbai, India banavar@sc.iitb.ac.in

Arpita Sinha, Assistant Professor, Systems and Control Engineering, IIT Bombay, Mumbai, India asinha@sc.iitb.ac.in

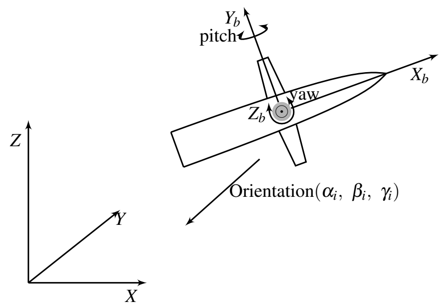


Fig. 1: Schematic of vehicle and transformation from body-fixed frame to earth-fixed frame

The pitch and yaw rates assume values from the discrete set, $\omega_{y_{i_b}} \in \{-\omega_i, 0, +\omega_i\}$ and $\omega_{z_{i_b}} \in \{-\omega_i, 0, +\omega_i\}$, where $\omega_i$ is constant and $\omega_i > 0$. The agents are considered identical and hence $v_i = v_j$ and $\omega_i = \omega_j$. The kinematic model of the $i^{th}$ vehicle is given by

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{bmatrix} = R_{z_{i_3}}(\gamma_i) R_{y_{i_2}}(\beta_i) R_{z_{i_1}}(\alpha_i) \begin{bmatrix} v_i \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \dot{\alpha}_i \\ \dot{\beta}_i \\ \dot{\gamma}_i \end{bmatrix} = \frac{-1}{s(\beta_i)} \begin{bmatrix} c(\alpha_i)c(\beta_i) & s(\alpha_i)c(\beta_i) & -s(\beta_i) \\ s(\alpha_i)s(\beta_i) & -c(\alpha_i)s(\beta_i) & 0 \\ -c(\alpha_i) & -s(\alpha_i) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_{y_{i_b}} \\ \omega_{z_{i_b}} \end{bmatrix} \tag{2}$$

where $(\dot{x}_i, \dot{y}_i, \dot{z}_i)$ are the velocity components in the inertial frame and $(v_i, 0, 0)$ are the velocity components in the body-fixed frame, $R_{z_{i_3}}(\gamma)$, $R_{y_{i_2}}(\beta)$, $R_{z_{i_1}}(\alpha)$ are rotation matrices parametrized by the Euler angles $\alpha$, $\beta$, $\gamma$ and $c(.) = \cos(.)$ and $s(.) = \sin(.)$.

### B. Sensors

The sensor of each agent has a conical view with the half angle of the cone being $\phi \in (0, \pi)$ as shown in Figure 2. This field-of-view is termed as the *windshield*. The range of view within this angle is assumed infinite. It is also assumed that another vehicle cannot occlude the view of the agent if it appears within the windshield. The field-of-view, when seen from inside the agent, appears like a disc. This disc can be divided into four quadrants as shown in Figure 3. The sensors do not give the actual distance between agents. They only give a discrete output based on the quadrant from which the assigned agent(the target) moves out.

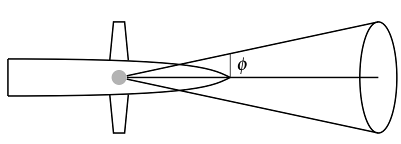


Fig. 2: The conical field-of-view of the agent

Let the output set be $O$ and the sensor measurement of the $i^{th}$ vehicle be $(o_{y_i}, o_{z_i}) \in O$. $(o_{y_i}, o_{z_i})$ takes the following values according to the quadrant from which agent $j$ escapes (with reference to Fig. 3):-

$$(o_{y_i}, o_{z_i}) = \begin{cases} (1,1) & \text{first quadrant, } y^- \text{ and } z^+ axes \\ (-1,1) & \text{second quadrant, } y^+ \text{ and } z^+ axes \\ (-1,-1) & \text{third quadrant, } y^+ \text{ and } z^- axes \\ (1,-1) & \text{fourth quadrant, } y^- \text{ and } z^- axes \\ (0,0) & \text{does not escape} \end{cases} \tag{3}$$

