2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Approximate Zero Polynomials of Polynomial Matrices and Linear Systems

Nicos Karcanias and George Halikias

Abstract— **This paper introduces the notions of approximate and optimal approximate zero polynomial of a polynomial matrix by deploying recent results on the approximate GCD of a set of polynomials [1] and the exterior algebra [4] representation of polynomial matrices. The results provide a new definition for the "approximate", or "almost" zeros of polynomial matrices and provide the means for computing the distance from non-coprimeness of a polynomial matrix. The computational framework is expressed as a distance problem in a projective space. The general framework defined for polynomial matrices provides a new characterization of approximate zeros and decoupling zeros [2], [4] of linear systems and a process leading to computation of their optimal versions. The use of restriction pencils provides the means for defining the distance of state feedback (output injection) orbits from uncontrollable (unobservable) families of systems, as well as the invariant versions of the "approximate decoupling polynomials".**

I. INTRODUCTION

The notion of almost zeros and almost decoupling zeros for a linear system has been introduced in [4] and their properties have been linked to mobility of poles under compensation. The basis of that definition has been the representation of the Plücker embedding by using the Grassmann polynomial vectors [3]. This process has introduced new system invariant and led to the definition of "almost zeros" of a set of polynomials as the minima of a function associated with the polynomial vector [2]. Here we develop the concept further by introducing the notion of "approximate zero polynomials" using the exterior algebra framework introduced in [4], and then by deploying the results on the approximate GCD defined in [1]. The notion of "approximate zero polynomials" (AZP) of a polynomial matrix and "optimal" AZP are defined in terms of an optimization expressing the computation of the distance of a point in a projective space from the intersection of two varieties. The first is the Grassmann variety [3], [13] and the second is the given degree GCD variety of the projective space. The results on polynomial matrices are then used to define the "approximate input, output decoupling zero polynomials" and "approximate zero polynomial" of a linear system.

Defining the distance of a system described by the pair (A,B) (pair (A,C)) from the family of uncontrollable (unobservable) systems has been a subject under consideration for some time [15]. It is worth pointing that although the controllability (observability) properties are invariant under state feedback (output injection), their "strength" (measured with different criteria) is not. This raises the question of whether invariant measures can be defined. Here we introduce a new framework for evaluating such distances that allows the computation of the specific system (A,B) $((A,C))$, as well as the state feedback (output injection) orbits $(A+BL,B)$ $((A+KC,C))$ from the uncontrollable (unobservable) systems. The latter is a new dimension to the problem and it is complemented by the definition of the corresponding decoupling zero polynomials.

We are using the exterior algebra framework by deploying the Plücker embedding [3] to associate polynomial vectors to polynomial matrices; thus we have a framework that allows a proper definition of the notion of "approximate matrix divisor" of polynomial models, as well as the notion of the distance of a polynomial matrix from families of non-coprime matrices. It is shown that the characterisation and computation of an "approximate matrix divisor" is equivalent to a distance problem of a general set of polynomials from the intersection of two varieties, a GCD (defined by the degree of the desirable GCD) and the dynamic Grassmann variety that is defined by the Forney order [8] of the polynomial matrix. The notion of approximate matrix divisor introduced here refers to a family of square matrices all having the same polynomial as determinant.

The results introduce a computational framework that potentially can provide the means for defining "approximate zero polynomials" for linear systems and introduce new measures of distance of systems from uncontrollability, unobservability using the "strength" associated with a given approximate solution. The characterisation of distance from uncontrollability, un-observability uses the algebraic matrix pencil characterisation [6], [7] which is based on the properties of Grassmann vectors and associated Plücker matrices of the corresponding pencils [9]. Using the algebraic feedback free criteria introduced by the restriction pencils [13], [14], a new notion of distance that is invariant under feedback is introduced, which expresses distance from state feedback orbit (uncontrollability case), output injection orbit (unobservability case). The use of Grassmann vectors implies that the general results on the "strength" of approximation, defined in [1] for polynomial vectors, yield lower bounds for the corresponding approximate polynomials of polynomial matrices.

N. Karcanias is with the Systems and Control Research Centre, School of Engineering and Mathematical Sciences, City University, Northampton Square, London EC1V 0HB, UK n.karcanias@city.ac.uk

G. Halikias is with the the Systems and Control Research Centre, School of Engineering and Mathematical Sciences, City University, Northampton Square, London EC1V 0HB, UK g.halikias@city.ac.uk

II. DEFINITIONS AND PRELIMINARY RESULTS

Consider the linear system $S(A,B,C,D)$:

$$S(A,B,C,D): \ \dot{\underline{x}} = A\underline{x} + B\underline{u}, \ \underline{y} = C\underline{x} + D\underline{u} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times p}$. It is assumed that (A,B) is controllable and (A,C) is observable. Alternatively, $S(A,B,C,D)$ is defined by the transfer function matrix represented in terms of left, right coprime matrix fraction descriptions (LCMFD, RCMFD), as

$$G(s) = D_l(s)^{-1}N_l(s) = N_r(s)D_r(s)^{-1} \tag{2}$$

where $N_l(s)$, $N_r(s) \in \mathbb{R}^{m\times p}[s]$, $D_l(s) \in \mathbb{R}^{m\times m}[s]$ and $D_r(s) \in \mathbb{R}^{p\times p}[s]$. We shall denote by N a left annihilator of B, i.e. $N \in \mathbb{R}^{(n-p)\times n}$, $NB = 0$ and by M a right annihilator of C, i.e. $M \in \mathbb{R}^{n\times(n-m)}$, $CM = 0$, where N, M have full rank.

