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Sampled-Data Control of Two-level Quantum Systems Based on Sliding Mode Design

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Abstract— **This paper presents a sampled-data control approach based on sliding mode design for the robust control of a two-level quantum system, where the control law can be designed offline and then be used online for a quantum system with bounded uncertainties in the system Hamiltonian. Such a control law consists of a periodic sampling process and a Lyapunov control design. A sufficient condition on the relationships between related parameters in the control system is established and the Lyapunov control design is demonstrated by simulation, which can be used to show the required robustness of the quantum system in the presence of uncertainties.**

I. INTRODUCTION

Controlling quantum phenomena is becoming an important task in different research areas [1]-[7]. The development of quantum control theory can provide systematic methods and a theoretical framework for analyzing and synthesizing quantum control problems. Several control methods such as optimal control [8]-[10], learning control [3] and Lyapunov design [11]-[14] have been applied to the quantum domain. Recently, the robust control problem for quantum systems has been recognized as a key issue in developing practical quantum technology [15]-[18] since it is unavoidable that many types of uncertainties (including noise, disturbance, decoherence, etc.) exist for most practical quantum systems. Several methods have been proposed for the robust control of quantum systems. For example, James *et al.* [19] have formulated and solved a quantum robust control problem using the H^{∞} method for linear quantum stochastic systems. A risk-sensitive control approach has been applied to quantum systems [20]. In [21]-[23], we develop a sliding mode control approach to enhance the robustness of quantum systems. In particular, two approaches based on sliding mode design [24] have been proposed for the control of quantum systems in [21] and potential applications of sliding mode control to quantum information processing have been presented. Ref. [22] presents a detailed sliding mode control method for two-level quantum systems to deal with bounded uncertainties in the system Hamiltonian. This paper will focus on a robust control problem for two-level quantum systems based on sliding mode design and will propose a sampled-data control approach [25] to enhance the performance of a quantum system with bounded uncertainties. The sampled-data control scheme in this paper involves a fixed sampling period T. However, the approach in [22] involves at least two measurement periods T and T_1 ($T_1 \ll T$). This means that the approach of [22] may require measurements which are very close together and this may be difficult to achieve in practice. In this sense, the sampled-data control approach in this paper is more practicable than the control method in [22].

The objective of this paper is to design a control law for a two-level quantum system to guarantee the required robustness when bounded uncertainties exist in the system Hamiltonian. The required robustness is defined as follows: maintaining the system's state in a sliding mode domain $\mathscr{D}$ in which the system's state has a high fidelity with the sliding mode state $|0\rangle$ (an eigenstate of the free Hamiltonian of the two-level quantum system); once the system's state collapses out of $\mathscr{D}$ when making a measurement (sampling), driving it back to $\mathscr{D}$ within a short time period $(1-\beta)T$ and maintaining the state in $\mathscr{D}$ for the following time period βT (where $0 < \beta \leq 1$, T is the sampling period and generally we choose β satisfying $\frac{1-\beta}{\beta} \ll 1$).

This paper is organized as follows. Section II introduces the robust control approach based on sliding mode design for two-level quantum systems. In Section III, we present the main results involving the control method, the main theorem and an illustrative example. Section IV gives the proof of the main theorem. Concluding remarks are given in Section V.

II. CONTROL PROBLEM FORMULATION

The quantum control model under consideration can be described as (we have assumed $\hbar = 1$ by using atomic units)

$$\begin{array}{l} i|\dot{\psi}(t)\rangle = (H_0 + H_\Delta + H_u)|\psi(t)\rangle, \\ |\psi(t=0)\rangle = |\psi_0\rangle. \end{array} \tag{1}$$

Here, the quantum state $|\psi(t)\rangle$ corresponds to a two-dimensional complex unit vector in a Hilbert space, the free Hamiltonian is $H_0 = \frac{1}{2}\sigma_z$, the uncertainties are $H_\Delta = \delta(t)I_z + \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$ ($\delta(t), \varepsilon_x(t), \varepsilon_y(t) \in \mathbf{R}$), the control Hamiltonian is $H_u = \sum_{k=x,y,z} u_k(t)I_k$, ($u_k(t) \in \mathbf{R}$, $I_k = \frac{1}{2}\sigma_k$) and the Pauli matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ take the following form:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Furthermore, we assume that the uncertainties are bounded: $|\delta(t)| \leq \delta$ $(0 \leq \delta < 1)$, $\sqrt{\varepsilon_x^2(t) + \varepsilon_y^2(t)} \leq \varepsilon$ $(\varepsilon > 0)$.

To deal with the uncertainties H_Δ, we have proposed a sliding mode control approach where the eigenstate $|0\rangle$ is identified as the sliding mode S [22]. We further define a sliding mode domain $\mathscr{D} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \geq 1 - p_0, 0 < p_0 < 1\}$, where p_0 is a given constant. We aim to drive and then maintain a two-level quantum system's state in the sliding mode domain $\mathscr{D}$ using a sampled-data control method [25]. However, the uncertainties H_Δ may take the system's state away from $\mathscr{D}$. Since the sampling process (a measurement operation) unavoidably makes the sampled system's state change, we expect that the control laws will guarantee that the system's state remains in $\mathscr{D}$ except that the sampling process may take it away from $\mathscr{D}$ with a small probability (not greater than p_0).

