2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Embedded optimal control problems

Nikolaj Nordkvist, Peter E. Crouch, Anthony M. Bloch, and Amit K. Sanyal

***Abstract*— In this paper we define a class of optimal control problems which we denote "embedded optimal control problems". These are not true optimal control problems since the control system is not locally controllable on the manifold on which it is defined. Despite this, they allow for a well defined associated optimal control problem which does not admit abnormal extremals. We apply Pontryagin's maximum principle to the embedded optimal control problem to derive the generating differential equations for the normal and abnormal extremals. We show that the normal extremal generating equations in a sense contain the extremal generating equations for the associated optimal control problem. We show that this is not the case for the abnormal extremal generating equations. This has applications to the study of the optimal control of systems constrained to a given submanifold of a configuration space, for example the sphere or hypersphere. We apply the theory to three examples in order to illustrate its applicability and to show how it relates to well known results.**

I. INTRODUCTION

This paper studies so-called embedded optimal control problems. These problems are not truly optimal control problems, in the sense that the control system is not locally controllable on the manifold on which the problem is defined. Treating the problem as if it is an optimal control problem nonetheless we apply Pontryagin's maximum principle and thereby we obtain generating equations for the normal and abnormal extremals. We show that the normal extremal generating equations in fact contain the extremal generating equations for an associated optimal control problem, which does not admit abnormal extremals, and that the abnormal extremal generating equations do not provide such an interpretation. This has several possible applications. For one it provides an alternative method to analyze true optimal control problems, an additional task then being to construct an embedded optimal such that an associated optimal control problem coincides with the optimal control problem under consideration. If the space on which the embedded optimal control problem is defined is linear (for example $\mathbb{R}^3$), but the associated optimal control problem is on a nonlinear space (for example the sphere S^2), then the embedded optimal control problem will be simpler to analyze, globally and locally, than the associated optimal control problem.

This paper is an extension of our earlier work [1], [2], [3], [4], the first paper introducing the so-called "symmetric representation", which in turn is motivated by the Euler equations for an n-dimensional rigid body as treated in, for example, [5], [6], [7], [8]. The symplectic subflows of the symmetric representation are studied in detail in [4]. In [9] an analogous variational analysis was carried out for Stiefel manifolds, which in the extreme cases give the geodesic curves on an ellipsoid and the Euler equations on $SO(n)$, respectively. Euler's fluid equations for incompressible inviscid flow were obtained in [10], [11] as the solutions to an optimal control problem and a representation of the equations resembling the symmetric representation was obtained. In the recent article [12] a generalization of many of these problems was obtained by formulating them as a Clebsch optimal control problem. A discretization of the Clebsch optimal control problem is analyzed in [13] and applications as globally defined symplectic integrators for mechanical systems are presented.

The paper is organized as follows. In section II we introduce the embedded optimal control problem and discuss how the equations generating the normal extremals contain the extremal generating equations for an associated optimal control problem. Section III presents three examples of embedded optimal control problems and a detailed analysis of their normal extremals. Finally, in section IV, we summarize the results in a conclusion and discuss future directions of this work.

II. RESULTS ON EMBEDDED OPTIMAL CONTROL PROBLEMS

Let Q denote an n-dimensional manifold. Let $\mathfrak{X}(Q)$ denote the space of smooth vector fields on Q. Consider $D \subset \mathfrak{X}(Q)$ and $\mathcal{D} \subset TQ$ defined at every $q \in Q$ by

$$\mathcal{D}(q) = \operatorname{span}\{X(q),\ X \in D\}.$$

This is an example of a generalized distribution; if the rank of $\mathcal{D}$ is constant on Q it is a distribution in the classical sense.

Consider the control system

$$\dot{q} = \sum_{i=1}^{m} X_i(q)u_i, \tag{1}$$

where $q \in Q$, $u = (u_1, \ldots, u_m) \in U$ with $U \subset \mathbb{R}^m$ an open neighborhood of $0 \in \mathbb{R}^m$. Let N denote an invariant submanifold of (1) and let $\ell : Q \times U \to \mathbb{R}$ be a cost function. For each choice of invariant submanifold $N \subsetneq Q$ we introduce the concept of an embedded optimal control problem as follows:

Nikolaj Nordkvist, Mathematics and Sciences Division, Leeward Community College, Pearl City, HI 96782 nikolaj@hawaii.edu
Peter E. Crouch, Electrical Engineering, University of Hawai'i at Mānoa, Honolulu, HI 96822 pcrouch@hawaii.edu
Anthony M. Bloch, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 abloch@umich.edu
Amit K. Sanyal, Mechanical and Aerospace Engineering, New Mexico State University, Las Cruces, NM 88003 asanyal@nmsu.edu

Problem A (Embedded optimal control problem): Minimize

$$\int_0^T \ell\big(q(t), u(t)\big)\mathrm{d}t,$$

subject to $\dot{q} = \sum_{i=1}^m X_i(q)u_i$, $q \in Q$, $u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m$, and with fixed endpoints $q(0) = q_0 \in N$ and $q(T) = q_T \in N$.