The family of frequency assignment problems has a common formulation that allows a unifying treatment in terms of the Abstract Determinantal Assignment Problem. Thus :

(i) Pole assignment by state feedback: Pole assignment by state feedback $L \in \mathbb{R}^{n\times p}$ reduces to

$$p_L(s) = \det\{sI - A - BL\} = \det\{B(s)\tilde{L}\} \tag{3}$$

where $B(s) = [sI - A, -B]$ is defined as the system controllability pencil and $\tilde{L} = [I_n, L^t]^t$. The zeros of $B(s)$ are the input decoupling zeros of the system [6].

(ii) Design observers: The design of an n-state observer by an output injection $T \in \mathbb{R}^{n\times m}$ reduces to

$$p_T(s) = \det\{sI - A - TC\} = \det\{\tilde{T}C(s)\} \tag{4}$$

where $C(s) = [sI - A^t, -C^t]^t$ is the observability pencil and $\tilde{T} = [I_n, T]$ represents output injection. The zeros of $C(s)$ define the output decoupling zeros [6].

(iii) Zero assignment by squaring down: Given a system with $m > p$ and $\underline{c} \in \mathbb{R}^p$ the vector of the variables which are to be controlled, then $\underline{c} = Hy$ where $H \in \mathbb{R}^{p\times m}$ is a *squaring down post-compensator*, and $G'(s) = HG(s)$ is the *squared down transfer function matrix* [5]. A right MFD for $G'(s)$ is defined $G'(s) = HN_r(s)D_r(s)^{-1}$, $G(s) = N_r(s)D_r(s)^{-1}$. Finding H such that $G'(s)$ has assigned zeros is defined as the *zero assignment by squaring down* problem [5], and the zero polynomial of $S(A,B,HC,HD)$ is

$$z_K(s) = \det\{HN_r(s)\} \tag{5}$$

Remark 1: The zeros of $M(s)$ are fixed zeros of all polynomial combinants $f(s)$. The input (output) decoupling zeros are fixed zeros under state feedback (output injection) and nonsquare zeros are fixed zeros under all squaring down compensators. For the case of polynomial matrices, the zeros are expressed as zeros of matrix divisors [7], or as the roots of the GCD of a polynomial multi-vector [3], [4]. The latter formulation allows the development of a framework for defining "almost zeros" in a way that also permits the quantification of the strength of approximation. ■

A. The Abstract Determinantal Assignment Problem (DAP):

This problem is to solve equation (7) below with respect to the constant matrix H:

$$\det(HN(s)) = f(s) \tag{6}$$

where $f(s)$ is the polynomial of an appropriate d-degree. DAP is a multilinear nature problem of a determinantal character. If $M(s) \in \mathbb{R}^{p\times r}[s]$, $r \le p$ such that $\operatorname{rank}\{M(s)\} = r$ and let $\mathscr{H}$ be a family of full rank $r\times p$ constant matrices having a certain structure then DAP is reduced to solve equation (7) with respect to $H \in \mathscr{H}$

$$f_M(s,H) = \det(HM(s)) = f(s) \tag{7}$$

where $f(s)$ is a real polynomial of some degree d.

Notation [3]: Let $Q_{k,n}$ be the set of lexicographically ordered, strictly increasing sequences of k integers from $1,2,\dots,n$. If $\{\underline{x}_{i_1},\dots,\underline{x}_{i_k}\}$ is a set of vectors of a vector space $\mathscr{V}$, $\omega = (i_1,...,i_k) \in Q_{k,n}$, then $\underline{x}_{i_1} \wedge \dots \wedge \underline{x}_{i_k} = \underline{x}_\omega \wedge$ denotes the exterior product and by $\wedge^r\mathscr{V}$ we denote the r-th exterior power of $\mathscr{V}$. If $H \in F^{m\times n}$ and $r \le \min\{m,n\}$, then by $C_r(H)$ we denote the r-th compound matrix of H [3].

If $\underline{h}_i^t, \underline{m}_i(s), i \in \underline{r}$, denote the rows of H, columns of $M(s)$ respectively, then

$$C_r(H) = \underline{h}_1^t \wedge \dots \wedge \underline{h}_r^t = \underline{h}^t \wedge \in \mathbb{R}^{l\times\sigma} \tag{8}$$

$$C_r(M(s)) = \underline{m}_1(s) \wedge \dots \wedge \underline{m}_r(s) = \underline{m}\wedge \in \mathbb{R}^\sigma[s], \ \sigma = \binom{p}{r} \tag{9}$$

and by Binet-Cauchy theorem [3] we have that [4]:

$$f_M(s,H) = C_r(H)C_r(M(s)) = \langle \underline{h}\wedge, \underline{m}(s)\wedge \rangle = \sum_{\omega\in Q_{r,p}} h_\omega m_\omega(s)$$

$\omega = (i_1,\dots,i_r) \in Q_{r,p}$, and h_ω, $m_\omega(s)$ are the coordinates of $\underline{h}\wedge$, $m(s)\wedge$, respectively. Note that h_ω is the $r\times r$ minor of H which corresponds to the ω set of columns of H and h_ω is a multilinear function of the entries h_{ij} of H.