III. Main Results

A. Control method

The basic method we use is illustrated in Fig. 1. For any sampling time nT $(n = 0,1,2,\dots)$, if the measured value corresponds to $|0\rangle$, we apply zero control and wait for the next measurement at time $(n+1)T$; otherwise, apply a control law designed using the Lyapunov methodology to drive the system's state from $|1\rangle$ back to a subset of $\mathscr{D}$ at $t \in [nT,(n+1-\beta)T]$, then apply zero control for $t \in ((n+1-\beta)T,(n+1)T)$ and sample again at the time $(n+1)T$. For every time point nT $(n = 0,1,2,\dots)$, whether the Lyapunov control law should be used depends on the measurement result ($|0\rangle$ or $|1\rangle$). The sampling period T and the Lyapunov control law can be designed offline in advance.

Using a similar argument to Theorem 1 in [22], we have the following result.

Lemma 1: For a two-level quantum system with the initial state $|\psi(0)\rangle = |0\rangle$ at the time $t = 0$, the system evolves to $|\psi(t)\rangle$ under the action of $H(t) = [1+\delta(t)]I_z + \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$ (where $|\delta(t)| \leq \delta$, $0 \leq \delta < 1$, $\sqrt{\varepsilon_x^2(t)+\varepsilon_y^2(t)} \leq \varepsilon$ and $\varepsilon > 0$). If $t \in [0,T]$, where

$$T = \frac{\arccos(1-2p_0)}{\varepsilon}, \tag{3}$$

the system's state will remain in $\mathscr{D} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \geq 1-p_0\}$ (where $0 < p_0 < 1$). When one makes a projective measurement with the measurement operator σ_z at the time t, the probability of failure $p = |\langle 1|\psi(t)\rangle|^2$ is not greater than p_0.

Hence, we use T defined in (3) as the sampling period to guarantee the required performance. If the measurement result corresponds to $|1\rangle$, a control law is required to drive the system's state back to a subset of $\mathscr{D}$. Different control law design approaches can be used to accomplish this task. Here we employ a Lyapunov method to design the control law. In quantum control, several Lyapunov functions have been constructed, such as state distance-based and average value-based approaches [11]-[14], [26]. Here we select a Lyapunov function based on the Hilbert-Schmidt distance between a state $|\psi\rangle$ and the sliding mode state $|\phi_j\rangle$ [12], [26]; i.e.,

$$V(|\psi\rangle,S) = \frac{1}{2}(1-|\langle\phi_j|\psi\rangle|^2).$$

The control law can be selected as (for details, see [22], [26]):

$$u_k = K_k f_k(\Im[e^{i\angle\langle\psi|\phi_j\rangle}\langle\phi_j|I_k|\psi\rangle]), \quad (k = x,y,z) \tag{4}$$

where $\Im[a+bi] = b$ $(a,b \in \mathbf{R})$, $\angle c$ denotes the argument of a complex number c, $K_k > 0$ may be used to adjust the control amplitude and $f(\cdot)$ satisfies $xf(x) \geq 0$. Define $\angle\langle\psi|\phi_j\rangle = 0^\circ$ when $\langle\psi|\phi_j\rangle = 0$.

To guarantee the required robustness, the Lyapunov control should drive the system's state into a subset $\mathscr{E}$ of $\mathscr{D}$. The subset $\mathscr{E}$ can be defined as $\mathscr{E} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \geq 1-\alpha p_0, 0 < p_0 < 1, 0 \leq \alpha \leq 1\}$. The main theorem (Theorem 2) in the following subsection will give a sufficient condition on the relationships between α, p_0 and β to guarantee the required robustness.

B. Main Theorem

Theorem 2: For a two-level quantum system with the initial state $|\psi(0)\rangle$ satisfying $|\langle\psi(0)|1\rangle|^2 \leq \alpha p_0$ $(0 \leq \alpha \leq 1)$ at the time $t = 0$, the system evolves to $|\psi(t)\rangle$ under the action of $H(t) = [1+\delta(t)]I_z + \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$ (where $\sqrt{\varepsilon_x^2(t)+\varepsilon_y^2(t)} \leq \varepsilon$, $\varepsilon > 0$, $|\delta(t)| \leq \delta$ and $0 \leq \delta < 1$). If $t \in [0,\beta T]$ and

$$\alpha \leq \frac{1-\cos[(1-\beta)\arccos(1-2p_0)]}{2p_0} \tag{5}$$

where $0 < \beta \leq 1$ and

$$T = \frac{\arccos(1-2p_0)}{\varepsilon}, \tag{6}$$

the system's state will remain in $\mathscr{D} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \geq 1-p_0\}$ (where $0 < p_0 < 1$). When one makes a projective measurement with the measurement operator σ_z at the time t, the probability of failure $p = |\langle 1|\psi(t)\rangle|^2$ is not greater than p_0.