The embedded optimal control problem is well posed when (1) restricted to N is accessible from q_0. When this is the case we can impose on q_T that it lies in a small neighborhood of q_0 and thus belongs to the set of states reachable from q_0 in time T. The associated optimal control problem is now given by:

Problem B (Associated optimal control problem): Minimize

$$\int_0^T \ell\big(q(t), u(t)\big)\mathrm{d}t,$$

subject to $\dot{q} = \sum_{i=1}^m X_i|_N(q)u_i$, $q \in N$, $u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m$, and with fixed endpoints $q(0) = q_0 \in N$ and $q(T) = q_T \in N$.

If we require that $\dim(\mathrm{span}\{X_1(q_0), \dots, X_m(q_0)\}) = m$ or $\dim(\mathrm{span}\{X_1(q_T), \dots, X_m(q_T)\}) = m$, then the associated optimal control problem does not admit abnormal extremals. In what follows we make the standing assumption that q_0 and q_T satisfy this constraint. We do this to reduce the complexity by eliminating the possibility of abnormal extremals for the associated optimal control problem.

If $\dim(N) = m$, $\dim(\mathrm{span}\{X_1(q), \dots, X_m(q)\}) = m$ for all $q \in N$, and $U = \mathbb{R}^m$ we can construct a cost function $L : TN \to \mathbb{R}$ satisfying $L(q, \dot{q}) = \ell(q, u)$, with the identification $\dot{q} = \sum_{i=1}^m X_i(q)u_i$, so the associated optimal control problem is in this case equivalent to a variational problem, and vice versa.

When $N \subset Q$ is an embedded submanifold of Q the inclusion $i : N \hookrightarrow Q$ is an embedding. The pullback bundle $i^*(T^*Q)$ is defined as the vector bundle over N whose fiber over $n \in N$ is given by $T^*_{i(n)}Q$. Since $T^*_n i : T^*_{i(n)}Q \to T^*_n N$, the dual of the tangent map of i is globally defined when restricted to $i^*(T^*Q)$. Furthermore since $i : N \hookrightarrow Q$ is an embedding we have that $T_n i : T_n N \to T_{i(n)}Q$ is injective for all $n \in N$ and therefore $T^*_n i : T^*_{i(n)}Q \to T^*_n N$ is surjective; that is $T^*i|_{i^*(T^*Q)}$ is surjective on fibers. When applying Pontryagin's maximum principle to the embedded optimal control problem the pullback bundle $i^*(T^*Q) \subset T^*Q$ must be invariant under the resulting Hamilton's equations due to the fact that N is a an invariant manifold of (1).

Consider two manifolds M_A and M_B with dynamical systems

$$\begin{aligned} \dot{x} &= f_A(x), && x \in M_A, && \text{system A,} \\ \dot{y} &= f_B(y), && y \in M_B, && \text{system B.} \end{aligned}$$

System B is called a homomorphic image of system A if there exists a differentiable surjective map $\Psi : M_A \to M_B$, called a homomorphism, such that

$$T\Psi \circ f_A = f_B \circ \Psi.$$

Clearly this means that if $x(t)$ is a solution to system A with initial condition $x(0) = x_0$ then $\Psi\big(x(t)\big)$ is a solution to system B with initial condition $y(0) = \Psi(x_0)$; all solutions to system B can be generated from solutions to system A by this procedure. The following proposition shows that the normal extremal generating equations for the embedded optimal control problem are related to the extremal generating equations for the associated optimal control problem via such a homomorphism:

Proposition 1: Assume that N is an embedded submanifold of Q and that $u \mapsto \frac{\partial \ell}{\partial u}(q, u)$ is a diffeomorphism for all $q \in Q$. The *normal* and *abnormal extremal* generating equations on T^*Q, as given by Hamilton's equations prescribed by Pontryagin's maximum principle for the embedded optimal control problem, have $i^*(T^*Q) \subset T^*Q$ as an invariant manifold. The *extremal* generating equations on T^*N, as given by Hamilton's equations prescribed by Pontryagin's maximum principle for the associated optimal control problem, are the homomorphic image of the normal extremal generating equations on T^*Q (restricted to $i^*(T^*Q)$,) as given by Hamilton's equations prescribed by Pontryagin's maximum principle for the embedded optimal control problem. The map $T^*i|_{i^*(T^*Q)} : i^*(T^*Q) \to T^*N$ provides a homomorphism.