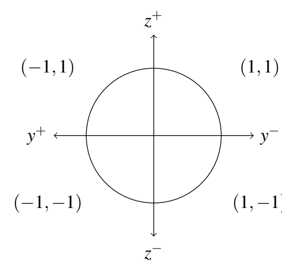


Fig. 3: Field-of-view is a disc when seen from agent

### C. Controls

The output of the sensors actuate the controllers for necessary action. The control law is defined as:

$$(\omega_{y_{i_b}}, \omega_{z_{i_b}}) = (o_{z_i}, o_{y_i})\omega_i \tag{4}$$

## III. Graph Theory and Lyapunov Candidate

For simplicity and without any loss of generality, we assume that $(i+1)$ *mod n* is assigned to $i$. This cyclic pursuit can be denoted in terms of a *cyclic graph*, $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the agents being its nodes and the directed edge $e_{i,i+1} \in \mathscr{E}(\mathscr{G})$, the edge set. Let the distance between agents $i$ and $i+1$ be denoted by $l_{i,i+1}$. As agent $i$ catches up with agent $i+1$, $i$ and $i+1$ move as one entity, which is denoted by $i+1$. This is called *merging*. The merging operation is triggered when the distance between the pursued and the pursuer reduces to the *merging radius* $\rho$ $(>0)$, that is,

$$e_{i,i+1} \in \mathscr{E}(\mathscr{G}), \; l_{i,i+1} \leq \rho$$

After merging, agent $i-1$, which was pursuing $i$, starts pursuing $i+1$. The node $i$ is deleted and the edges $e_{i-1,i}$ and $e_{i,i+1}$ are deleted from $\mathscr{E}(\mathscr{G})$ and a new edge $e_{i-1,i+1}$ comes into effect. The number of nodes reduces.

Let $V : \mathbb{R}^{3n} \to \mathbb{R}$ be a function which is defined as

$$V = \sum_{e_{i,j} \in \mathscr{E}(\mathscr{G})} l_{i,j} \tag{5}$$

Since $V$ is the sum of distances, it will be always positive and will only go to zero when edges do not exist. So $V$ is a suitable candidate for being the Lyapunov function of the system. $V$ is also called a *graph compatible Lyapunov function* as it is based on the digraph $\mathscr{G}$ ([9]). The time-derivative of $V$ is given by

$$\dot{V} = \sum_{e_{i,j} \in \mathscr{E}(\mathscr{G})} \dot{l}_{i,j} \tag{6}$$

**Lemma III.1.** *(Lemma 2, [9]) A graph-compatible Lyapunov function V is rendezvous positive definite if and only if its assignment graph $\mathscr{G}$ is connected.* ■

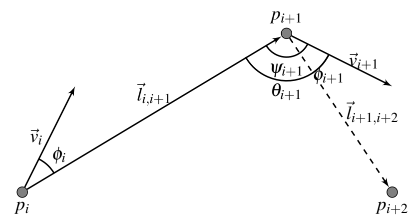


Fig. 4: Two consecutive agents in cyclic pursuit

As shown in Fig. 4, which is in 3D space,

- $p_i,\ p_{i+1},\ p_{i+2}$ are the positions of agents $i$, $i+1$ and $i+2$ respectively measured in in the earth-fixed frame.
- $l_{i,i+1},\ l_{i+1,i+2}$ are the line-of-sight distances of agent $i$ and $i+1$ respectively.
- $\phi_{i+1}$ is the angle between the velocity vector $\vec{v}_{i+1}$ and $\vec{l}_{i+1,i+2}$.
- $\theta_{i+1}$ is the smaller angle between $\vec{l}_{i,i+1}$ and $\vec{l}_{i+1,i+2}$.
- $\psi_{i+1}$ is the smaller angle between $\vec{v}_{i+1}$ and $\vec{l}_{i,i+1}$.

