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Feedback stabilisation of locally controllable systems

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Abstract— **It is shown that, for real analytic control systems, small-time local controllability from an equilibrium implies the existence of a locally asymptotically stabilising piecewise analytic feedback.**

I. INTRODUCTION

The motivation for the present work is to establish a connection between small-time local controllability and state feedback stabilisation for nonlinear control systems. To this end, we rely on results proven by Grasse [14] to relax the assumption of global controllability in Sussmann's construction of a piecewise analytic feedback [24]. Important open problems related to stabilisation, such as the connection between geometric control and Lyapunov-theoretic techniques, the relation between asymptotic and finite-time stabilisation for general nonlinear control systems, or the obstructions to the existence of stabilising feedbacks of a given regularity, can potentially be addressed using ideas along the lines of this paper.

In the sequel we refer to "small-time local controllability" simply as "local controllability" and to "state feedback" simply as "feedback".

A. Stabilisation of locally controllable systems.

For one-dimensional systems it can be shown that local controllability implies local asymptotic stabilisation using piecewise constant feedback controls [10, p. 342], whereas for two-dimensional systems Kawski [17] has proven that local controllability implies the existence of a Hölder continuous asymptotically stabilising feedback. However, the technique of Kawski's proof is specific to the two-dimensional nature of the state space and, indeed, in dimension three it can be shown by counterexample [11] that there exist locally controllable systems that cannot be asymptotically stabilised by continuous feedback (underactuated controllable driftless systems belong to this category). If the dimension of the state space is not two or three, Coron [12] has shown that certain sufficient conditions for local controllability imply the existence of an almost smooth time-varying periodic feedback that stabilises the system in small time. Our observation in the present paper is that local controllability from an equilibrium implies the existence of a locally asymptotically stabilising piecewise analytic feedback. The results we just mentioned relate local controllability to different types of stabilisation and the relation between the different types of feedback involved is an important issue that requires further investigation.

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B. Stabilisation of asymptotically controllable systems

Already in [7, p. 181], the observation is made that asymptotic controllability is an obvious necessary condition for asymptotic feedback stabilisation and it has been shown [2], [9] that under additional hypotheses—which authors usually incorporate in the *definition* of asymptotic controllability—it becomes sufficient as well, provided that a feedback is suitably defined. Robustness issues associated with the feedbacks constructed in [2] and [9] have been addressed in [3], [18], [20], [21] . The literature on the existence of control-Lyapunov functions with special properties and, consequently, of classes of stabilising controls for asymptotically controllable systems, under various sets of assumptions, is vast. A useful overview of the main results, along with entry points to the literature, can be found in [4], while [13], [22], [27] contain recent results for nonlinear control systems of a general form.

In the remainder of the paper we deal exclusively with locally controllable systems. The relation between local controllability from a point and asymptotic controllability to the same point, regardless of optimality considerations, is currently not well understood and it is an interesting problem to find conditions under which the two notions become equivalent. One advantage of establishing the stabilisability of locally controllable systems is that there exist general sufficient conditions of a (Lie) algebraic nature for local controllability [5], [25] and in certain cases—for example, homogeneous systems—locally controllable systems can be completely characterised [1]. To the best of our knowledge, computable criteria for asymptotic controllability exist only for a narrow class of systems, namely, parametrised time-varying linear ones [28]. On the other hand, as mentioned above, there are results that show the existence of *robust* feedback stabilisers for asymptotically controllable systems, whereas the possibility of modifying our proof to obtain a robust stabilising feedback has not yet been explored.

II. BACKGROUND

A. Basic definitions

All manifolds are by definition second-countable Hausdorff topological spaces equipped with a maximal real analytic atlas. If M denotes a manifold, TM denotes the tangent bundle of M and this notation is applied recursively to denote, for example, the tangent bundle of TM by TTM. To avoid cumbersome expressions, we sometimes write $\Gamma^{\omega}E$ for the space of real analytic sections of a vector bundle (E, π, M). We begin by making precise what is meant by a "control system" [14], [15], [24].