Proof: That $i^*(T^*Q) \subset T^*Q$ is invariant follows trivially from the fact that N is a an invariant manifold of (1).

If N is an embedded submanifold of Q this means that locally there exists a preferred coordinate system $\big(V, \phi = (x, y)\big)$, where V is an open subset of Q, satisfying $\phi|_{V \cap N} : V \cap N \to \mathbb{R}^n \times \{0\}$, where n is the dimension of N. We denote by (x, y, p_x, p_y) the local coordinates on T^*V induced by the preferred coordinates $\phi = (x, y)$; that is, if $\alpha \in T^*_q V$ then $p_{x_j} = \alpha\left(\frac{\partial}{\partial x_j}\right)$ and $p_{y_k} = \alpha\left(\frac{\partial}{\partial y_k}\right)$. Locally $i^*(T^*Q)$ is described by the coordinates (x, p_x, p_y) and the map $T^*i|_{i^*(T^*Q)} : i^*(T^*Q) \to T^*N$ is in these coordinates given by $(x, p_x, p_y) \mapsto (x, p_x)$.

In a preferred coordinate system $\big(V, \phi = (x, y)\big)$ we have that $\dot{q} = \sum_{i=1}^m X_i(q)u_i$ is given by

$$\dot{x} = \sum_{i=1}^m \tilde{X}_i(x, y)u_i, \qquad \dot{y} = \sum_{i=1}^m \tilde{Y}_i(x, y)u_i, \tag{2}$$

where $\tilde{Y}_i(x, 0) = 0$. According to Pontryagin's maximum principle the Hamiltonian for the extremals of the embedded optimal control problem is

$$\begin{gathered} \hat{H}(x, y, p_x, p_y, u) = \\ \left\langle (p_x, p_y), \Big(\sum_{i=1}^m \tilde{X}_i(x, y)u_i, \sum_{i=1}^m \tilde{Y}_i(x, y)u_i \Big) \right\rangle \\ -p_0 \ell(x, y, u), \end{gathered} \tag{3}$$

and the extremal generating control u^* satisfies

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\hat{H}(x,y,p_x,p_y,u^*+sv)=$$
$$\left\langle(p_x,p_y),\Big(\sum_{i=1}^m\tilde{X}_i(x,y)v_i,\sum_{i=1}^m\tilde{Y}_i(x,y)v_i\Big)\right\rangle$$
$$-p_0\left\langle\frac{\partial\ell}{\partial u}(x,y,u^*),v\right\rangle=0, \tag{4}$$

for all $v\in U$, where $\langle\cdot,\cdot\rangle$ is the usual dot product. The constant p_0 is $p_0=1$ (in general it just needs to be nonzero) for normal extremals and $p_0=0$ for abnormal extremals. According to Pontryagin's maximum principle the normal extremals of the embedded optimal control problem on Q locally satisfy

$$\begin{aligned}
\dot{x}=&\frac{\partial H^*}{\partial p_x}=\sum_{i=1}^m X_i(x,y)u_i^*,\\
\dot{y}=&\frac{\partial H^*}{\partial p_y}=\sum_{i=1}^m \tilde{Y}_i(x,y)u_i^*,\\
\dot{p}_x=&-\frac{\partial H^*}{\partial x}\\
=&-\left\langle(p_x,p_y),\Big(\sum_{i=1}^m\frac{\partial\tilde{X}_i}{\partial x}(x,y)u_i^*,\sum_{i=1}^m\frac{\partial\tilde{Y}_i}{\partial x}(x,y)u_i^*\Big)\right\rangle\\
&+\frac{\partial\ell}{\partial x}(x,y,u^*),\\
\dot{p}_y=&-\frac{\partial H^*}{\partial y},
\end{aligned}$$

where $H^*(x,y,p_x,p_y)=\hat{H}(x,y,p_x,p_y,u^*)$. If $q(t)\in N$ (and thus $y(t)=0$) the optimal control u^* satisfies

$$\begin{aligned}
0=&\left\langle(p_x(t),p_y(t)),\Big(\sum_{i=1}^m\tilde{X}_i(x(t),0)v_i,\sum_{i=1}^m\tilde{Y}_i(x(t),0)v_i\Big)\right\rangle\\
&-\left\langle\frac{\partial\ell}{\partial u}(x(t),0,u^*),v\right\rangle\\
=&\left\langle(p_x(t),p_y(t)),\Big(\sum_{i=1}^m\tilde{X}_i(x(t),0)v_i,0\Big)\right\rangle\\
&-\left\langle\frac{\partial\ell}{\partial u}(x(t),0,u^*),v\right\rangle\\
=&\left\langle(p_x(t),0),\Big(\sum_{i=1}^m\tilde{X}_i(x(t),0)v_i,0\Big)\right\rangle\\
&-\left\langle\frac{\partial\ell}{\partial u}(x(t),0,u^*),v\right\rangle,
\end{aligned}$$