All angles are considered to be positive. Thus,

$$\dot{l}_{i,i+1} = -v_{i+1}\cos(\psi_{i+1}) - v_i\cos(\phi_i) \tag{7}$$

Assuming all agents to have unit speed, from (7)

$$\dot{V} = -\sum_{i=1}^{n}(\cos(\phi_i)+\cos(\psi_i)) \tag{8}$$

A necessary condition for rendezvous to occur is that $\dot{V}$ should be strictly less than zero.

**Theorem III.2.** *Unit speed cyclic pursuit of n agents with kinematics given by (1) and (2) will rendezvous if the agents maintain their targets within the windshield and the windshield angle $\phi$ satisfies*

$$0<\phi<\begin{cases}\pi/2 & n=2\\ \min\{\pi/n, cos^{-1}(\frac{n-1}{n})\} & n\geq 3\end{cases} \tag{9}$$

*Proof.* To prove this theorem we proceed in the following manner: we first rule out the condition $\phi\in[\pi/n,\pi]$. Next we find a condition on $\phi$ for $n=2$. Then using some lemmas we find the condition on $\phi$ for $n\geq 3$, thus proving the theorem.

**Lemma III.3.** *(Lemma 4, [2]) Let ABC and ACD be two triangles in $\mathbb{R}^3$ with a common side AC. Define the three angles*

$$\alpha=\angle BAC \quad \beta=\angle CAD \quad \gamma=\angle BAD \quad (0\leq\alpha,\beta,\gamma\leq\pi).$$

*Then*

$$\gamma\leq\alpha+\beta \tag{10}$$

*with equality holding only if A, B, C, D are coplanar.* ■

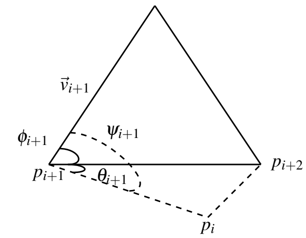


Fig. 5: Triangle inequality of angles. $\triangle$s formed by $p_i$, $p_{i+1}$, $p_{i+2}$ and the velocity vector $\vec{v}_{i+1}$

Consider the triangles formed by the points $p_i$, $p_{i+1}$, $p_{i+2}$ and the velocity vector $\vec{v}_{i+1}$ as shown in Figure 5. Applying *Lemma III.3* we have

$$\psi_{i+1}\leq\min\{(\theta_{i+1}+\phi_{i+1}),(2\pi-(\theta_{i+1}+\phi_{i+1}))\} \tag{11}$$

**Lemma III.4.** *For any integer $n\geq 2$, the windshield angle $\phi=\pi/n$ permits trajectories for which $\dot{V}=0$.*

*Proof.* This lemma is similar to the one described by [9]. Consider a case where the $n$ agents lie on a convex polygon in the X-Y plane. Let the target of each agent is on the right hand side boundary of latter's windshield. Thus, $\theta_i=\left(\frac{n-2}{n}\right)\pi$ and $\phi_i=\phi=\frac{\pi}{n}$. From lemma(III.3) we have $\psi_i=\theta_i+\phi_i=\left(\frac{n-1}{n}\right)\pi,\ \forall i$. So, from (8), $\dot{V}=0$. Hence, trajectories with $\dot{V}=0$ also exist when $\phi=\frac{\pi}{n}$. ■

When $\phi\in(\pi/n,\pi]$ for the same case as above, we have $\dot{V}>0$. Hence, trajectories with $\dot{V}>0$ also exist when $\phi\in(\frac{\pi}{n},\pi]$. So we rule out the domain $\phi\in[\frac{\pi}{n},\pi]$.

We now state another result on a closed polygonal line that we employ in the proof of our main result. If $\beta_i$ is the angle formed by the polygonal line at the $i$-th vertex such that $0<\beta_i<\pi$ and $\alpha_i$ be the supplementary angle of $\beta_i$ then the following lemma holds.