DAP Linear sub-problem: Set $\underline{m}(s) \wedge \underline{p}(s) \in \mathbb{R}^\sigma[s]$, $f(s) \in \mathbb{R}[s]$. Determine the existence of $\underline{k} \in \mathbb{R}^\sigma$, $\underline{k} \neq 0$, such that

$$f_M(s,H) = \underline{k}^t\underline{p}(s) = \sum k_i p_i(s) = f(s), \ \ i \in \underline{\sigma} \tag{10}$$

DAP Multilinear sub-problem: Assume that $\mathscr{K}$ is the family of solution vectors $\underline{k}$ of (5). Determine if there exists $H^t = [h_1,...,h_r]$, $H^t \in \mathbb{R}^{p\times r}$, such that

$$\underline{h}_1 \wedge \dots \wedge \underline{h}_r = \underline{h}\wedge = \underline{k}, \quad k \in K \tag{11}$$

Lemma 1 [3]: Let $\underline{k} \in \mathbb{R}^\sigma$, $\sigma = \binom{p}{r}$ and let k_ω, $\omega = (i_1,...,i_r) \in Q_{r,p}$ be the Plücker coordinates of a point in $P_{\sigma-1}(\mathbb{R})$. Necessary and sufficient condition for the existence of $H \in \mathbb{R}^{r\times p}$, $H = [\underline{h}_1,\dots,\underline{h}_r]^t$, such that

$$\underline{h}\wedge = \underline{h}_1 \wedge \dots \wedge \underline{h}_r = \underline{k} = [\dots,k_\omega,\dots]^t \tag{12}$$

is that the coordinates k_ω satisfy the quadratics

$$\sum_{k=1}^{r+1} (-1)^{\nu-1} k_{i_1,\dots,i_{r-1},j_\nu^k j_1,\dots,j_{\nu-1},j_{\nu+1},j_{r+1}} = 0 \tag{13}$$

where $1 \le i_1 < i_2 < \dots < i_{r-1} \le n$, $1 \le j_1 < j_2 < \dots < j_{r+1} \le n$. ■

The quadratics defined by equation (13) are known as the *Quadratic Plücker Relations* (QPR) [3] and they define the Grassmann variety $\Omega(r,p)$ of $P_{\sigma-1}(\mathbb{R})$.

III. GRASSMANN INVARIANTS OF LINEAR SYSTEMS

Consider $T(s) \in \mathbb{R}^{p\times r}[s]$, $T(s) = [\underline{t}_1(s),\ldots,\underline{t}_r(s)]$, $p \geq r$, $\operatorname{rank}\{T(s)\} = r$, $\mathscr{X}_t = \operatorname{Range}_{\mathbb{R}(s)}(T(s))$. If $T(s) = M(s)D(s)^{-1}$ is a RCMFD of $T(s)$, then $M(s)$ is a polynomial basis for $\mathscr{X}_t$. If $Q(s)$ is a greatest right divisor of $M(s)$ then $T(s) = \tilde{M}(s)Q(s)D(s)^{-1}$, where $\tilde{M}(s)$ is a least degree polynomial basis of $\mathscr{X}_t$ [7]. A *Grassmann Representative* (GR) for $\mathscr{X}_t$ is defined by [4]

$$\underline{t}(s)\wedge = \underline{t}_1(s)\wedge\ldots\wedge\underline{t}_r(s) = \underline{\tilde{m}}_1(s)\wedge\ldots\wedge\underline{\tilde{m}}_r(s)\cdot z_t(s)/p_t(s) \tag{14}$$

where $z_t(s) = \det\{Q(s)\}$, $p_t(s) = \det\{D(s)\}$ are the zero, pole polynomials of $T(s)$ and $\underline{\tilde{m}}(s) = \underline{m}_1(s)\wedge\ldots\wedge\underline{\tilde{m}}_r(s) \in \mathbb{R}^{\sigma}[s]$, $\sigma = \binom{p}{r}$, is also a GR of $\mathscr{X}_t$. Since $\tilde{M}(s)$ is a least degree polynomial basis for $\mathscr{X}_t$, the polynomials of $\underline{\tilde{m}}(s)\wedge$ are coprime and $\underline{\tilde{m}}(s)\wedge$ is a *reduced polynomial* GR ($\mathbb{R}$ - $\mathbb{R}[s]$- GR) of $\mathscr{X}_t$. If $\delta = \deg\{\underline{\tilde{m}}(s)\wedge\}$, then δ is the Forney dynamical order [8] of $\mathscr{X}_t$. $\underline{\tilde{m}}(s)\wedge$ may be expressed as

$$\underline{\tilde{m}}(s)\wedge = p(s) = p_0 + p_1 s + \ldots + p_\delta s^\delta = P_\delta \cdot \underline{e}_\delta(s) \tag{15}$$

where $P_\delta \in \mathbb{R}^{\sigma\times(\delta+1)}$ is a basis matrix for $\underline{\tilde{m}}(s)\wedge$ and $\underline{e}_\delta(s) = [1,s,...,s^\delta]^t$. All $\mathbb{R}[s]$-GRs of $\mathscr{X}_t$ differ only by a nonzero scalar factor $a \in \mathbb{R}$ and if $\|\underline{p}_\delta\| = 1$, we define the canonical $\mathscr{R}[s]$-GR $\underline{g}(\mathscr{X}_t)$ and the basis matrix P_δ is the *Plücker matrix* of $\mathscr{X}_t$ [4].

Theorem 1: $\underline{g}(\mathscr{X}_t)$, or the associated Plücker matrix P_δ, is a complete (basis free) invariant of $\mathscr{X}_t$. ■

If $M(s) \in \mathbb{R}^{p\times r}[s]$, $p \geq r$, $\operatorname{rank}\{M(s)\} = r$, is a polynomial basis of $\mathscr{X}_t$, then $M(s) = \tilde{M}(s)Q(s)$, where $\tilde{M}(s)$ is a least degree basis and $Q(s)$ is a greatest right divisor of the rows of $M(s)$ and thus

$$\underline{m}(s)\wedge = \underline{\tilde{m}}(s)\wedge\cdot\det(Q(s)) = P_\delta \underline{e}_\delta(s) z_m(s) \tag{16}$$

A number of Plücker type matrices are:

(a) Controllability Plücker Matrices: For the pair (A,B), $\underline{b}(s)^t\wedge$ denotes the exterior product of the rows of $B(s) = [sI-A, -B]$ and $P(A,B)$ is the basis matrix of $\underline{b}(s)^t\wedge$, then $P(A,B)$ is the *controllability Plücker matrix* and its rank characterises the controllability properties. For the linear system an equivalent "state feedback-free" characterisation of controllability [14] is provided by the input-restricted pencil $R(s) = sN - NA \in \mathbb{R}^{(n-p)\times n}[s]$ which is invariant of the state feedback orbit and its elementary divisors define the set of input-decoupling zeros of the system. If $\underline{r}(s)^t\wedge$ is the exterior product of the rows of $R(s)$ and $P(N,NA)$ is the basis matrix of $\underline{r}(s)^t\wedge$, then $P(N,NA)$ will be referred to as the *restricted controllability Plücker matrix*.