for all $v\in U$, meaning that u^* is independent of $p_y(t)$. This then gives that for $q(t)\in N$ we have

$$\begin{aligned}
\dot{p}_x(t)=&-\left\langle(p_x(t),p_y(t)),\Big(\sum_{i=1}^m\frac{\partial\tilde{X}_i}{\partial x}(x(t),0)u_i^*,0\Big)\right\rangle\\
&+\frac{\partial\ell}{\partial x}(x(t),0,u^*)\\
=&-\left\langle(p_x(t),0),\Big(\sum_{i=1}^m\frac{\partial\tilde{X}_i}{\partial x}(x(t),0)u_i^*,0\Big)\right\rangle\\
&+\frac{\partial\ell}{\partial x}(x(t),0,u^*),
\end{aligned}$$

and since u^* is independent of $p_y(t)$ we thus see that in the normal extremal generating equations $(x(t),p_x(t))$ evolve independently of $p_y(t)$. The equations that $(x(t),p_x(t))$ satisfy coincide with those giving the extremals for the associated optimal control problem on N (which does not admit abnormal extremals as this is a fully actuated optimal control problem). ■

Proposition 1 shows how the extremal generating equations for the associated optimal control problem are contained in the normal extremal generating equations for the embedded optimal control problem. Thus the normal extremal generating equations for the embedded optimal control problem can be derived and analyzed instead of the extremal generating equations for the associated optimal control problem. If the space Q is linear but the space N is nonlinear this approach will be much simpler to carry out from a global perspective. If $\mathcal{D}$ is completely integrable then the extremal generating equations for the embedded optimal control problem give the solution to any associated optimal control problem for N being a maximal integral manifold of $\mathcal{D}$. This means that the embedded optimal control problem provides a foliation of solutions to all associated optimal control problems on leaves of $\mathcal{D}$.

The following proposition deals with the abnormal extremal generating equations for the embedded optimal control problem:

Proposition 2: Assume that N is an embedded submanifold of Q. Furthermore assume that $u\mapsto\frac{\partial\ell}{\partial u}(q,u)$ is a diffeomorphism for all $q\in Q$ and that $\frac{\partial\ell}{\partial u}(q,0)=0$. Let $\alpha(t)\in i^*(T^*Q)$ be a solution to the *abnormal extremal* generating equations as given by Hamilton's equations as prescribed by Pontryagin's maximum principle for the embedded optimal control problem. Then $\beta(t)=T^*i\big(\alpha(t)\big)\in T^*N$ is not a solution to the *extremal* generating equations, as given by Hamilton's equations prescribed by Pontryagin's maximum principle for the associated optimal control problem, unless trivially $\beta(t)=\big(x(t),p_x(t)\big)=(x_0,0)$.

Proof: In a preferred coordinate system $\big(V,\phi=(x,y)\big)$ we have that $\dot{q}=\sum_{i=1}^m X_i(q)u_i$ is given by (2) for $\tilde{Y}_i(x,0)=0$. According to Pontryagin's maximum principle the Hamiltonian for the abnormal extremals of the embedded optimal control problem is given by (3) with $p_0=0$ and the abnormal extremal generating control u^* satisfies (4). If $q(t)\in N$ (and thus $y(t)=0$) the optimal control u^* is seen to solve

$$0 = \left\langle (p_x(t), p_y(t)), \Big(\sum_{i=1}^{m} \tilde{X}_i(x(t),0)v_i, 0 \Big) \right\rangle$$
$$= \left\langle (p_x(t), 0), \Big(\sum_{i=1}^{m} \tilde{X}_i(x(t),0)v_i, 0 \Big) \right\rangle,$$

for all $v \in U$ which is only satisfied for $p_x(t) = 0$.

The Hamiltonian giving the extremals for the associated optimal control problem is

$$\hat{H}(x, p_x, u) = \left\langle p_x, \sum_{i=1}^{m} \tilde{X}_i(x,0)u_i \right\rangle - \ell(x,0,u).$$

The extremal generating control u^* thus satisfies

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \hat{H}(x, p_x, u^* + sv) = \left\langle p_x, \sum_{i=1}^{m} \tilde{X}_i(x,0)v_i \right\rangle - \left\langle \frac{\partial \ell}{\partial u}(x,0,u^*), v \right\rangle = 0,$$

for all $v \in U$. Thus if $p_x = 0$ the assumption gives that $u^* = 0$ which in return gives $x(t)$ constant. ■

Proposition 2 shows that the abnormal extemal generating equations for the embedded optimal control problem are not interesting in the sense that they do not contain any information about the extremals for the associated optimal control problem.