Definition 1: Let M be a manifold and Ω a separable metric space. A ***real analytic control system*** (or ***control system***) on M is a map $f : M \times \Omega \to TM$ such that, for any $\omega \in \Omega$ fixed, the map $f_{\omega} : M \ni x \mapsto f(x,\omega) \in TM$ is a real analytic vector field on M. The manifold M is called the ***state space*** of the control system.

If we let $\mathscr{U}_{\text{meas}}$ denote the set of measurable functions $u : I \subset \mathbb{R} \to \Omega$, where I is an interval, then the class of ***admissible controls*** for a control system f is defined to be the subset $\mathscr{U}_f \subset \mathscr{U}_{\text{meas}}$ containing the functions u with the property that the differential equation $\dot{x}(t) = f(x(t), u(t))$ satisfies Caratheodory's theorem on the existence and uniqueness of solutions. If Tf_{ω}: $TM \times \Omega \to TTM$ (fix ω to compute Tf_{ω} and consider the resulting map on $TM \times \Omega$) is continuous, a property we always assume, then $\mathscr{U}_f$ contains the set $\mathscr{U}_{\text{step}}$ of piecewise constant maps with a finite number of discontinuities [14], [15] [23, p. 474 Thm 54, p. 480 Prop. C3.4].

Given a control system f, we define the ***time-reversed system*** to be the control system $-f$. The name stems from the fact that, if γ is the trajectory of f corresponding to a control $u : [0,T] \to \Omega$, then $\tilde{\gamma}(t) = \gamma(T-t)$ is the trajectory of $-f$ corresponding to the control $\tilde{u}(t) = u(T-t)$ and, in colloquial terms, we can say that the trajectories of $-f$ are those of f run backwards. Observe that $\mathscr{U}_f = \mathscr{U}_{-f}$ (i.e., the minus sign does not affect the conditions for existence and uniqueness of solutions).

Next, we recall the definition of small-time local controllability. Let $p \in M$ and $t \in \mathbb{R}$ with $t > 0$. If $t \mapsto x(t; p, u)$ is the trajectory that corresponds to an admissible control u then the ***reachable set*** of f from p at time t is the set

$$R_f(p,t) = \left\{ q \in M \mid q = x(t; p, u) \text{ for some } u \in \mathscr{U}_f \right\}.$$

Definition 2: A control system $f : M \times \Omega \to TM$ is ***small-time locally controllable*** (or ***locally controllable***) from a point $p \in M$ if, for every $T > 0$, the point p is in the interior of the set

$$R_f(p, [0,T)) \triangleq \bigcup_{t \in [0,T)} R_f(p,t).$$

B. Normal reachability

This section is concerned with a particular property of locally controllable control systems which is essential for proving the existence of a piecewise analytic feedback. In geometric control theory, there is an obvious appeal in knowing that certain control-theoretic tasks can be accomplished using piecewise constant controls. This is due on the one hand to the simplicity of the latter and on the other to the direct geometric interpretation of a control system with piecewise constant controls as a family of vector fields on a manifold. In our case, in order to be able to construct a piecewise analytic feedback, we need not only know that the points in the reachable set of a locally controllable system can be reached using a piecewise constant control, but also that the concatenation of flows that corresponds to the piecewise constant control has rank equal to the dimension of the state space, as a map from the space of switching times to the state space of the control system. This last property is called normal reachability and it has been shown [14] to be true for a wide class of control systems. In the remainder of the present section we make the aforementioned ideas precise.

Recall from above that $f_{\omega}(x) = f(x,\omega)$ and, therefore, if a control system $f : M \times \Omega \to TM$ is given, then f_{ω} denotes the vector field obtained by fixing the control to the value ω. Also, if

$$\Phi^f : \mathbb{R} \times M \supset U \ni (t,p) \mapsto \Phi^f(t,p) \in M$$

is the flow (defined on some open subset U) of a vector field f on a manifold M, we write Φ^f_p for the map $t \mapsto \Phi^f(t,p)$ [8, p. 95].