Remark 1: The proof of Theorem 2 will be presented in Section IV. Using Lemma 1 and Theorem 2, we aim to maintain the system's state in $\mathscr{D}$ by implementing periodic sampling with the sampling period T in (3). This theorem provides a sufficient condition to guarantee the required robustness. Given p_0, β, we can select α satisfying (5) in Theorem 2. If the sampled result is $|1\rangle$, we apply a Lyapunov control law designed to drive the system's state into $\mathscr{E}$. The sampling period and the Lyapunov control law can be designed in advance.

C. Illustrative example

Now we present an illustrative example to demonstrate the proposed method. Assume $p_0 = 0.01$. Consider the case: $\varepsilon = 0.2$, $\beta = 0.95$. From the simulation in [22], we find the fact that the Lyapunov control is also not sensitive to small uncertainties in the system Hamiltonian. More simulation results suggest that the robustness for the Lyapunov control can be enhanced if we select the terminal condition $|\langle 1|\psi(t)\rangle|^2 \leq \eta\alpha p_0$ (where $0 < \eta < 1$) instead of $|\langle 1|\psi(t)\rangle|^2 \leq \alpha p_0$. Here, we select $\eta = 0.8$. Hence, we design the sampling period $T = 1.002$ using (3). Using Theorem 2, we select $\alpha = 0.0025$. We design the Lyapunov control using (4) and the terminal condition $|\langle 1|\psi(t)\rangle|^2 \leq \eta\alpha p_0 = 2\times 10^{-5}$ with the control Hamiltonian $H_u = \frac{1}{2}u(t)\sigma_y$. Using (4), we select $u(t) = K(\Im[e^{i\angle\langle\psi(t)|0\rangle}\langle 0|\sigma_y|\psi(t)\rangle])$ and $K = 500$. Let the time stepsize be given by $\delta t = 10^{-6}$. We can obtain the probability curve of $|0\rangle$ shown in Fig. 2 and the control value shown in

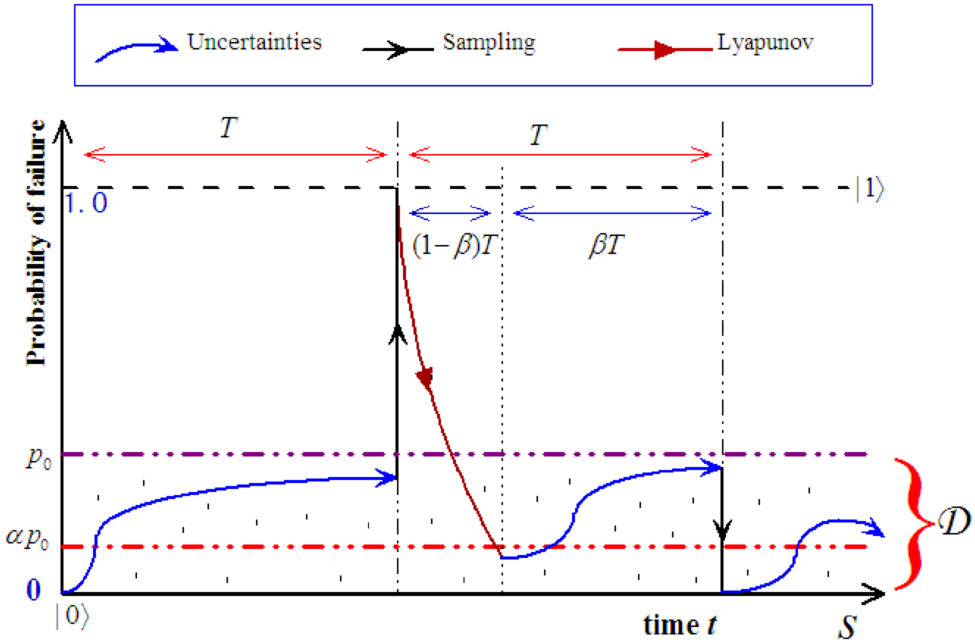

Fig. 1. The proposed sampled-data control scheme for a two-level quantum system based on sliding mode design. In this figure, the labels "Lyapunov", "sampling" and "uncertainties" refer to the evolution process of the quantum system under the Lyapunov control law, the sampling process and uncertainties in the system Hamiltonian, respectively.

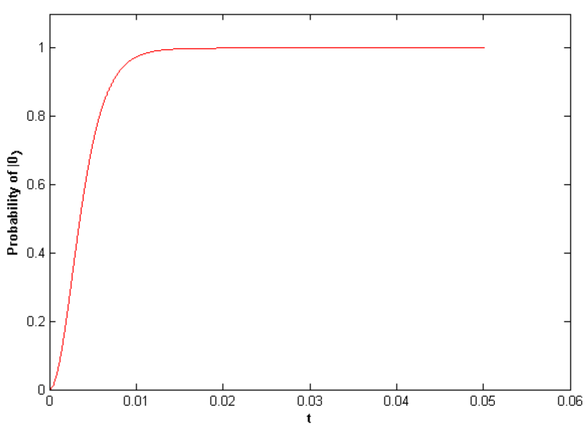

Fig. 2. The probability of $|0\rangle$ under the Lyapunov control law.