Let Φ be an action of a Lie group G on Q. The infinitesimal generator of the action Φ is the vector field on Q defined by

$$u_Q(q) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(tu)}(q), \quad q \in Q,$$

for $u \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. A *Clebsch optimal control problem*, see [12], is defined as an embedded optimal control problem with control system

$$\dot{q} = u_Q(q), \quad u \in \mathfrak{g}$$

and cost function $\ell(q,u) = \ell(u)$. Since a group orbit $\mathrm{orb}(q_0)$ is invariant under the flow of this system N can be chosen as any group orbit. The equations generating the normal extremals for Clebsch optimal control problems can despite the very general framework be described in closed form which involve the cotangent lift of Φ, the momentum map, and the functional derivative of ℓ. Since these details are beyond the focus of this paper we refer to [12] for a thorough analysis.

III. Examples

In this section we treat three different embedded optimal control problems in detail. First we discuss an embedded optimal control problem with $Q = \mathfrak{gl}(n)$ and $N = G$ a matrix Lie group and then we analyze two different embedded optimal control problems with $Q = \mathbb{R}^3$ and $N = S^2$. The first two problems can be expressed as Clebsch optimal control problems, which is not the case for the last problem.

A. $Q = \mathfrak{gl}(n)$ and $N = G \subset Gl(n)$ a matrix Lie group

Consider the control system

$$\dot{q} = qu, \tag{5}$$

where $q \in Q = \mathfrak{gl}(n)$ and $u \in \mathfrak{g}$, $\mathfrak{g} \subset \mathfrak{gl}(n)$ being the Lie algebra of a matrix Lie group $G \subset Gl(n)$. Then $N = G$ is an invariant manifold and $q\mathfrak{g} = T_qG$ for all $q \in G$ so the system (5) is accessible on G. For this system consider the cost function

$$\ell(q,u) = \langle u, \Sigma(u) \rangle,$$

where $\langle A, B \rangle = \mathrm{tr}(A^T B)$ is the trace inner product on $\mathfrak{gl}(n)$ and Σ is a positive definite self-adjoint operator. We notice that this is a Clebsch optimal control problem with the action being right matrix multiplication.

To find the normal extremal generating equations for this embedded optimal control problem we apply Pontryagin's maximum principle. We define the Hamiltonian $H : \mathfrak{gl}(n) \times \mathfrak{gl}(n) \times \mathfrak{g} \to \mathbb{R}$ as

$$H(q,p,u) = \langle p, qu \rangle - \tfrac{1}{2} \langle u, \Sigma(u) \rangle = \left\langle \mathbb{P}(q^T p), u \right\rangle - \tfrac{1}{2} \langle u, \Sigma(u) \rangle,$$

where $p \in \mathfrak{gl}(n)$ is a costate and $\mathbb{P} : \mathfrak{gl}(n) \to \mathfrak{g}$ is the orthogonal projection onto $\mathfrak{g}$. The optimizing control u^* must satisfy

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} H(q,p,u^* + s\tilde{u}) = 0, \quad \forall \tilde{u} \in \mathfrak{g},$$

giving

$$u^* = \Sigma^{-1}\big(\mathbb{P}(q^T p)\big).$$

The maximum principle then gives the normal extremals as generated by Hamilton's equations with Hamiltonian $H(q,p,u^*(q,p)) = \frac{1}{2}\left\langle \mathbb{P}(q^T p), \Sigma^{-1}\big(\mathbb{P}(q^T p)\big)\right\rangle$, which are the *symmetric representation:*

$$\dot{q} = qu, \quad \dot{p} = -pu^T, \quad u = \Sigma^{-1}\big(\mathbb{P}(q^T p)\big), \tag{6}$$

where $(q,p) \in \mathfrak{gl}(n) \times \mathfrak{gl}(n)$. We notice that the sets $Gl(n) \times \mathfrak{gl}(n)$, $G \times \mathfrak{gl}(n)$, and $G \times G^T$ are *invariant manifolds*. It is not surprising that $G \times \mathfrak{gl}(n)$ is an invariant manifold as this is the pullback bundle $i^*(T^*Q)$ for this example.