Definition 3: Given a control system $f : M \times \Omega \to TM$ and $p, q \in M$, we say that q is ***normally reachable*** from p via f if there exist $k \in \mathbb{N}$, $(\omega_1, \ldots, \omega_k) \in \Omega^k$ and $(s_1, \ldots, s_k) \in \mathbb{R}^k$ such that $q = \Phi^{f_{\omega_k}}_{s_k} \circ \cdots \circ \Phi^{f_{\omega_1}}_{s_1}(p)$ and the map $(t_1, \ldots, t_k) \mapsto \Phi^{f_{\omega_k}}_{t_k} \circ \cdots \circ \Phi^{f_{\omega_1}}_{t_1}(p)$ has rank equal to $\dim M$ at $(s_1, \ldots, s_k)$.

We can now state the following important theorem that will be used later in the paper. The theorem is a combination of results, all of which can be found in [14], stated in a form suitable for our purpose.

Theorem 4: If a control system $f : M \times \Omega \to TM$ is locally controllable from $p \in M$, then the time-reversed system $-f$ is also locally controllable from p and every point in $R_{-f}(p; [0,T))$ is normally reachable from p.

As stated earlier, the theorem guarantees that we can actually reach an open neighbourhood of each point in the reachable set of a locally controllable system.

C. Semianalytic and subanalytic sets

The notion of a piecewise analytic feedback that we employ in the present paper was first introduced by Sussmann [24] who proved that such a feedback exists for globally controllable nonlinear systems. As in [24], to prove the existence of a piecewise analytic feedback for locally controllable systems we rely heavily on the properties of subanalytic sets and we refer the reader to the paper by Sussmann and the references therein for the necessary background. In the present section we confine ourselves to recalling the definitions necessary to state the main result of the paper.

Subanalytic sets are images of semianalytic sets and so we begin by defining the latter. Let M be a manifold and U an open subset of M. Let also $C^{\omega}(U;\mathbb{R})$ denote the ring of real analytic functions defined on U, $\mathbb{Z}_{>0}$ denote the set of positive integers and set

$$\mathcal{S}(C^{\omega}(U;\mathbb{R})) = \left\{ X \subset U \mid X = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in U \mid f_{ij}\, \sigma\, 0\}, \right.$$
$$\left. f_{ij} \in C^{\omega}(U;\mathbb{R}),\ \sigma \in \{=,>\},\ p,q \in \mathbb{Z}_{>0} \right\}.$$

Given the notation above, semianalytic and, subsequently, subanalytic sets are defined as follows.

Definition 5: (i) A subset X of M is ***semianalytic*** if every point $p \in M$ has a neighbourhood U such that $X \cap U \in \mathcal{S}(C^{\omega}(U;\mathbb{R}))$.

(ii) A subset X of a manifold M is ***subanalytic*** if, for all p in M, there exists a neighbourhood U of p such that $X \cap U = \mathrm{pr}_1(A)$, where pr_1 denotes the projection to the first factor, A is a relatively compact semianalytic subset of $M \times N$ and N is a manifold.

In other words, subanalytic sets are locally projections of semianalytic sets.

D. Piecewise analytic feedbacks

The purpose of the present section is to define what a piecewise analytic feedback is, building on a series of more basic definitions. We also recall a theorem on subanalytic stratifications which is essential in constructing a piecewise analytic feedback. We follow closely the development in [24].

Definition 6: (i) An ***analytic stratification*** of a manifold M is a partition $\mathscr{P}$ into connected real analytic submanifolds (called ***strata***) such that

a) $\mathscr{P}$ is locally finite,

b) $\overline{S} = \bigcup_{\substack{T \in \mathscr{P} \\ T \cap \overline{S} \neq \emptyset}} T, \forall S \in \mathscr{P}$; that is, the closure $\overline{S}$ of every stratum S is the union of the stata T in the partition $\mathscr{P}$ that have nonempty intersection with $\overline{S}$, and

c) $(T \subset \overline{S}) \wedge (T \neq S) \Rightarrow \mathrm{codim} T > \mathrm{codim} S$; that is, if a stratum T is contained in the closure $\overline{S}$ and it is not the whole stratum S, then the codimension of T is larger than the codimension of S.

(ii) A ***subanalytic stratification*** is an analytic stratification whose strata are subanalytic sets.

(iii) A stratification is called ***compatible*** with a family $\mathscr{A}$ of subsets of M if every $\mathscr{A}$ is a union of strata.