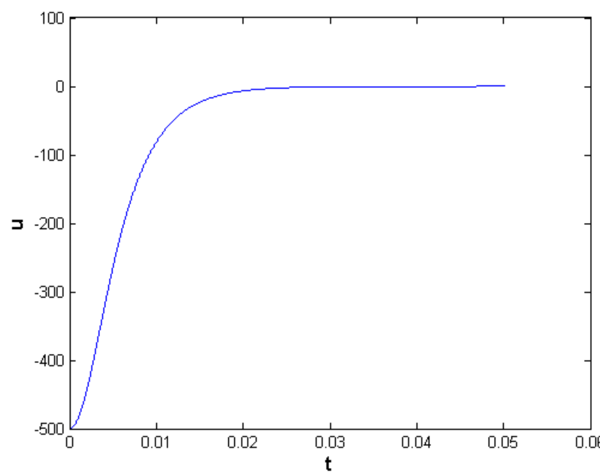

Fig. 3. The control value $u(t)$.

Fig. 3. For the noise $\varepsilon(t)I_x$ or $\varepsilon(t)I_y$ where $\varepsilon(t)$ obeys a uniform distribution in $[-0.2, 0.2]$, more simulation results show that the system's state is also driven into $\mathscr{E}$ using the Lyapunov control law in Fig. 3.

IV. Proof of the Main Theorem

In practical applications, we often use the density operator ρ to describe the quantum state of a quantum system. For a two-level quantum system, the state ρ can be represented in terms of the Bloch vector $\mathbf{r} = (x, y, z) = (\mathrm{tr}\{\rho\sigma_x\}, \mathrm{tr}\{\rho\sigma_y\}, \mathrm{tr}\{\rho\sigma_z\})$:

$$\rho = \frac{1}{2}(I + \mathbf{r}\cdot\sigma). \qquad (7)$$

The dynamical equation for ρ can be written as

$$\dot{\rho} = -i[H, \rho] \qquad (8)$$

where $[A, B] = AB - BA$. After we represent the state ρ with the Bloch vector, the pure states (satisfying $\rho = |\psi\rangle\langle\psi|$) for a two-level quantum system correspond to the surface of the Bloch sphere, where $(x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. An arbitrary pure state $|\psi\rangle$ for a two-level quantum system can be represented as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle, \qquad (9)$$

where $|0\rangle$ and $|1\rangle$ are eigenstates of H_0.

Assume that the state at time t is ρ_t. If we sample using projective measurements on this system, the probability p that it will collapse into $|1\rangle$ (the probability of failure) is

$$p = \langle 1|\rho_t|1\rangle = \frac{1-z_t}{2}. \quad (10)$$

To prove Theorem 2, we first prove two lemmas (Lemma 3 and Lemma 4).

Lemma 3: For a two-level quantum system with the initial state $(x_0, y_0, z_0) = (0,0,1)$, the system evolves to (x_t^A, y_t^A, z_t^A) and (x_t^B, y_t^B, z_t^B) under the action of $H^A = [1+\delta(t)]I_z + \varepsilon\cos\gamma_0 I_y + \varepsilon\sin\gamma_0 I_x$ (constant $\varepsilon > 0$ and $|\delta(t)| \le \delta$) and $H^B = \varepsilon\cos\gamma_0 I_y + \varepsilon\sin\gamma_0 I_x$, respectively. For arbitrary $t \in [0, \frac{\pi}{2\sqrt{4+\varepsilon^2}}]$, $z_t^A \ge z_t^B$.

Proof: For the system with Hamiltonian $H^A = [1+\delta(t)]I_z + \varepsilon\cos\gamma_0 I_y + \varepsilon\sin\gamma_0 I_x$, using $\dot{\rho} = -i[H,\rho]$ and (7), we obtain the following state equations

$$\begin{pmatrix} \dot{x}_t^A \\ \dot{y}_t^A \\ \dot{z}_t^A \end{pmatrix} = \begin{pmatrix} 0 & -[1+\delta(t)] & \varepsilon\cos\gamma_0 \\ 1+\delta(t) & 0 & -\varepsilon\sin\gamma_0 \\ -\varepsilon\cos\gamma_0 & \varepsilon\sin\gamma_0 & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix}, \quad (11)$$

where $(x_0^A, y_0^A, z_0^A) = (0,0,1)$.