When expressing the control as $u = q^{-1}\dot{q}$ the cost function for the associated optimal control can be rewritten as

$$\ell(q,u) = \langle u, \Sigma(u) \rangle = \left\langle q^{-1}\dot{q}, \Sigma\left(q^{-1}\dot{q}\right) \right\rangle.$$

Therefore the associated optimal control problem is equal to the variational problem on G with the left-invariant Lagrangian

$$L(q,\dot{q}) = \left\langle q^{-1}\dot{q}, \Sigma\left(q^{-1}\dot{q}\right) \right\rangle.$$

Since the Lagrangian is left-invariant the extremals are given by the Euler-Poincaré equations, see, e.g., [1], [14], as

$$\dot{q} = qu, \qquad \dot{M} = \mathrm{ad}_u^*(M), \qquad u = \Sigma^{-1}(M). \tag{7}$$

Letting $\zeta, \xi, \eta \in \mathfrak{g}$ we get

$$\begin{aligned}\left\langle \zeta, \mathrm{ad}^*_\xi(\eta)\right\rangle &= \langle \mathrm{ad}_\xi(\zeta), \eta\rangle = \langle \xi\zeta - \zeta\xi, \eta\rangle \\ &= \left\langle \zeta, \xi^T\eta - \eta\xi^T\right\rangle = \left\langle \zeta, \mathbb{P}([\eta, -\xi^T])\right\rangle.\end{aligned}$$

Therefore $\mathrm{ad}^*_\xi(\eta) = \mathbb{P}([\eta, -\xi^T])$ and (7) can be rewritten as

$$\dot{q} = qu, \qquad \dot{M} = \mathbb{P}([M, -u^T]), \qquad u = \Sigma^{-1}(M).$$

Let $\mathfrak{g}^\perp \subset \mathfrak{gl}(n)$ be the orthogonal subspace to $\mathfrak{g}$ of $\mathfrak{gl}(n)$ with respect to the trace inner product. In general we have

$$\begin{aligned}\langle A, [B, C]\rangle &= \langle A, BC - CB\rangle \\ &= \left\langle [A, -B^T], C\right\rangle.\end{aligned}$$

Letting $A \in \mathfrak{g}^\perp$ and $B, C \in \mathfrak{g}$ be arbitrary we therefore obtain the commutation relation between $\mathfrak{g}^\perp$ and $\mathfrak{g}^T$ as

$$[\mathfrak{g}^\perp, \mathfrak{g}^T] \subset \mathfrak{g}^\perp. \tag{8}$$

Along solutions of the symmetric representation (6) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(q^Tp) &= \mathbb{P}(\dot{q}^Tp + q^T\dot{p}) = \mathbb{P}(u^Tq^Tp - q^Tpu) \\ &= \mathbb{P}([q^Tp, -u^T]) = \mathbb{P}([\mathbb{P}(q^Tp), -u^T]),\end{aligned}$$

where we have used (8) in the last step. Proposition 1 gives that the extremal generating equations for the associated optimal control problem are contained in the normal extremal generating equations for the embedded optimal control problem. The above calculations show this explicitly for this example. In particular the extremal generating equations for the associated optimal control problem and the normal extremal generating equations for the embedded optimal control problem are related by $M = \mathbb{P}(q^Tp)$.

B. $Q = \mathbb{R}^3$ and $N = S^2$

First we consider the control system

$$\dot{q} = u \times q, \tag{9}$$

where $q, u \in \mathbb{R}^3$ and $\times$ is the cross product. For this system we have $\dot{q} \perp q$ and thus $N = S^2$ is an invariant manifold. Since $T_qS^2 = \mathbb{R}^3 \times q$ the control system (9) is seen to be accessible on S^2. We consider the cost function

$$\ell(q, u) = \tfrac{1}{2}(u_1^2 + u_2^2 + u_3^2) = \tfrac{1}{2}u \cdot u.$$

We remark that this is in fact a Clebsch optimal control problem. The action for this is that of left multiplication of $SO(3)$ and an element of $\mathbb{R}^3$ is identified with an element of $\mathfrak{so}(3)$ via $a \mapsto a^\times$, where $a^\times b = a \times b$.