Definition 7: A ***piecewise analytic vector field*** on a manifold M is a quadruple $V = \left(\Sigma, (\Sigma_1, \Sigma_2), \{V_S\}_{S \in \Sigma_1}, E\right)$ where

(i) Σ is an analytic stratification of M,

(ii) (Σ_1, Σ_2) is a partition of Σ; strata in Σ_1 are said to be of the first kind and strata in Σ_2 of the second kind,

(iii) for every $S \in \Sigma_1$, V_S is an analytic vector field on S,

(iv) E is a map which assigns to every point p in a stratum $S \in \Sigma_2$ a stratum $E(p) \in \Sigma_1$,

(v) for every $p \in S \in \Sigma_1$, the integral curve γ of V_S through p is either defined for all $t \geq 0$ or, if the integral curve is defined up to some time $T > 0$ and $\gamma([0,T))$ is contained in a compact subset of M, then $\lim_{t \to T^-} \gamma(t)$ exists, and

(vi) for every $p \in S \in \Sigma_2$, there exists a unique integral curve γ of $V_{E(p)}$ such that $\lim_{t \to 0^+} \gamma(t) = p$.

Before we comment on the previous definition, we first relate, in the obvious manner, a piecewise analytic vector field with the notion of feedback as is known from control theory. For a piecewise analytic vector field to be a feedback for a control system it has to be "realisable" by means of the control system. This is the content of the next definition.

Definition 8: A ***piecewise analytic feedback*** for a control system $f : M \times \Omega \to TM$ is a piecewise analytic vector field V on M such that, for every $p \in S \in \Sigma_1$, there exists $\omega \in \Omega$, such that $V_S(p) = f(p, \omega)$.

The next theorem is fundamental for showing that a piecewise analytic feedback, with all its defining properties, exists for locally or globally controllable systems.

Theorem 9: [24] Let $\mathscr{A}$ be a locally finite family of nonempty subanalytic subsets of M and, for each $A \in \mathscr{A}$, let $F(A)$ be a finite collection of vector fields on M. Then there exists a subanalytic stratification $\mathscr{P}$ of M compatible with $\mathscr{A}$ and having the following property: whenever $\mathscr{P} \ni S \subset A \in \mathscr{A}$ and $X \in F(A)$, then either X is everywhere tangent to S or X is nowhere tangent to S.

III. LOCAL CONTROLLABILITY AND FEEDBACK STABILISATION

A. Main result

Given a control system $f : M \times \Omega \to TM$, we say that a point $p \in M$ is an ***equilibrium*** for f if there exists $\omega \in \Omega$ such that $f(p, \omega) = 0$. We also say that f ***admits a uniform bound*** if, given a positive real number T, there exists a compact set containing all trajectories defined on $[0, T]$ and starting at p [19].

Theorem 10: Let $f : M \times \Omega \to TM$ be a control system locally controllable from an equilibrium $p \in M$. Suppose f admits a uniform bound, Ω is compact, and $\{f(p, \omega) \mid \omega \in \Omega\}$ is convex for every $p \in M$. There exists a locally asymptotically stabilising piecewise analytic feedback.

Proof: If $\xi = (X_1, \dots, X_k)$ is a finite sequence of vector fields, $|\xi|$ denotes the number of elements in ξ, i.e. $|\xi| = k$, and if $\tau = (t_1, \dots, t_k)$ is a k-tuple of real numbers, $|\tau|$ is equal to the sum of the components of τ, i.e. $|\tau| = t_1 + \cdots + t_k$. Given such a $\tau \in \mathbb{R}^n$, we denote the cubical neighbourhood of τ of side 2ε by $C^n_\varepsilon(\tau)$; in other words,

$$C^n_\varepsilon(\tau) = \{(s_1, \dots, s_k) \in \mathbb{R}^n \mid |t_i - s_i| \le \varepsilon, i \in \{1, \dots, n\}\}.$$

The non-negative orthant of $\mathbb{R}^n$ will be denoted by $\mathbb{R}^n_{\ge 0}$ and it is, by definition, the set $\{\tau = (t_1, \dots, t_n) \in \mathbb{R}^n \mid t_i \ge 0, i \in \{1, \dots, n\}\}$. For a collection of vector fields ξ and a k-tuple τ as above, Φ^ξ_τ stands for the composition $\Phi^{X_1}_{t_1} \circ \cdots \circ \Phi^{X_k}_{t_k}$ and, similarly to the case of a single vector field, Φ^ξ_p is the map $\tau \mapsto \Phi^\xi(\tau, p)$.