**Lemma III.5.** *(Lemma 3, [2]) In any closed polygonal line with n segments in $\mathbb{R}^3$, the inequalities $\sum_{i=1}^{n}\beta_i\leq(n-2)\pi$ and $\sum_{i=1}^{n}\alpha_i\geq 2\pi$, hold. Equality occurs iff the polygonal line is a planar convex polygon.* ■

When $n=2$, $\psi_1=\phi_2$ and $\psi_2=\phi_1$. So from (8),

$$\dot{V}=-\sum_{i=1}^{2}(\cos(\phi_i)+\cos(\psi_i))$$

For $\dot{V}<0$, we should have

$$-\sum_{i=1}^{2}(\cos(\phi_i)+\cos(\psi_i)) < 0$$
$$\Rightarrow\cos(\phi_1)+\cos(\phi_2) > 0 \tag{12}$$

(12) holds if $\phi<\pi/2$ imposing $\cos(\phi_1)>0$ and $\cos(\phi_2)>0$. Thus, for $n=2$, when $\phi<\pi/2$, $\dot{V}<0$. We now check for $n\geq 3$. From *Lemma III.4* and the fact that $\phi_i<\phi$ we have,

$$0<\phi_i<\phi<\pi/n \tag{13}$$

For any closed polygonal line we can always consider the smaller angle as interior angle. So

$$0 < \theta_i < \pi \tag{14}$$

From (13) and (14) we have

$$0 < \theta_i + \phi_i < \pi + \pi/n \tag{15}$$

Hence, from (11) and (15) we have

$$-\cos\psi_i < -\cos(\theta_i + \phi_i) \tag{16}$$

From (8) and (16) we have

$$\dot{V} \leq -\sum_{i=1}^{n}(\cos\phi_i + \cos(\theta_i + \phi_i)) =: \dot{V}_1 \tag{17}$$

$\dot{V}_1$ can now be written as the sum of two functions, $f$ and $g$ as follows

$$f := -\sum_{i=1}^{n}\cos(\theta_i + \phi_i) \tag{18}$$

$$g := -\sum_{i=1}^{n}\cos\phi_i \tag{19}$$

Note that $-n \leq f, g \leq n$. From (15) we can have two sets:-

$$\Theta_{out} = \{(\theta_1, \ldots, \theta_n) | \exists\ i, (\theta_i + \phi_i) \in [0, \pi/2)\} \tag{20}$$

$$\Theta_{in} = \{(\theta_1, \ldots, \theta_n) | \forall\ i, (\theta_i + \phi_i) \in [\pi/2, \pi + \pi/n)\} \tag{21}$$

We have two disjoint sets defined as above as the behavior of $f$ is different in $\Theta_{out}$ and $\Theta_{in}$. We now analyse the behavior of $f$ in $\Theta_{out}$. If $(\theta_1, \ldots, \theta_n) \in \Theta_{out}$ then atleast for one $i$, $(\theta_i + \phi_i)$ must belong to $[0, \pi/2)$. Let for $i = k$, $\theta_k + \phi_k \in [0, \pi/2)$. So, $-\cos(\theta_k + \phi_k) < 0$. Hence,

$$f < n - 1 \tag{22}$$

Now the behavior of $f$ has to be analysed in $\Theta_{in}$. Here, we state a lemma similar to the lemma stated in [9].

**Lemma III.6.** *Unit speed cyclic pursuit of n agents satisfying (13) has the property that the function $f(\Theta, \Phi)$ has a single stationary point in $\Theta_{in}$.*

*Proof.* Keeping $\phi_i$'s constant, $f$ only depends on $\theta_i$'s. Let

$$h := \sum_{i=1}^{n}\theta_i$$

From *Lemma III.5* the $\theta_i$'s are constrained by the inequality:

$$h = \sum_{i=1}^{n}\theta_i \leq (n-2)\pi \tag{23}$$

The above inequality can be converted to an equality constraint as follows

$$\mathscr{H} := h + \beta^2 - (n-2)\pi = 0 \tag{24}$$

where $\beta \in \mathbb{R}$ is termed a slack variable. The equality constraint is incorporated into the optimization problem with the help of a Lagrange multiplier $(\lambda)$ as follows:

$$\begin{aligned} f_{mod}(\theta_1, \ldots, \theta_n, \beta, \lambda) &= f - \lambda\mathscr{H} \\ &= f - \lambda\left(\sum_{i=1}^{n}\theta_i + \beta^2 - (n-2)\pi\right) \end{aligned} \tag{25}$$