We apply Pontryagin's maximum principle to find the normal extremals of the embedded optimal control problem. The Hamiltonian is

$$\begin{aligned}H(q, p, u) &= p \cdot (u \times q) - \tfrac{1}{2}u \cdot u \\ &= (q \times p) \cdot u - \tfrac{1}{2}u \cdot u,\end{aligned}$$

and the optimal control u^* is calculated as

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} H(q, p, u^* + sv) = 0, \quad \forall v \in \mathbb{R}^3,$$

giving the optimal control as

$$u^* = q \times p.$$

The normal extremal generating equations are Hamilton's equations with Hamiltonian $H(q, p, u^*(q, p)) = \frac{1}{2}(q \times p) \cdot (q \times p)$ which are the differential equations:

$$\dot{q} = u \times q, \qquad \dot{p} = u \times p, \qquad u = q \times p.$$

We notice that that the sets $S^2 \times \mathbb{R}^3$ and $S^2 \times S^2$ are *invariant manifolds*. The pullback bundle $i^*(T^*Q)$ is $S^2 \times \mathbb{R}^3$ for this example and therefore this set is expected to be invariant. Calculating the time derivative of $q \times p$ along solutions of this system gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(q \times p) &= \dot{q} \times p + q \times \dot{p} \\ &= \Big((q \times p) \times q\Big) \times p + q \times \Big((q \times p) \times p\Big) \\ &= -(q \times p) \times (q \times p) = 0,\end{aligned}$$

where we have used the Jacobi identity in the second to last derivation. This shows that the optimal control $u = q \times p$ is constant $u = u_0 = q_0 \times p_0$ for this system. The solutions are thus given by

$$q(t) = \exp(tu_0^\times)q_0, \qquad p(t) = \exp(tu_0^\times)p_0, \tag{10}$$

where $\exp(tu_0^\times)z$ is a rotation of z around u_0 an angle of $t||u_0||$ (this is expressed in terms of the matrix exponential.)

Consider the geodesic problem on S^2. The Lagrangian for this problem is

$$L(q, \dot{q}) = \tfrac{1}{2}\dot{q} \cdot \dot{q},$$

for $q \in S^2$. Since $(u \times q) \cdot (u \times q) \le u \cdot u$ and $(u \times q) \cdot (u \times q) = u \cdot u$ if and only if $u \perp q$ we have that the associated optimal control problem of the embedded optimal control problem above is equivalent to the geodesic problem. This is the reason why the solutions to the embedded optimal control problem as given by (10) indeed are geodesic curves on S^2.

Consider instead the control system

$$\dot{q} = e_1 \times qu_1 + e_2 \times qu_2, \tag{11}$$

where $q \in \mathbb{R}^3$, $u_i \in \mathbb{R}$, $e_1 = (1, 0, 0)^T$, and $e_2 = (0, 1, 0)^T$. Since $\dot{q} \perp q$ we have that the sphere $N = S^2$ is an invariant manifold. The reason we consider this system is because $\mathrm{span}(e_1 \times q, e_2 \times q)$ is not equal to T_qS^2 for all $q \in S^2$ and it cannot be expressed as a Clebsch optimal control problem. Since $[e_1 \times q, e_2 \times q] = -e_3 \times q$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields and $e_3 = (0, 0, 1)^T$, the Lie algebra rank condition (see,e.g. [15]) is satisfied on S^2 and therefore the restriction of (11) to S^2 is accessible. The set where $\dim(\mathrm{span}\{e_1 \times q, e_2 \times q\}) \neq 2$ is given by $\{q \in \mathbb{R}^3 \mid q_3 = 0\}$. Therefore to satisfy the standing assumption the third components of the two end points cannot both be zero which will guarantee that the associated optimal control problem does not admit abnormal extremals. For this system we consider the cost function

$$\ell(q, u) = \tfrac{1}{2}(u_1^2 + u_2^2) = \tfrac{1}{2}u \cdot u,$$

where $\cdot$ denotes the Euclidean dot product. To find the normal extremals of the embedded optimal control problem we apply Pontryagin's maximum principle. The Hamiltonian is defined as

$$\begin{aligned} H(q,p,u) &= p \cdot \Big(B(q)u\Big) - \tfrac{1}{2} u \cdot u \\ &= (B(q)^T p) \cdot u - \tfrac{1}{2} u \cdot u, \end{aligned}$$

where the 3×2 matrix $B(q)$ is defined as $B(q) := [e_1 \times q, e_2 \times q]$. The optimal control control is seen to be

$$u^* = B(q)^T p = \begin{bmatrix} (e_1 \times q) \cdot p \\ (e_2 \times p) \cdot p \end{bmatrix}.$$

The maximum principle then gives the normal extremals as generated by Hamilton's equations with Hamiltonian $H(q,p,u^*(q,p)) = \frac{1}{2} p^T B(q) B(q)^T p$, which are the differential equations

$$\begin{aligned} \dot{q} &= e_1 \times q u_1 + e_2 \times q u_2, & \dot{p} &= e_1 \times p u_1 + e_2 \times p u_2, \\ u_1 &= (e_1 \times q) \cdot p, & u_2 &= (e_2 \times q) \cdot p, \end{aligned}$$

where $q, p \in \mathbb{R}^3$. We notice that the sets $S^2 \times \mathbb{R}^3$ and $S^2 \times S^2$ are *invariant manifolds*. Again it is not surprising that $S^2 \times \mathbb{R}^3$ is an invariant manifold as this is the pullback bundle $i^*(T^*Q)$ for this example.