We now consider $\delta(t)$ as a control input and select the performance measure as

$$J(\delta) = z_f. \quad (12)$$

Also, we introduce the Lagrange multiplier vector $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))^T$ and obtain the corresponding Hamiltonian function $\mathbb{H} = \mathbb{H}(\mathbf{r}(t), \delta(t), \lambda(t), t)$ as follows:

$$\mathbb{H} \equiv \lambda^T(t) \begin{pmatrix} 0 & -[1+\delta(t)] & \varepsilon\cos\gamma_0 \\ 1+\delta(t) & 0 & -\varepsilon\sin\gamma_0 \\ -\varepsilon\cos\gamma_0 & \varepsilon\sin\gamma_0 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}, \quad (13)$$

where $\mathbf{r}(t) = (x_t, y_t, z_t)$. That is

$$\begin{aligned} \mathbb{H}(\mathbf{r}(t), \delta(t), \lambda(t), t) &= [1+\delta(t)](\lambda_2(t)x_t - \lambda_1(t)y_t) \\ &\quad +\varepsilon\cos\gamma_0(\lambda_1(t)z_t - \lambda_3(t)x_t) \\ &\quad -\varepsilon\sin\gamma_0(\lambda_2(t)z_t - \lambda_3(t)y_t). \end{aligned} \quad (14)$$

According to Pontryagin's minimum principle [27], a necessary condition for $\delta^*(t)$ to minimize $J(\delta)$ is

$$\mathbb{H}(\mathbf{r}^*(t), \delta^*(t), \lambda^*(t), t) \le \mathbb{H}(\mathbf{r}^*(t), \delta(t), \lambda^*(t), t). \quad (15)$$

Hence, if we do not consider singular cases (i.e., $\lambda_2(t)x_t - \lambda_1(t)y_t \equiv 0$), the optimal control $\delta^*(t)$ should be chosen as follows:

$$\delta^*(t) = -\delta\,\mathrm{sgn}(\lambda_2(t)x_t - \lambda_1(t)y_t). \quad (16)$$

That is, the optimal control strategy for $\delta(t)$ is bang-bang control; i.e., $\delta^*(t) = \bar{\delta} = +\delta$ or $-\delta$. Now we consider $H^A = (1+\bar{\delta})I_z + \varepsilon\cos\gamma_0 I_y + \varepsilon\sin\gamma_0 I_x$, which leads to the following state equations

$$\begin{pmatrix} \dot{x}_t^A \\ \dot{y}_t^A \\ \dot{z}_t^A \end{pmatrix} = \begin{pmatrix} 0 & -(1+\bar{\delta}) & \varepsilon\cos\gamma_0 \\ 1+\bar{\delta} & 0 & -\varepsilon\sin\gamma_0 \\ -\varepsilon\cos\gamma_0 & \varepsilon\sin\gamma_0 & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix}, \quad (17)$$

where $(x_0^A, y_0^A, z_0^A) = (0,0,1)$. The corresponding solution is as follows

$$(x_t^A\ y_t^A\ z_t^A)^T = \begin{pmatrix} \frac{\varepsilon\cos\gamma_0}{\sqrt{(1+\bar{\delta})^2+\varepsilon^2}}\sin\omega t - \frac{(1+\bar{\delta})\varepsilon\sin\gamma_0}{(1+\bar{\delta})^2+\varepsilon^2}\cos\omega t + \frac{(1+\bar{\delta})\varepsilon\sin\gamma_0}{(1+\bar{\delta})^2+\varepsilon^2} \\ -\frac{\varepsilon\sin\gamma_0}{\sqrt{(1+\bar{\delta})^2+\varepsilon^2}}\sin\omega t - \frac{(1+\bar{\delta})\varepsilon\cos\gamma_0}{(1+\bar{\delta})^2+\varepsilon^2}\cos\omega t + \frac{(1+\bar{\delta})\varepsilon\cos\gamma_0}{(1+\bar{\delta})^2+\varepsilon^2} \\ \frac{\varepsilon^2}{(1+\bar{\delta})^2+\varepsilon^2}\cos\omega t + \frac{(1+\bar{\delta})^2}{(1+\bar{\delta})^2+\varepsilon^2} \end{pmatrix} \quad (18)$$

where $\omega = \sqrt{(1+\bar{\delta})^2+\varepsilon^2}$.

From (18), we know that z_t is a monotonically decreasing function in t when $t \in [0, \frac{\pi}{2\sqrt{4+\varepsilon^2}}]$. Hence, we only consider the case $t \in [0, t_f]$ where $t_f \in [0, \frac{\pi}{2\sqrt{4+\varepsilon^2}}]$.

Now consider the optimal control problem with a fixed final time t_f and a free final state $\mathbf{r}_f = (x_f, y_f, z_f)$. According to Pontryagin's minimum principle, $\lambda^*(t_f) = \frac{\partial}{\partial \mathbf{r}}\mathbf{r}^*(t_f)$. From this, it is straightforward to verify that $(\lambda_1(t_f), \lambda_2(t_f), \lambda_3(t_f)) = (0,0,1)$. Now let us consider another necessary condition $\dot{\lambda}(t) = -\frac{\partial \mathbb{H}(\mathbf{r}(t), \varepsilon(t), \lambda(t), t)}{\partial \mathbf{r}}$ which leads to the following relationships:

$$\dot{\lambda}(t) = (\dot{\lambda}_1(t)\ \dot{\lambda}_2(t)\ \dot{\lambda}_3(t))^T = \begin{pmatrix} 0 & -(1+\bar{\delta}) & \varepsilon\cos\gamma_0 \\ 1+\bar{\delta} & 0 & -\varepsilon\sin\gamma_0 \\ -\varepsilon\cos\varphi_0 & \varepsilon\sin\gamma_0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix}, \quad (19)$$

where $(\lambda_1(t_f), \lambda_2(t_f), \lambda_3(t_f)) = (0,0,1)$. The corresponding solution is

$$\begin{aligned} \lambda_1(t) &= -\frac{\varepsilon\cos\gamma_0}{\sqrt{(1+\bar{\delta})^2+\varepsilon^2}}\sin\omega(t_f - t) \\ &\quad -\frac{(1+\bar{\delta})\varepsilon\sin\gamma_0}{(1+\bar{\delta})^2+\varepsilon^2}\cos\omega(t_f-t) + \frac{(1+\bar{\delta})\varepsilon\sin\gamma_0}{(1+\bar{\delta})^2+\varepsilon^2}, \\ \lambda_2(t) &= \frac{\varepsilon\sin\gamma_0}{\sqrt{(1+\bar{\delta})^2+\varepsilon^2}}\sin\omega(t_f - t) \\ &\quad -\frac{(1+\bar{\delta})\varepsilon\cos\gamma_0}{(1+\bar{\delta})^2+\varepsilon^2}\cos\omega(t_f-t) + \frac{(1+\bar{\delta})\varepsilon\cos\gamma_0}{(1+\bar{\delta})^2+\varepsilon^2}, \\ \lambda_3(t) &= \frac{\varepsilon^2}{(1+\bar{\delta})^2+\varepsilon^2}\cos\omega(t_f-t) + \frac{(1+\bar{\delta})^2}{(1+\bar{\delta})^2+\varepsilon^2}. \end{aligned} \quad (20)$$

We obtain

$$\lambda_2(t)x_t - \lambda_1(t)y_t = \frac{\varepsilon^2(1+\bar{\delta})}{\omega^{3/2}}[\sin\omega t + \sin\omega(t_f - t) - \sin\omega t_f]. \quad (21)$$

It is easy to show that the quantity $(\lambda_2(t)x_t - \lambda_1(t)y_t) \ge 0$ occurring in (16) does not change its sign when $t_f \in [0, \frac{\pi}{2\sqrt{4+\varepsilon^2}}]$ and $t \in [0, t_f]$. Hence, the optimal control is $\delta^*(t) = \bar{\delta} = -\delta$.

Now, we exclude the possibility that there exists a singular case. Suppose that there exists a singular interval $[t_0, t_1]$ (where $t_0 \ge 0$ and we assume that $[t_0, t_1]$ is the first singular interval) such that when $t \in [t_0, t_1]$

$$h(t) = \lambda_2(t)x_t - \lambda_1(t)y_t \equiv 0. \quad (22)$$

We also have the following relationship

$$\ddot{h}(t) = \lambda_3(t)x_t - \lambda_1(t)z_t \equiv 0 \quad (23)$$

where we have used (11) and the following costate equation

$$\dot{\lambda}(t) = (\dot{\lambda}_1(t)\ \dot{\lambda}_2(t)\ \dot{\lambda}_3(t)\ )^T = \begin{pmatrix} 0 & -[1+\delta(t)] & \varepsilon\cos\gamma_0 \\ 1+\delta(t) & 0 & -\varepsilon\sin\gamma_0 \\ -\varepsilon\cos\gamma_0 & \varepsilon\sin\gamma_0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix}. \quad (24)$$

If $t_0 = 0$, we have $(x_0, y_0, z_0) = (0,0,1)$. By the principle of optimality [27], we may consider the case $t_f = t_1$. Using (22), (23) and $(\lambda_1(t_1), \lambda_2(t_1), \lambda_3(t_1)) = (0,0,1)$, we have $x_{t_1} = 0$ and $y_{t_1} = 0$. Using the relationship of $x_t^2 + y_t^2 + z_t^2 = 1$, we obtain $z_{t_1} = 1$ or -1. If $z_{t_1} = 1$, the initial state and the final state are the same state $|0\rangle$. However, if we use the control $\delta(t) = \bar{\delta}$, from (18) we have $z_{t_1}(\bar{\delta}) = \frac{\varepsilon^2}{(1+\bar{\delta})^2+\varepsilon^2}\cos\omega t_1 + \frac{(1+\bar{\delta})^2}{(1+\bar{\delta})^2+\varepsilon^2} < z_{t_1} = 1$. Hence, this contradicts the fact that we are considering the optimal case $\min z_f$. If $z_{t_1} = -1$, there exists $0 < \tilde{t}_1 < t_1$ such that $z_{\tilde{t}_1} = 0$. By the principle of optimality [27], we may consider the case $t_f = \tilde{t}_1$. From the two equations (22) and (23), we know that $z_{\tilde{t}_1}^2 = 1$ which contradicts $z_{\tilde{t}_1} = 0$. Hence, no singular condition can exist if $t_0 = 0$.