Theorem 2 [9]: $S(A,B)$ is controllable, iff $P(A,B)$ or equivalently $P(N,NA)$ has full rank. ■

(b) Observability Plücker Matrix: For the pair (A,C), $\underline{c}(s)\wedge$ denotes the exterior product of the columns of $C(s) = [sI-A^t, -C^t]^t$ and $P(A,C)$ is the basis matrix of $\underline{c}(s)\wedge$. $P(A,C)$ is the *observability Plücker matrix* and its rank characterises system observability. For the linear system an equivalent "output injection feedback-free" characterisation of observability [14] is provided by the output-restricted pencil $Q(s) = sM - AM \in \mathbb{R}^{n\times(n-m)}$ which is invariant of the output-injection orbit and its elementary divisors define the set of output-decoupling zeros of the system. If $\underline{q}(s)^t\wedge$ is the exterior product of the rows of $Q(s)$ and $P(M,AM)$ is the basis matrix of $\underline{q}(s)^t\wedge$, then $P(M,AM)$ will be referred to as the *restricted observability Plücker matrix*.

Theorem 3 [9]: $S(A,C)$ is observable, iff $P(A,C)$ or equivalently $M(M,AM)$ has full rank. ■

Remark 2: As far as the exact properties of controllability (observability) the pencils $B(s)$, $R(s)$ ($C(s)$, $Q(s)$) provide equivalent characterisations. The invariance of $R(s)$, $Q(s)$ under feedback has significant differences when it comes to characterising the "relative degree" of controllability, observability, respectively. The relative rank properties of the matrices $P(A,B)$, $P(N,NA)$ and $P(A,C)$, $P(M,AM)$ as defined by the singular values characterise respectively different system properties. In fact, rank properties of:

(i) $P(A,B)$, $P(A,C)$ provide an indication for relative controllability and observability respectively.
(ii) $P(N,NA)$, $P(M,AM)$ provide an indication for relative controllability and observability of the state feedback, output injection orbits respectively. ■

The vectors $\underline{b}^t\wedge$, $\underline{r}^t\wedge$, $\underline{c}^t\wedge$, and $\underline{q}^t\wedge$ are decomposable multivectors [3] and thus the corresponding matrix coefficients should satisfy special conditions (based on the QPRs [3]) and thus they are sub-families of the corresponding general sets of matrices. This leads to:

Proposition 1: The smallest singular values of the Plücker matrices may be used to provide lower bounds for the distance from the family uncontrollable (unobservable) systems. In particular, the smallest singular values of:

(i) $P(A,B)$ ($P(A,C)$) provide a lower bound for distance of the system $S(A,B,C)$ from the family of uncontrollable (unobservable) systems.
(ii) $P(N,NA)$ ($P(M,AM)$) provide a lower bound for distance of the state feedback (output injection) system orbit $S(A+BL,B)$ ($S(A+KC,C)$) from the family of uncontrollable (unobservable) systems. ■

Remark 3: For the cases $p=1$, or $p=n-1$, for (A,B) or $m=1$, or $m=n-1$ for (A,C) the lower bounds become exact. ■

(c) Column Plücker Matrices: For the transfer function $G(s)$, $m \geq p$, $\underline{n}(s)\wedge$ is the exterior product of the columns of the numerator $N_r(s)$, of a RCMFD and $P(N)$ is the basis matrix of $\underline{n}(s)\wedge$. Note that $d = \delta$, the Forney order of $\mathscr{X}_t$, if $G(s)$ has no finite zeros and $d = \delta + k$, where k is the number of finite zeros of $G(s)$, otherwise. If $N_r(s)$ is least degree, then $P_c(N)$ is the *column space Plücker matrix* .

Theorem 4 [10]: For a generic system with $m > p$, for which $p(m-p) > \delta+1$, where δ is the Forney order, $P_c(N)$ has full rank. ■

IV. Approximate GCD of Polynomial Sets

Consider a set $P=\{a(s),b_i(s)\in\mathbb{R}[s],\ i=1,2,\ldots,h$ of polynomials which has $h+1$ elements and with the two largest degrees (n,p), which is also denoted as $P_{h+1,n}$. The greatest common divisor (GCD) of P will be denoted by $\varphi(s)$. For any $P_{h+1,n}$ we define a vector representative $\underline{p}_{h+1}(s)$ and a basis matrix P_{h+1}. The classical approaches for the study of coprimeness and determination of the GCD makes use of the Sylvester Resultant, S_P, [11], [12]:

Theorem 5: For as set of polynomials $P_{h+1,n}$ with a resultant S_P the following properties hold true:

1) Necessary and sufficient condition for a set of polynomials to be coprime is that $\operatorname{rank}(S_P)=n+p$.
2) Let $\varphi(s)$ be the GCD of P. Then $\operatorname{rank}(S_P)=n+p-\deg\varphi(s)$.
3) If we reduce S_P, by using elementary row operations, to its row echelon form, the last non-vanishing row defines the coefficients of the GCD. ■

The results in [12] establish a matrix based representation of the GCD, which is equivalent to the standard algebraic factorisation of the GCD of polynomials. This new GCD representation provides the means to define the notion of the "approximate GCD" subsequently in a formal way, and thus allows the definition of the optimal solution.