IV. Conclusion

In this paper we defined the class of embedded optimal control problems and introduced an associated optimal control problem. An embedded optimal control problem is not locally controllable whereas its associated optimal control problem is. We applied Pontryagin's maximum principle to the embedded optimal control problem to obtain the generating equations for its normal and abnormal extremals. It was shown that the extremal generating equations for the associated optimal control problem are contained in the normal extremal generating equations for the embedded optimal control problem (in the sense that they are the homomorphic image of the normal extremal generating equations.) A similar description is not available for the abnormal extremals.

We applied the theory to three examples. The first two examples can be stated as Clebsch optimal control problems, but not the last. For the first example we showed explicitly how the normal extremal generating equations, the so-called symmetric representation, are related to the extremal generating equations for the associated optimal control problem. In this case the associated optimal control problem is equivalent to a left-invariant variational problem and the extremal equations can therefore easily be obtained from the Euler-Poincaré equations. The second example revealed its explicit normal extremal to extremal relationship through a close connection to the geodesic problem on S^2.

In [13] we are continuing parts of this work. In particular we examine how to express a mechanical system as a Clebsch optimal control problem and we analyze a discretization of the Clebsch optimal control problem. Combined this provides a method for globally defining variational (and thus symplectic) integrators for certain mechanical systems.

References

[1] A. M. Bloch, P. E. Crouch, J. E. Marsden, and T. S. Ratiu, "The symmetric representation of the rigid body equations and their discretization," *Nonlinearity*, vol. 15, no. 4, pp. 1309–1341, 2002.

[2] A. M. Bloch, P. E. Crouch, J. E. Marsden, and A. K. Sanyal, "Optimal control and geodesics on quadratic matrix Lie groups," *Foundations of Computational Mathematics*, vol. 8, no. 4, pp. 469–500, 2008.

[3] P. E. Crouch, N. Nordkvist, and A. K. Sanyal, "Optimal control and geodesics on matrix Lie groups," in *Proc. 9th Portuguese Conference on Automatic Control - CONTROLO' 2010*, Coimbra, Portugal, Sept. 2010.

[4] A. M. Bloch, P. E. Crouch, N. Nordkvist, and A. K. Sanyal, "Embedded geodesic problems and optimal control for matrix Lie groups," *Journal of Geometric Mechanics*, vol. 3, no. 2, pp. 197–223, 2011.

[5] Y. N. Federov and V. V. Kozlov, "Various aspects of n-dimensional rigid body dynamics," *American Mathematical Society Translations*, vol. 168, pp. 141–171, 1995.

[6] S. V. Manakov, "Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body," *Functional Analysis and Its Applications*, vol. 10, no. 4, pp. 328–329, 1976.

[7] A. S. Mischenko and A. T. Fomenko, "On the integration of the Euler equations on semisimple Lie algebras," *Sov. Math. Dokl.*, vol. 17, pp. 1591–1593, 1976.

[8] T. S. Ratiu, "The motion of the free n-dimensional rigid body," *Indiana University Mathematics Journal*, vol. 29, no. 4, pp. 609–629, 1980.

[9] A. M. Bloch, P. E. Crouch, and A. K. Sanyal, "A variational problem on Stiefel manifolds," *Nonlinearity*, vol. 19, pp. 2247–2276, 2006.

[10] A. M. Bloch, P. E. Crouch, D. D. Holm, and J. E. Marsden, "An optimal control formulation for inviscid incompressible ideal fluid flow," in *IEEE Conf. on Decision and Control*, Sydney, Australia, Dec. 2000, pp. 1273–1279.

[11] D. D. Holm, "Riemannian optimal control formulation of incompressible ideal fluid flow," 2009, preprint.

[12] F. Gay-Balmaz and T. S. Ratiu, "Clebsch optimal control formulation in mechanics," *Journal of Geometric Mechanics*, vol. 3, no. 1, pp. 41–79, 2011.

[13] A. M. Bloch, P. E. Crouch, and N. Nordkvist, "Continuous and discrete embedded optimal control problems and their application to the analysis of Clebsch optimal control problems and mechanical systems," in preparation.

[14] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*, 2nd ed. New York: Springer Verlag, 1999.

[15] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems*, ser. Texts in Applied Mathematics. New York: Springer Verlag, 2004, vol. 49.