The conditions for stationarity are

$$\begin{aligned} \sin(\theta_i + \phi_i) &= \lambda,\ \forall\ i \\ 2\lambda\beta &= 0 \\ \mathscr{H} &= 0 \end{aligned} \tag{26}$$

From $2\lambda\beta = 0$ we have the following possibilities:

$$(\beta = \lambda = 0),\ (\beta \neq 0,\ \lambda = 0),\ (\beta = 0,\ \lambda \neq 0)$$

For the first 2 solutions, when $\lambda = 0$, we have from (26),

$$\begin{aligned} (\theta_i + \phi_i) &= \pi, \quad \forall i \quad [\because (\theta_i + \phi_i) \in \Theta^-] \\ \Rightarrow \sum_{i=1}^{n}(\theta_i + \phi_i) &= n\pi \end{aligned} \tag{27}$$

From (13) and (23), we have

$$0 < \sum(\theta_i + \phi_i) < (n-1)\pi, \quad \forall i \tag{28}$$

As we are considering the $\Theta_{in}$ set, (28) reduces to

$$\frac{n\pi}{2} < \sum(\theta_i + \phi_i) < (n-1)\pi, \quad \forall i \tag{29}$$

It is clear that (27) violates (29). So, $\lambda = 0$ cannot be a solution. Then only possible solution is $\beta = 0$, $\lambda \neq 0$. Now, if $(\theta_i + \phi_i) \geq \frac{n-1}{n}\pi$, $\forall\ i$, then (29) is violated. For all $i$, $(\theta_i + \phi_i)$ cannot be less than $\frac{\pi}{2}$ as we are considering the set $\Theta_{in}$. So to satisfy (29), there exists at least one $i$ such that $(\theta_i + \phi_i) \in (\frac{\pi}{2}, \frac{n-1}{n}\pi)$. So $\sin(\theta_i + \phi_i) > 0$ and hence from (26) we have $\lambda > 0$ and therefore, $\sin(\theta_i^c + \phi_i) > 0$, $\forall i$. We are considering the $\Theta_{in}$ set. So, there is only one value of $(\theta_i^c + \phi_i)$ such that $\sin(\theta_i^c + \phi_i) > 0$. But this holds for all $i$. So, $(\theta_1^c + \phi_1) = (\theta_2^c + \phi_2) = \cdots = (\theta_n^c + \phi_n)$. Thus, $f(\Theta, \Phi)$ has a single stationary point in $\Theta_{in}$. ■

**Lemma III.7.** *The stationary point is a point of maxima in $\Theta_{in}$ and the maximum value obtained by $f$ is*

$$f \leq -n\cos\left(\frac{(n-2)\pi + n\phi}{n}\right) \tag{30}$$

At the single stationary point since $\sin(\theta_i^c + \phi_i) = \lambda > 0$, $\forall i$, $(\theta_1^c + \phi_1) = (\theta_2^c + \phi_2) = \ldots = (\theta_n^c + \phi_n)$. Thus, at the stationary point, $(\theta_i^c + \phi_i) = \frac{\sum_{i=1}^{n}(\theta_i^c + \phi_i)}{n}$. The second derivative of $f_{mod}$, $\nabla^2_{\theta_1, \cdots, \theta_n, \beta, \lambda} f_{mod}|_{(\theta_i^c + \phi_i,\ \forall i)}$ is negative definite. Thus, the stationary point is a point of maxima. Now

$$\begin{aligned} f = &\ -n\cos\left(\frac{\sum_{i=1}^{n}(\theta_i^c + \phi_i)}{n}\right) \\ \leq &\ -n\cos\left(\frac{(n-2)\pi + n\phi}{n}\right) \end{aligned}$$ ■

**Lemma III.8.** *For n agents in cyclic pursuit with unit speed, $f(\Theta, \Phi)$ satisfies*

$$f(\Theta, \Phi) \leq \max\left\{n - 1, \quad -n\cos\left(\frac{(n-2)\pi + n\phi}{n}\right)\right\} \tag{31}$$