Theorem 6: Consider $P=\{a(s),b_1(s),\ldots,b_h(s)\}$, $\deg a(s)=n$, $\deg b_i(s)\leq p\leq n$, $i=1,\ldots,h$ be a polynomial set, S_P the respective Sylvester matrix, $\varphi(s)=\lambda_k s^k+\cdots+\lambda_1 s+\lambda_0$ be the GCD of the set and let k be its degree. Then there exists transformation matrix $\Phi_\varphi\in\mathbb{R}^{(n+p)\times(n+p)}$ such that:

$$\bar{S}_{P^*}^{(k)}=S_P\Phi_\varphi=\left[\begin{array}{c|c}0_k & \bar{S}_{P^*}\end{array}\right] \tag{17}$$

or

$$S_P=\bar{S}_{P^*}^{(k)}\hat{\Phi}_\varphi=\left[\begin{array}{c|c}0_k & \bar{S}_{P^*}\end{array}\right]\hat{\Phi}_\varphi \tag{18}$$

where $\Phi_\varphi=\hat{\Phi}_\varphi^{-1}$, $\hat{\Phi}_\varphi$ being the Toeplitz form of $\varphi(s)$ [12] and

$$\bar{S}_{P^*}^{(k)}=\begin{bmatrix}0 & S_0^{(k)}\\ 0 & S_1^{(k)}\\ \vdots & \vdots\\ 0 & S_h^{(k)}\end{bmatrix}=[0\ \tilde{S}_P^{(k)}] \tag{19}$$

where $S_i^{(k)}$ are appropriate Toeplitz blocks. ■

The problem which is addressed next is the formal definition of the notion of the "approximate GCD" [1] and the evaluation of its strength. We shall denote by $\Pi(n,p;h+1)$ the set of all polynomial sets $P_{h+1,n}$ with the (n,p) the maximal two degrees and $h+1$ elements. If $P_{h+1,n}\in\Pi(n,p;h+1)$ we can define an (n,p)-ordered perturbed set

$$\begin{aligned} P'_{h+1,n} &= P_{h+1,n}-Q_{h+1,n}\in\Pi(n,p;h+1) && (20)\\ &= \{p'_i(s)=p_i(s)-q_i(s):\deg q_i(s)\leq\deg p_i(s)\} && (21)\end{aligned}$$

This process is described by Figure 1.

Lemma 2 [1]: For a set $P_{h+1,n}\in\Pi(n,p;h+1)$ and an $\omega(s)\in\mathbb{R}[s]$ with $\deg\omega(s)\leq p$, there always exists a family of (n,p)-ordered perturbations $Q_{h+1,n}$ and for every element of this family $P'_{h+1,n}=P_{h+1,n}-Q_{h+1,n}$ has a GCD divisible by $\omega(s)$. ■

Definition 1: Let $P_{h+1,n}\in\Pi(n,p;h+1)$ and $\omega(s)\in\mathbb{R}[s]$ be a given polynomial with $\deg\omega(s)=r\leq p$. If $\Sigma_\omega=\{Q_{h+1,n}\}$ is the set of all (n,p)-order perturbations

$$P'_{h+1,n}=P_{h+1,n}-Q_{h+1,n}\in\Pi(n,p;h+1) \tag{22}$$

with the property that $\omega(s)$ is a common factor of the elements of $P'_{h+1,n}$. If $Q^*_{h+1,n}$ is the minimal norm element of the set Σ_ω, then $\omega(s)$ is referred as an *r-order almost common factor* of $P_{h+1,n}$, and the norm of $Q^*_{h+1,n}$, denoted by $\|Q^*\|$, as the *strength* of $\omega(s)$. If $\omega(s)$ is the GCD of

$$P^*_{h+1,n}=P_{h+1,n}-Q^*_{h+1,n} \tag{23}$$

then $\omega(s)$ will be called an *r-order almost GCD* of $P_{h+1,n}$ with strength $\|Q^*\|$. A polynomial $\hat{\omega}(s)$ of degree r for which the strength $\|Q^*\|$ is a global minimum will be called the *r-order optimal almost GCD* (OA-GCD) of $P_{h+1,n}$. ■

The above definition suggests that any polynomial $\omega(s)$ may be considered as an "approximate GCD", as long as $\deg\omega(s)\leq p$. Important issues in the definition of approximate (optimal approximate) GCD are the parameterisation of the Σ_ω set, the definition of an appropriate metric for $Q_{h+1,n}$ and the solution of the optimization problem to define $Q^*_{h+1,n}$. The set of all resultants corresponding to $\Pi(n,p;h+1)$ set, will be denoted by $\Psi(n,p;h+1)$.

Remark 4: If $P_{h+1,n}$, $Q_{h+1,n}$, $P'_{h+1,n}\in\Pi(n,p;h+1)$ are sets of polynomials and S_P, S_Q, $\bar{S}'_P$ denote their generalised resultants, then these resultants are elements of $\Psi(n,p;h+1)$ then $S'_P=S_P-S_Q$. ■