Since the control system f is locally controllable from p, the time-reversed system $-f$ is also locally controllable from p by Theorem 4. By the same theorem, if q is a point in the reachable set $R_{-f}(p, [0,T])$ of $-f$, then q is normally reachable from p via $-f$ in time less than $T + \alpha$, where α is a positive real number. That this is the case follows from the fact that p is an equilibrium for f and, therefore, $R_{-f}(p, [0,T]) \subset R_{-f}(p, [0, T+\alpha))$. Theorem 4 can then be applied directly to the set $R_{-f}(p, [0, T+\alpha))$. Normal reachability of q from p via $-f$ in time less than $T + \alpha$ means precisely that we can find a finite sequence of vector fields $\xi_q = (X_1, \dots, X_{|\xi_q|})$ and $\sigma_q = (s_1, \dots, s_{|\xi_q|}) \in \mathbb{R}^{|\xi_q|}_{\ge 0}$ such that $|\sigma_q| < T + \alpha$, each X_i, $i \in \{1, \dots, |\xi_q|\}$, is of the form $-f_\omega$, and $\Phi^{\xi_q}(\sigma_q, p) \triangleq \Phi^{X_1}_{s_1} \circ \cdots \circ \Phi^{X_{|\xi_q|}}_{s_{|\xi_q|}} = q$, with the map $\Phi^{\xi_q}_p$ having rank equal to the dimension of M at σ_q. Since the rank of the map $\Phi^{\xi_q}_p$ is $\dim M$ at σ_q, the set $\Phi^{\xi_q}_p(C^{|\xi_q|}_{\varepsilon_q}(\sigma_q) \cap \mathbb{R}^{|\xi_q|}_{\ge 0})$ contains a neighbourhood of q, for some (in fact, any) positive ε_q. For every $\tau \in C^{|\xi_q|}_{\varepsilon_q}(\sigma_q) \cap \mathbb{R}^{|\xi_q|}_{\ge 0}$, we can define a curve

$$\eta_\tau : [0, |\tau|] \ni t \mapsto \eta_\tau(t) \in \mathbb{R}^{|\xi_q|}$$

such that $t \mapsto \left(\Phi^{\xi_q}_p \circ \eta_\tau\right)(t)$ is the concatenation of trajectories of the control system $-f$ that connects p and $\Phi^{\xi_q}_p(\tau)$ (see Figure 1). Let A_q denote the set of points in $\mathbb{R}^{|\xi_q|}_{\ge 0}$ which are of the form $\eta_\tau(t)$, $\tau \in C^{|\xi_q|}_{\varepsilon_q}(\sigma_q) \cap \mathbb{R}^{|\xi_q|}_{\ge 0}$, $t \in [0, |\tau|]$. The set A_q can be written as a finite union of compact semianalytic sets: $A_q = A^1_q \cup \cdots \cup A^{|\xi_q|}_q$. The geometric meaning of the A^i_q is shown in Figure 2; they are sets of increasing dimension and every A^i_q is a rectangular "neighbourhood" of the i-th segment of the curve η_τ. The explicit description of the sets A^i_q in terms of inequalities—inequalities that stem form the fact that $\tau \in C^{|\xi_q|}_{\varepsilon_q}(\sigma_q) \cap \mathbb{R}^{|\xi_q|}_{\ge 0}$ and $t \in [0, |\tau|]$—is as follows [24, p. 45]: the set A^i_q consists of those points $(t_1, \dots, t_{|\xi_q|})$ that satisfy

(i) $t_j = 0$, for $j \le |\xi_q| - i$,
(ii) $0 \le t_j \le b_j$, for $j = |\xi_q| + 1 - i$, and