*Proof.* $f$ is a continuous function in $\Theta_{out} \cup \Theta_{in}$. From (22) and (30), (31) follows. ■

In continuation of the proof of *Theorem III.2*, for $\dot{V}_1 < 0$, $-g$ has to be greater than $f$. To ensure this $-g$ has to be greater than $\max\left\{n-1, \quad -n\cos\left(\frac{(n-2)\pi+n\phi}{n}\right)\right\}$. From (19), we have,

$$\max\left\{n-1, \quad -n\cos\left(\frac{(n-2)\pi+n\phi}{n}\right)\right\} \;<\; \sum_{i=1}^{n}\cos(\phi_i) \tag{32}$$

From (13) $0 \;<\; \phi_i \;<\; \phi \;<\; \frac{\pi}{n}$. So, $\cos\phi \;<\; \cos\phi_i$. So, to satisfy (32) for all $\phi_i$'s the following must be true.

$$\max\left\{n-1, \quad -n\cos\left(\frac{(n-2)\pi+n\phi}{n}\right)\right\} \;<\; n\cos(\phi)$$
$$\Rightarrow \phi \;<\; \min\left\{\cos^{-1}\left(\frac{n-1}{n}\right), \frac{\pi}{n}\right\}$$

Hence, for $n \;\geq\; 3$, and $\dot{V}_1$ to be less than zero,

$$\phi \;<\; \min\left\{\cos^{-1}\left(\frac{n-1}{n}\right), \quad \frac{\pi}{n}\right\} \tag{33}$$

As from (17), $\dot{V} < \dot{V}_1$, so for the same range of $\phi$, $\dot{V} \;<\; 0$ also. Hence, the condition on $\phi$ is proved for $n \geq 3$. ■

### A. Condition on angular speed

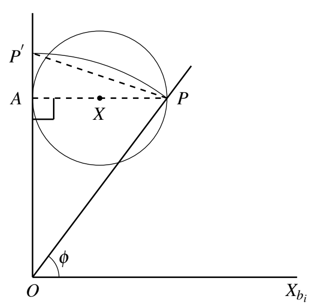


Fig. 6: Condition on $\omega$

Conside Fig. 6. Let agent $i$ be at $O$ and $i+1$ be at the point $P$ on the boundary of the field-of-view of $i$. Assume $OP \geq \rho$. $\vec{v}_{i+1}$ can be in any direction. So $i+1$ can move out of field-of-view in any direction. Consider frame-of-reference with $O$ as the origin and $X_{b_i}$ as the $x$-axis. $\vec{v}_i$ is along the $x$-axis ($OX_{b_i}$). The relative velocity of $i+1$ with respect to $i$ is $\vec{v}_{i,i+1} = \vec{v}_{i+1} - \vec{v}_i$. The locus of permissible $\vec{v}_{i,i+1}$ at $P$ is a sphere of radius $v$ passing through $P$. Consider a plane passing through $OX_{b_i}$ containing $P$. The great circle of the sphere passing through $P$ will lie on this plane and the vector from $P$ to centre of the great circle will be $-\vec{v}_i$. Let us draw $OP^{'} \perp OX_{b_i}$. Assume $PX = |\vec{v}_i|\triangle t$ such that the great circle is scaled by $\triangle t$ and is made tangent to $OP^{'}$. The centre of this circle is at $X$. From the geometry shown in Fig. 6, we have

$$AP \;=\; OP\cos(\phi) \;\Rightarrow\; OP\cos(\phi) \;=\; 2v\triangle t$$

$$\begin{aligned} OP\cos(\phi) &\;\geq\; \rho\cos(\phi) \\ \Rightarrow \qquad \triangle t &\;\geq\; \frac{\rho}{2v}\cos(\phi) \end{aligned}$$