Theorem 7: Let $P_{h+1,n}\in\Pi(n,p;h+1)$ be a set, $S_P\in\Psi(n,p;h+1)$ be the corresponding generalized resultant and let $\upsilon(s)\in\mathbb{R}[s]$, $\deg\upsilon(s)=r\leq p$, $\upsilon(0)\neq 0$. Any perturbation set $Q_{h+1,n}\in\Pi(n,p;h+1)$, i.e. $P'_{h+1,n}=P_{h+1,n}-Q_{h+1,n}$, which has $\upsilon(s)$ as common divisor, has a generalized resultant $S_Q\in\Psi(n,p;h+1)$ that is expressed as

$$S_Q=S_P-\bar{S}_{P^*}^{(r)}\hat{\Phi}_\upsilon=\left[\begin{array}{c|c}0_r & \bar{S}_{P^*}\end{array}\right]\hat{\Phi}_\upsilon \tag{24}$$

where $\hat{\Phi}_\upsilon$ is the Toeplitz representation of $\upsilon(s)$ and $\bar{S}_{P^*}\in\mathbb{R}^{(p+hn)\times(n+p-r)}$ the (n,p)-expanded resultant of a $P^*\in\Pi(n-r,p-r;h+1)$. Furthermore, if the parameters of $\bar{S}_{P^*}$ are such that $\bar{S}_{P^*}$ has full rank, then $\upsilon(s)$ is a GCD of set $P'_{h+1,n}$. ■

Remark 5: The result provides a parameterisation of all perturbations $Q_{h+1,n}\in\Pi(n,p;h+1)$ which yield sets $P'_{h+1,n}$ having a GCD with degree at least r and divided by the given polynomial $\upsilon(s)$. The free parameters are the coefficients of the$P^*_{h+1,n-r}\in\Pi(n-r,p-r;h+1)$ set of polynomials. For a set of parameters, $\upsilon(s)$ is a divisor of $P'_{h+1,n}$; for generic sets, $\upsilon(s)$ is a GCD of $P'_{h+1,n}$. ■

The evaluation of strength of "approximate GCD" has to relate to the coefficients of the polynomials and the Frobenius norm is an appropriate choice.

Corollary 1: Let $P_{h+1,n}\in\Pi(n,p;h+1)$ and $\upsilon(s)\in\mathbb{R}[s]$, $\deg\upsilon(s)=r\leq p$. The polynomial $\upsilon(s),\upsilon(0)\neq 0$ is an r-order almost common divisor of $P_{h+1,n}$ and its strength is defined as a solution of the following minimization problem:

$$f(P,P^*) = \min_{\forall P^*} \left\| S_P - \left[\begin{array}{c|c} 0_r & \bar{S}_{P^*} \end{array} \right] \hat{\Phi}_\upsilon \right\|_F \qquad (25)$$

where $P^* \in \Pi(n,p;h+1)$. Furthermore $\upsilon(s)$ is an r-order almost GCD of $P_{h+1,n}$ if the minimal corresponds to a coprime set P^* or to full rank S_{P^*}. ■

The optimization problem defining the strength of any order approximate GCD is now used to investigate the "best" amongst all approximate GCDs of a degree r. We consider polynomials $\upsilon(s)$, $\upsilon(0) \neq 0$.

Optimisation Problem [1]: This can be expressed as

$$\begin{aligned} f_1(P,P^*) &\triangleq \|\hat{\Phi}_\upsilon\|_F \cdot f(P,P^*) & (26) \\ &= \min_{\forall P^*} \{ \|S_P - \left[\begin{array}{c|c} 0_r & \bar{S}_{P^*} \end{array} \right] \hat{\Phi}_\upsilon \|_F \cdot \|\Phi_\upsilon\|_F \} & (27) \\ &= \min_{\forall P^*} \|S_P \Phi_\upsilon - \left[\begin{array}{c|c} 0_r & \bar{S}_{P^*} \end{array} \right] \|_F & (28) \end{aligned}$$

where P, Φ_υ have the structure defined by $\upsilon(s)$ of degree r.

Theorem 8 [1]: Consider the set of polynomials $P \in \Pi(n,p;h+1)$ and S_P be its Sylvester matrix. Then,

1) For a certain approximate GCD $\upsilon(s)$ of degree k, the perturbed set $\tilde{P}$ corresponding to minimal perturbation applied on P, such that $\upsilon(s)$ becomes an exact GCD, is defined by:

$$S_{\tilde{P}} = \tilde{S}'_P \hat{\Phi}_\upsilon = \left[\begin{array}{c|c} 0_k & \hat{S}^2_P \end{array} \right] \hat{\Phi}_\upsilon \qquad (29)$$

2) The strength of an arbitrary $\upsilon(s)$ of degree k is $f(P,P^*) = \min_{\forall P^*} \left\| \tilde{S}'_P \Phi_\upsilon \right\|_F$.
3) The *optimal approximate GCD* of degree k is a $\varphi(s)$ defined by solving $f(P,P^*) = \min_{\forall P^* \ \deg \varphi(s)=k} \left\{ \left\| \tilde{S}'_P \Phi_\varphi \right\|_F \right\}$.

■

The optimization problem defined in the above Theorem is non-convex. Computational algorithms for for calculating the optimal approximate GCD are currently under investigation.

V. GRASSMANN INVARIANTS, APPROXIMATE ZERO POLYNOMIALS AND DISTANCE PROBLEMS

The characterisation of the "approximate GCD" and its "optimal" version provides the means to define the respective approximate zero polynomials for different classes of linear systems properties, which cover the cases: (a) *Approximate zero polynomial* based on $\underline{n}(s)\wedge$; (b) *Approximate input decoupling zero polynomial* based on $\underline{b}(s)\wedge$; (c) *Approximate invariant input decoupling polynomial* based on $\underline{r}(s)\wedge$; (d) *Approximate output decoupling polynomial* based on $\underline{c}(s)\wedge$; (e) *Approximate invariant output decoupling polynomial* based on $\underline{q}(s)\wedge$.

Note that such polynomial multi-vectors have to satisfy the corresponding set of QPRs and this makes the computation of the approximate polynomials a more difficult problem. We shall develop the results for the case of a general polynomial matrix.