If $t_0 > 0$, using (16) we must select $\delta(t) = \bar{\delta}$ when $t \in [0, t_0]$. From (21), we know that there exist no $t_0 \in (0, t_f)$ satisfying $\lambda_2(t_0)x_{t_0} - \lambda_1(t_0)y_{t_0} = 0$. Hence, there exist no singular cases for our problem. From the previous analysis, $\delta(t) = -\delta$ is the optimal control when $t \in [0, \frac{\pi}{2\sqrt{4+\varepsilon^2}}]$.

For the system with Hamiltonian $H^B = \varepsilon\cos\gamma_0 I_y + \varepsilon\sin\gamma_0 I_x$, using $\dot{\rho} = -i[H,\rho]$ and (7), we obtain the following state equations

$$\begin{pmatrix} \dot{x}_t^B \\ \dot{y}_t^B \\ \dot{z}_t^B \end{pmatrix} = \begin{pmatrix} 0 & 0 & \varepsilon\cos\gamma_0 \\ 0 & 0 & -\varepsilon\sin\gamma_0 \\ -\varepsilon\cos\gamma_0 & \varepsilon\sin\gamma_0 & 0 \end{pmatrix} \begin{pmatrix} x_t^B \\ y_t^B \\ z_t^B \end{pmatrix}, \quad (25)$$

where $(x_0^B, y_0^B, z_0^B) = (0,0,1)$. The corresponding solution is as follows:

$$\begin{pmatrix} x_t^B \\ y_t^B \\ z_t^B \end{pmatrix} = \begin{pmatrix} \cos\gamma_0\sin\varepsilon t \\ -\sin\gamma_0\sin\varepsilon t \\ \cos\varepsilon t \end{pmatrix}. \quad (26)$$

We define $F(t)$ and $f(t)$ as follows:

$$F(t) = z_t^A - z_t^B = \frac{\varepsilon^2}{(1-\delta)^2+\varepsilon^2}\cos\omega t + \frac{(1-\delta)^2}{(1-\delta)^2+\varepsilon^2} - \cos\varepsilon t, \quad (27)$$

$$f(t) = \dot{F}(t) = -\frac{\varepsilon^2}{\sqrt{(1-\delta)^2+\varepsilon^2}}\sin\omega t + \varepsilon\sin\varepsilon t. \quad (28)$$

Now, we consider $t \in [0, \frac{\pi}{2\sqrt{4+\varepsilon^2}}]$ and obtain

$$\dot{f}(t) = \varepsilon^2(\cos\varepsilon t - \cos\omega t) \geq 0. \quad (29)$$

It is clear that $\dot{f}(t) = 0$ only when $t = 0$. Hence $f(t)$ is a monotonically increasing function and

$$\min_t f(t) = f(0) = 0.$$

Hence, we have

$$f(t) \geq 0. \quad (30)$$

From this, it is clear that $F(t)$ is a monotonically increasing function and

$$\min_t F(t) = F(0) = 0.$$

Hence $F(t) \geq 0$ when $t \in [0, \frac{\pi}{2\sqrt{4+\varepsilon^2}}]$. Therefore, we can conclude that $z_t^A \geq z_t^B$ for arbitrary $t \in [0, \frac{\pi}{2\sqrt{4+\varepsilon^2}}]$.

■

We now present another lemma.

Lemma 4: For a two-level quantum system with the initial state $(x_0, y_0, z_0) = (0,0,1)$, suppose the system evolves to (x_t, y_t, z_t) under the action of $H = \varepsilon(\cos\gamma I_y + \sin\gamma I_x)$ (γ is a constant). Then, z_t is independent of γ.

Proof: For $H = \omega(\sin\gamma I_x + \cos\gamma I_y)$, from (26), we have

$$z_t = \cos\varepsilon t.$$

It is clear that z_t is independent of γ. ■

Remark 2: Since z_t is independent of γ, it is enough to consider a special case $\gamma = \frac{\pi}{2}$ when analyzing z_t under $H = \varepsilon(\cos\gamma I_y + \sin\gamma I_x)$.

Now we can prove Theorem 2.

Proof: For $H^A = [1+\delta(t)]I_z + \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$, using $\dot{\rho} = -i[H^A, \rho]$ and (7), we obtain the following state equations

$$\begin{pmatrix} \dot{x}_t^A \\ \dot{y}_t^A \\ \dot{z}_t^A \end{pmatrix} = \begin{pmatrix} 0 & -[1+\delta(t)] & \varepsilon_y(t) \\ 1+\delta(t) & 0 & -\varepsilon_x(t) \\ -\varepsilon_y(t) & \varepsilon_x(t) & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix}. \quad (31)$$

We first consider $z_0 = \cos\theta_0 = 1 - 2\alpha p_0$, where $\theta_0 \in (0, \frac{\pi}{2})$.