In the time interval $\triangle t$ the agent $i+1$ can be anywhere on the circumference of the circle with center at $X$. In order to bring $i+1$ back into the field-of-view, agent $i$ has to rotate. If the rotation of the agent $i$ is such that the windshield boundary $OP$ moves to $OP^{'}$ in a time less than or equal to time interval $\triangle t$ then agent $i+1$ can be brought back to the field-of-view of $i$, irrespective of the direction of motion of $i+1$. The angular velocity $\omega^{'}$ ( about the axis perpendicular to this plane) required to do so is computed as

$$|\omega^{'}|\triangle t \;\geq\; \frac{\pi}{2} - \phi$$

If $|\omega^{'}|(\rho\cos(\phi))/2v$ is greater than or equal to $\frac{\pi}{2} - \phi$ then the above inequality is always satisfied. So

$$\begin{aligned} |\omega^{'}|\tfrac{\rho}{2v}\cos(\phi) &\;\geq\; \tfrac{\pi}{2} - \phi \\ \Rightarrow \quad |\omega^{'}| &\;\geq\; \left(\frac{\frac{\pi}{2}-\phi}{\cos(\phi)}\right)\frac{2v}{\rho} \end{aligned} \tag{34}$$

The numerator of $\left(\frac{\frac{\pi}{2}-\phi}{\cos(\phi)}\right)$ decreases at a rate more than the denominator with $\phi$ varying from 0 to $\pi/2$. So $\left(\frac{\frac{\pi}{2}-\phi}{\cos(\phi)}\right)$ is maximum when $\phi = 0$. So

$$|\omega^{'}| \;\geq\; (\pi v)/\rho \tag{35}$$

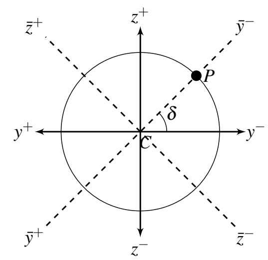


Fig. 7: Minimum $\omega$

Consider Fig. 7. When agent $i+1$ moves out of the field-of-view of agent $i$ from the point $P$, agent $i$ starts turning due to the control applied along $y^{-}$ and $z^{+}$ to bring back the target into field-of-view. $\bar{y}^{-}$ is the plane discussed above and shown in Fig. 6. $\bar{z}^{+}$ is the axis about which $\omega^{'}$ needs to be applied. The angular velocity along $\bar{y}^{-}$ and $\bar{z}^{+}$ is

$$\begin{bmatrix}\omega_{\bar{y}^{-}} \\ \omega_{\bar{z}^{+}}\end{bmatrix} \;=\; \begin{bmatrix}\cos(\delta) & -\sin(\delta) \\ \sin(\delta) & \cos(\delta)\end{bmatrix}\begin{bmatrix}\omega_y \\ \omega_z\end{bmatrix} \tag{36}$$

where $\delta$ is the angle of rotation of $y-z$ axes to coincide with $P$ (Fig. 7). The effective angular velocity that brings the target back in is about $\bar{z}^{+}$-axis and is given by

$$|\omega_{\bar{z}^{+}}| \;=\; |\omega_y\sin(\delta) + \omega_z\cos(\delta)| \tag{37}$$

From (37) we obtain the minimum value of $|\omega_{\bar{z}^{+}}|$ when $\delta = n\pi, n = 0, \pm 1, \ldots$ and it is equal to $\omega$. Hence,

$$\omega \;\geq\; |\omega^{'}| \;\geq\; (\pi v)/\rho \tag{38}$$

If the agents have their windshield angle $\phi$ and angular speed $\omega$ such that condition (9) and (38) are satisfied then rendezvous is guaranteed. In the following section we have presented some simulation results.

## IV. Simulation Results for Rendezvous in 3D

We have considered a "five-agent" system. The agents start from random points in space with the respective target agent inside the field-of-view. The forward speed for each agent is 10units/second for all $i$ and $\rho = 0.15$. The total distance of the system is $V = \sum_{i=1}^{n} l_{i,i+1}$ where $l_{i,i+1}$ is the distance between $i$ and $i+1$. We assume that initially each agent has been assigned its target in order and has the target in its field-of-view. Since $n = 5$, the limits on $\phi$ and $\omega$ are $\frac{\pi}{5}$ and 209.4 respectively. Considering $\phi = 0.2\frac{\pi}{5}$ and $\omega = 210$rad/second we find the agents converge to a point. See Fig. 8 and 9.