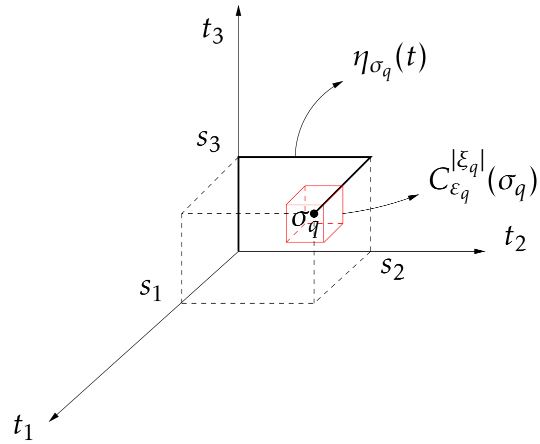

Fig. 1: The curve η_τ is used to express a concatenation of flows for a control system as one single curve.

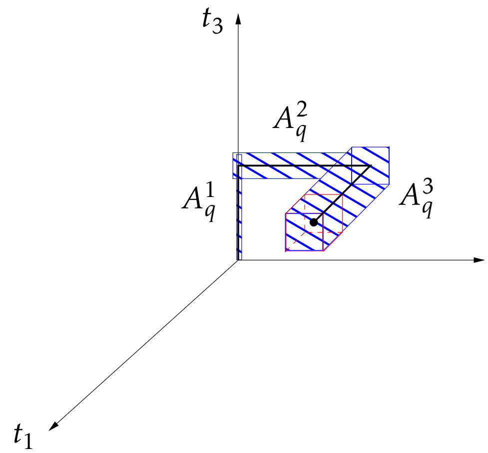

Fig. 2: A graphical representation of the sets A^i_q used in the proof of Theorem 10.

(iii) $a_j \le t_j \le b_j$, for $j > |\xi_q| + 1 - i$,

where $a^i_q = \max\left(\sigma_{q,i} - \varepsilon_q, 0\right)$, $b^i_q = \sigma_{q,i} + \varepsilon_q$, and $\sigma_{q,i}$ denotes the i-th component of σ_q. Since the sets A_q, A^i_q are compact semianalytic, their images $B_q = \Phi^{|\xi_q|}_p(A_q)$, $B^i_q = \Phi^{|\xi_q|}_p(A^i_q)$ are compact subanalytic sets and the set B_q contains a neighbourhood of the point q.

Consider now a strictly decreasing sequence (T_i) of positive real numbers with $T_i \to 0$ and $T_1 = T$. The assumptions of the theorem imply that the sets $R_{-f}(p, [0, T_i])$ form a decreasing sequence of compact sets [19, p. 242]. For the ease of notation, set $K_i = R_{-f}(p, [0, T_i])$ and $\mathring{K}_i = R_{-f}(p, [0, T_i))$. Each point q in $R_i \triangleq K_i \setminus \operatorname{int} K_{i+1}$ is contained in a set B_q and, because R_i is compact, every R_i can be covered with finitely many sets $B_{q_1}, \dots, B_{q_{k_i}}$. We reindex the points q_i that correspond to the sets that form the finite covers to create a sequence $B_{q_1}, B_{q_2}, \dots$ of sets. Using the indices i, j, and m uniquely defined by the relation

$$j = i + |\xi_{q_1}| + \cdots + |\xi_{q_{m-1}}|, \qquad (1)$$

with $1 \leq i \leq |\xi_{q_m}|$, we define the sequence of sets $D_j = B_{q_m}^{|\xi_{q_m}|+1-i}$ and we set $H_j = D_j \setminus \bigcup_{i \neq j} D_i$. The sequence $\left(H_j\right)_{j\geq 1}$ gives, by restriction of certain sets H_j if necessary, a locally finite partition of the punctured neighbourhood $\mathring{K}_1 \setminus \{p\}$ of p into relatively compact subanalytic sets. To each set H_j, $j \geq 1$, we assign the vector field $Y_j = -X_i$ and to $H_0 \triangleq \{p\}$ we assign the zero vector field; each vector field X_i is of the form $-f_\omega$ and, therefore, Y_j is of the form f_ω. Applying Theorem 9 to the family $\mathscr{H} = \{H_j\}_{j\geq 1}$ of subanalytic subsets, with $F(H_j) = \{Y_j\}$ (see the notation in the theorem), gives a stratification Σ of $\mathring{K}_1$, compatible with the family $\mathscr{H}$. The stratification Σ is partitioned into Σ_1 and Σ_2 (see Definition 7) in the following way: every stratum $S \in \Sigma$ is a subset of an H_j, if Y_j is everywhere tangent to S then $S \in \Sigma_1$, otherwise $S \in \Sigma_2$.