Define $\varepsilon(t) = \sqrt{\varepsilon_x^2(t) + \varepsilon_y^2(t)}$ and $\varepsilon_x(t) = \varepsilon(t)\sin\gamma_t$, $\varepsilon_y(t) = \varepsilon(t)\cos\gamma_t$. This leads to the following equation

$$(\dot{x}_t^A\ \dot{y}_t^A\ \dot{z}_t^A\ )^T = \begin{pmatrix} 0 & -[1+\delta(t)] & \varepsilon(t)\cos\gamma_t \\ 1+\delta(t) & 0 & -\varepsilon(t)\sin\gamma_t \\ -\varepsilon(t)\cos\gamma_t & \varepsilon(t)\sin\gamma_t & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix} \quad (32)$$

where $(x_0^A, y_0^A, z_0^A) = (\sin\theta_0\cos\varphi_0, \sin\theta_0\sin\varphi_0, \cos\theta_0)$ and $\varphi_0 \in [0, 2\pi]$.

Define $N(t) = -\varepsilon(t)\sin\theta_0\cos(\gamma_t + \varphi_0)$. From (32), we have

$$\dot{z}_t^A|_{t=0} = \lim_{t\to 0} N(t). \quad (33)$$

Now we take an arbitrary evolution state (except $|1\rangle$) starting from $|0\rangle$ as a new initial state. For $H^B = \varepsilon I_x$, the initial state can be represented as $(x_0', y_0', z_0) = (0, -\sin\theta_0, \cos\theta_0)$, where $\theta_0 \in (0, \pi)$. We have

$$\begin{pmatrix} \dot{x}_t^B \\ \dot{y}_t^B \\ \dot{z}_t^B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varepsilon \\ 0 & \varepsilon & 0 \end{pmatrix} \begin{pmatrix} x_t^B \\ y_t^B \\ z_t^B \end{pmatrix}. \quad (34)$$

Hence,

$$\dot{z}_t^B|_{t=0} = -\varepsilon\sin\theta_0. \quad (35)$$

It is clear that for any t

$$-\varepsilon \sin\theta_0 \leq N(t) \leq \varepsilon \sin\theta_0. \quad (36)$$

Therefore, for $\Delta t \to 0$, we have

$$z^A_{\Delta t} \geq z^B_{\Delta t}. \quad (37)$$

When $\sin\theta_0 = 0$, using Lemma 3 and Lemma 4, we can also obtain the same conclusion $z^A_{\Delta t} \geq z^B_{\Delta t}$.

From (34), we know that $z^B_t = \cos(\theta_0 + \varepsilon t)$. When $0 < t < \frac{\pi - 2\theta_0}{2\varepsilon}$, z^B_t decreases monotonically in t. We now define $g(t) = z^A_t - z^B_t$ and assume that there exist $t = t_1 \in [0, \frac{\pi-2\theta_0}{2\varepsilon})$ such that $z^A_{t_1} < z^B_{t_1}$. That is, $g(t_1) < 0$. Since $g(t)$ is continuous in t and $g(0) = 0$, there exists a time $t^* = \sup\{t | 0 \leq t < t_1, g(t) = 0\}$ satisfying $g(t) < 0$ for $t \in (t^*, t_1]$. However, we have established that for any $z^A_t = z^B_t$ and $\Delta t \to 0$, $z^A_{t+\Delta t} \geq z^B_{t+\Delta t}$, which contradicts $g(t) < 0$ for $t \in (t^*, t_1]$. Hence, we have the following relationship for $t \in [0, \frac{\pi-2\theta_0}{2\varepsilon})$

$$z^A_t \geq z^B_t. \quad (38)$$

From (10), it is clear that the probabilities of failure satisfy $p^A_t = \frac{1-z^A_t}{2} \leq p^B_t = \frac{1-z^B_t}{2}$. That is, the probability of failure p^A_t is not greater than p^B_t for $t \in [0, \frac{\pi-2\theta_0}{2\varepsilon})$.

Since $z^B_t = \cos(\theta_0 + \varepsilon t)$, we have $\Delta z^B_{\beta T} = \cos\theta_0 - \cos(\theta_0 + \varepsilon\beta T)$, where

$$T = \frac{\arccos(1 - 2p_0)}{\varepsilon}. \quad (39)$$

Using the relationship (38), we have

$$z^A_{\beta T} \geq 1 - 2\alpha p_0 + \cos(\theta_0 + \varepsilon\beta T) - \cos\theta_0 = M.$$

Now let

$$p = \frac{1 - z^A_{\beta T}}{2} \leq \frac{1-M}{2} \leq p_0.$$

Using the fact $\theta_0 = \arccos(1 - 2\alpha p_0)$, we have the following relationship

$$\alpha \leq \frac{1 - \cos[(1-\beta)\arccos(1-2p_0)]}{2p_0}. \quad (40)$$

■

V. CONCLUSIONS

This paper proposes a sampled-data control scheme to deal with the uncertainties in the system Hamiltonian for two-level quantum systems. The control law consisting of a periodic sampling process and a Lyapunov control law can be designed offline. We give a sufficient condition for the design of the control law to guarantee the required robustness. The proposed sampled-data control approach can be extended to more general finite-level quantum systems with uncertainties and has potential applications to robust quantum information processing.

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