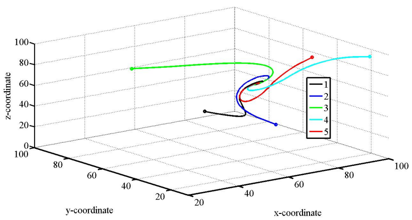


Fig. 8: Five agents with $\phi = 0.2\pi/5, \omega = 210, v = 10$

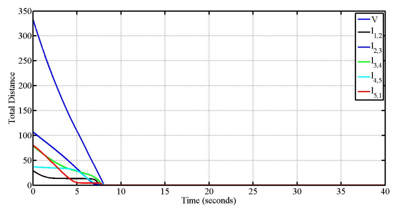


Fig. 9: Total distance ($V$) and individual distance for five agents with $\phi = 0.2\pi/5, \omega = 210, v = 10$

*1) Case A : Different values of $\phi$:* We carried out simulations for $\phi = 0.2\frac{\pi}{5}$, $0.5\frac{\pi}{5}$, $0.9\frac{\pi}{5}$, $1.02\frac{\pi}{5}$, $1.3\frac{\pi}{5}$ with $v = 10$, $\omega = 210$, $\rho = 0.15$. Fig. 10 shows a comparison between the sum of the distances between the agents ($V$) for different values of the windshield angle.

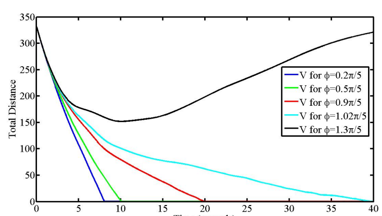


Fig. 10: Value of total distance ($V$) for different values of $\phi$

From Fig. 10 it is observed that the time for rendezvous increases with increase in the windshield angle. With the increase in the windshield angle the control becomes coarse. The target-agent also stays for a longer time in the field-of-view of the pursuing agent. To check whether rendezvous ocurred if condition (9) was violated, we carried out a few simulations. It was seen that for $\phi$ violating the condition (9) the agents still rendezvous. The condition on $\phi$ is a sufficient condition. However, for sufficiently larger value of $\phi$ the agents diverge.

*2) Case B : Different values of $\omega$:* We carried out a few simulations with $\omega = 2,\ 20,\ 210,\ 2000$ and $v = 10$, $\phi = 0.2\frac{\pi}{5}$, $\rho = 2$. See Fig. 11. It was observed that for $\omega$ satisfying the condtion (38) rendezvous was achieved. With $\omega$ below the lower bound the agents did not merge but they did not diverge out. It was also observed that for $\omega$ violating (38) $\dot{V}$ was having positve value also.

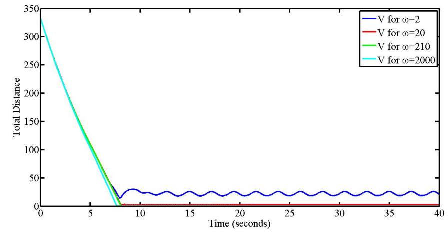


Fig. 11: Value of $V$ different values of $\omega$

The results of the simulation also show that with both $\phi$ and $\omega$ satisfying (9) and (38) respectively rendezvous is guaranteed.

## V. Summary and Future Scope of Work

Using a system of autonomous agents, each of which has a forward speed, and yaw and pitch control, we have shown that without communication, and with minimal sensing and quantized control we are able to achieve rendezvous. We have obtained a sufficient condition on the windshield angle $\phi$ and angular speed $\omega$ which guarantees rendezvous. The results have been demonstrated through simulations.

As an extension to the present work, some strategies for collison avoidance between agents $i$ and $j$, $(j \neq i+1)$ can be investigated. One could also investigate if the agents can rendezvous at a specific location in space.

## VI. ACKNOWLEDGMENTS


The authors acknowledge useful discussions with Anup Menon.


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