Corollary 2: Let $\Pi(n,p;h+1)$ be the set of all polynomial sets $P_{h+1,n}$ with $h+1$ elements and with the two higher degrees (n,p), $n \geq p$ and let S_P be the Sylvester resultant of the general set $P_{h+1,n}$. The variety of P^{N-1} which characterise all sets $P_{h+1,n}$ having a GCD with degree d, $0 < d \leq p$ is defined by the set of equations $C_{n+p-d+1}(S_P) = 0$. ■

The above defines a variety $\Delta_d(n,p;h+1)$ described by the polynomial equations in the coefficients of the vector $\underline{p}_{h+1,n}$, or the point $P_{h+1,n}$ of P^{N-1}, and will be called the *d-GCD variety* of P^{N-1}. This characterises all sets in $Pi(n,p;h+1)$ with a GCD of degree d. The definition of the the "optimal GCD" is thus a problem of finding the distance of a given set $P_{h+1,n}$ from the variety $\Delta_d(n,p;h+1)$. For any $P_{h+1,n} \in \Pi(n,p;h+1)$ this distance is defined by

$$d(P,\Delta) = \min_{\forall P^*,\varphi} \left\| S_P - \left[\begin{array}{c|c} 0_k & \bar{S}_{P^*} \end{array} \right] \hat{\Phi}_\varphi \right\|_F \qquad (30)$$

$\varphi(s) \in \mathbb{R}[s]$, $P^* \in \Pi(n-k,p-k;h+1)$, $\deg \varphi(s) = k$, the k-distance of $P_{h+1,n}$ from the the k-GCD variety $\Delta_k(n,p;h+1)$ and $\tilde{\varphi}(s)$ emerges as a solution to an optimisation problem and it is the *k-optimal approximate GCD* and the value $d(P,\Delta)$ is its *k-strenght*. For polynomial matrices we can extend the scalar definition of the approximate GCD as follows:

Definition 2: Consider the coprime polynomial matrix $T(s) \in \mathbb{R}^{q\times r}[s]$ and let $\Delta T(s) \in \mathbb{R}^{q\times r}[s]$ be an arbitrary matrix such that

$$T(s) + \Delta T(s) = \widehat{T}(s) = \widetilde{T}(s) R(s) \qquad (31)$$

where $R(s) \in \mathbb{R}^{r\times r}[s]$. Then $R(s)$ will be called an *approximate matrix divisor* of $T(s)$. ■

The above definition may be interpreted using exterior products as an extension of the problem defined for polynomial vector sets. The difference between general sets of vectors and those generated from polynomial matrices by taking exterior products is that the latter must satisfy the decomposability conditions [3] and in turn they define another variety of the Grassmann type.

Consider now the set of polynomial vectors $\Pi(n,p;h+1)$ and let $\Pi^\wedge(n,p;h+1)$ be its subset of the decomposable polynomial vectors $\underline{p}(s) \in \mathbb{R}^\sigma[s]$, which correspond to the $q\times r$ polynomial matrices with degree n. The set $\Pi^\wedge(n,p;h+1)$ is defined as the Grassmann variety $G(q,r;\mathbb{R}[s])$ of the projective space $P^{\sigma-1}(\mathbb{R}[s])$. The way we can extend the scalar results is based on:

(i) Parameterise the perturbations that move a general set $P_{\sigma,n}$, to a set $P'_{\sigma,n} = P_{\sigma,n} + Q_{\sigma,n} \in \Delta_k(n,p;\sigma)$ where initially $Q_{\sigma,n}$ and $P'_{\sigma,n}$ are free.

(ii) For the scalar results to be transferred back to the polynomial matrices the sets $P'_{\sigma,n}$ have to be decomposable multi-vectors which are denoted by $\Pi^\wedge(n,p;\sigma)$. The latter set will be referred to as the *n-order subset* of the Grassmann variety $G(q,r;\mathbb{R}[s])$ and the sets $P'_{\sigma,n}$ must be such that

$$P'_{\sigma,n} \in \Pi(n,p;\sigma) \bigcap \Delta_k(n,p;\sigma) = \Delta_k^\wedge \Pi(n,p;\sigma) \qquad (32)$$

where $\Delta_k^\wedge \Pi(n,p;\sigma)$ is the decomposable subset of $\Delta_k(n,p;\sigma)$. Parameterising all sets $P'_{\sigma,n}$ provides the means for posing a distance problem as before. This is clearly a constrained distance problem since now we have to consider the intersection variety defined by the corresponding set of QPRs and the equations of the GCD variety. Some preliminary results on this problem are stated below:

Lemma 3: The following properties hold true:

1) $\Pi^{\wedge}(n,p;h+1)$ is proper subset $\Pi(n,p;h+1)$ if $r \neq 1$ and $q \neq r-1$.
2) $\Pi^{\wedge}(n,p;h+1) = \Pi(n,p;h+1)$ if either $r=1$ or $q=r-1$.
3) The set $\Delta_k^{\wedge}\Pi(n,p;\sigma)$ is always nonempty. ■

The result is a direct implication of the decomposability conditions for multivectors [3].

Theorem 9: Let $P_{\sigma,n} \in \Pi^{\wedge}(n,p;\sigma)$ and denote by $d(P,\Delta_k)$, $d(P,\Delta_k^{\wedge})$ the distance from $\Delta_k(n,p;\sigma)$ and $\Delta_k^{\wedge}(n,p;\sigma)$ respectively. The following hold true:

1) If $q=r-1$ or $r=1$, then the solutions of the two optimisation problems are identical and $d(P,\Delta_k) = d(P,\Delta_k^{\wedge})$.
2) If $q \neq r-1$ and $r \neq 1$, then $d(P,\Delta_k) \leq d(P,\Delta_k^{\wedge})$. ■

Remark 6: For polynomial matrices this distance problem is defined on the set $P_{h+1,n}$ of $\Pi(n,p;h+1)$ from the intersection of the varieties $\Delta_d(n,p;h+1)$ and $G(q,r;\mathbb{R}[s])$. ■

The above suggests that the Grassmann distance problem has to be considered only when $q \neq r-1$ and $r \neq 1$. The Grassmann distance problem requires the study of some additional topics linked to algebraic geometry and exterior algebra such as: **(i)** Parameterisation of all decomposable sets P with a fixed order n; **(ii)** Characterisation of the set $\Delta_k^{\wedge}(n,p;\sigma)$ and its properties. For the special case $r=1$, $q=r-1$ the distance $d(P,\Delta_k)$ is reduced to that of the polynomial vector case since we guarantee decomposability.

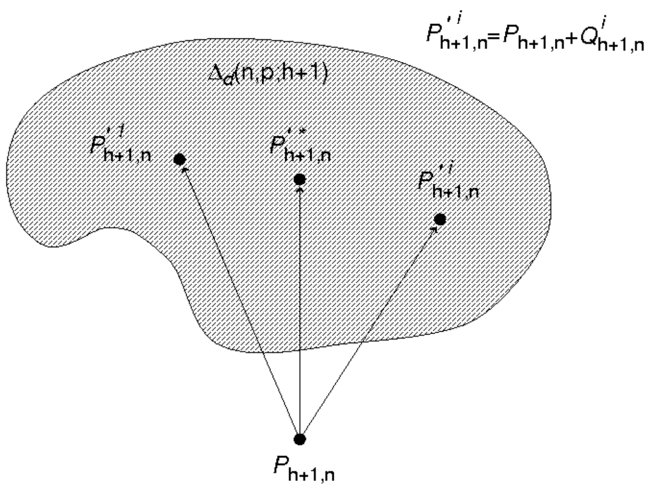

Fig. 1. *The notion of "approximate GCD"*

VI. CONCLUSIONS

The paper uses the recently introduced notion of "approximate GCD" of a set of polynomials [1] and the characterization of controllability and observability properties in terms of exterior products and associated Plücker matrices of controllability and observability pencils [9] to define distance from the set of uncontrollable, unobservable systems, as well as the corresponding approximate decoupling polynomials; furthermore, the use of the restriction pencils $R(s)$ and $Q(s)$ allows the definition of the distance of the state feedback, output injection orbits from uncontrollable, unobservable families respectively. The main distinctive feature of the approach, is the definition of distance of the orbits of systems from the uncontrollable, unobservable sets, as well as the definition of the approximate decoupling polynomials. The paper also extends the notion of approximate GCD of a set of polynomials to the case of *approximate matrix divisors*. It has been shown that this problem is equivalent to a distance problem from the intersection of two varieties and it is much harder than the polynomial vectors case. Our approach is based on the optimal approximate GCD and when this is applied to linear systems introduces new system invariants with significance in defining system properties under parameter variations on the corresponding model. The optimization problem is non-convex and developing methodology for computing this distance is a problem of current research.

REFERENCES

[1] N. Karcanias, S. Fatouros, M. Mitrouli and G. Halikias, Approximate greatest common divisor of many polynomials, generalised resultants, and strength of approximation, *Comput & Maths with Applications*, vol. 51(12), 2006, pp 1817-1830.

[2] N. Karcanias, C. Giannakopoulos and M. Hubbard, Almost zeros of a set of polynomials of $\mathbb{R}[s]$, *Int. J. Control*, vol. 38(6), 1983, pp 1213-1238.

[3] M. Marcus, *Finite dimensional multilinear algebra (in two parts)*, Marcel Deker, New York, 1973.

[4] N. Karcanias and C. Giannakopoulos, On Grassmann invariants and almost zeros of linear systems and the determinantal zero, pole assignment problem, *Int. J. Control*, vol 40, 1984.

[5] N. Karcanias and C. Giannakopoulos, Necessary and Sufficient Conditions for Zero Assignment by Constant Squaring Down, *Linear Algebra and Its Applications, Special Issue on Control Theory*, vol. 122/123/124, pp 415-446, 1989.

[6] H.H. Rosenbrock, *State Space and Multivariable Theory*, Nelson, London, 1970.

[7] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.

[8] G.D. Forney, Minimal Bases of Rational Vector Spaces with Applications to Multivariable Systems, *SIAM J. Control*, vol. 13, pp 493-520, 1975.

[9] N. Karcanias and J. Leventides, Grassman Invariants, Matrix Pencils and Linear System Properties, *Linear Algebra & Its Applications*, vol. 241-243, pp 705-731, 1996.

[10] J. Leventides and N. Karcanias N, Structured Squaring Down and Zero Assignment, *Int. J. Control*, vol. 81, pp 294-306, 2008.

[11] S. Barnett, *Matrices Methods and Applications*, Clarendon Press, Oxford, 1990.

[12] S. Fatouros and N. Karcanias, Resultant Properties of GCD of polynom and a Factorisation Represent of GCD, *Int. J. of Control*, vol. 76, pp 1666-1683, 2003.

[13] S. Jaffe and N. Karcanias, Matrix pencil characterisation of almost (A,B)-invariant subspaces: A classification of geometric concepts, *Int. J. Control*, vol. 33, pp 51-93, 1981.

[14] N. Karcanias and P. Macbean, Structural invariants and canonical forms of linear multivariable systems, *3rd IMA Intern Conf. On Control Theory*, Academic Press, pp 257-282, 1981.

[15] D.L. Boley and W.S. Lu, Measuring how far a controllable system is from an uncontrollable one, *IEEE Tran. Auto. Contr.*, vol AC-31(3), pp 249-251, 1986.