We now show that trajectories of the piecewise analytic feedback starting in a neighbourhood of p converge to p. More specifically, let q be an arbitrary point in $\mathring{K}_1 \setminus \{p\}$; then q belongs to H_j, for some $j > 0$, and there exists a time σ_q^1 such that $\gamma_1(\sigma_q^1) \in H_k$, for some $k > j$, where γ_1 is the integral curve of Y_j through q. If γ_2 denotes the integral curve of Y_k through $\gamma_1(\sigma_q^1)$, there exists a time σ_q^2 such that $\gamma_2(\sigma_q^2) \in H_\ell$, for some $\ell > k$ and iterating this process shows, by definition of the sets H_j, that the trajectory of the piecewise analytic feedback that starts at q approaches p asymptotically.

Recall that an equilibrium p is Lyapunov stable if, for any neighbourhood U of p, there exists a neighbourhood V of p such that all trajectories starting in V converge to p without leaving the set U. Because $K_i \to \{p\}$, for any neighbourhood U of p, there exists $\lambda \in \mathbb{Z}_{>0}$ such that $K_\ell \subset U$. If we set $V = K_{\ell+\mu}$, for sufficiently large $\mu \in \mathbb{Z}_{>0}$, then the sets U and V satisfy the conditions for Lyapunov stability and, therefore, the closed-loop system with the piecewise analytic feedback we constructed above is Lyapunov stable.

The verification that there is a well-defined map E so that $(\Sigma, (\Sigma_1, \Sigma_2), \{V_S\}_{S\in\Sigma_1}, E)$ is a piecewise analytic feedback for f is the same as in [24]. ∎

Remark 11: The proof of Theorem 10 can be modified to obtain a piecewise analytic feedback that steers a neighbourhood of p to p in *finite* time. In that case, however, a weaker form of stability has to be substituted for Lyapunov stability and it is a question of ongoing research whether Lyapunov stability and stabilisation in finite time (using a piecewise analytic feedback) can be reconciled.

B. Corollary

The theory of linear control systems tells us that controllability of the unstable eigenvalues of a system implies the existence of a linear asymptotically stabilising feedback (in fact, the implication is an equivalence). In other words, it suffices to be able to control the unstable dynamics in order to stabilise a system. Theorem 10 can be used to obtain an analogous result for nonlinear control systems and the following corollary formalises this idea.

Let $f : M \times \Omega \to TM$ be a control system with $\dim M = n$. We first perform some constructions with vector fields following [26]. Let $\mathscr{C}$ be the Lie algebra of vector fields generated by the vector fields f_ω, $\omega \in \Omega$. Let $\mathscr{D} \subset \mathscr{C}$ be the derived algebra which can be shown to be the set of all finite real-linear combinations of vector fields from the set

$$[f_{\omega_1}, [f_{\omega_2}, \dots, [f_{\omega_{k-1}}, f_{\omega_k}]]], \qquad k \geq 1,\ \omega_1, \dots, \omega_k \in \Omega.$$

Let $\mathscr{X}_0$ be the family of vector fields

$$\mathscr{X}_0 = \Big\{\lambda_1 f_{\omega_1} + \cdots + \lambda_k f_{\omega_k} \mid k \geq 1,\ \omega_1, \dots, \omega_k \in \Omega,\ \lambda_1 + \cdots + \lambda_k = 0\Big\}.$$

Then take $\mathscr{C}_0 = \mathscr{X}_0 + \mathscr{D}$; this is a family of vector fields that can be shown to be closed under Lie bracket. Moreover, this family of vector fields is invariant under the vector fields f_ω, $\omega \in \Omega$, in the sense that $[f_\omega, X] \in \mathscr{C}_0$ for every $X \in \mathscr{C}_0$. Denote by C_0 the distribution generated by the family of vector fields $\mathscr{C}_0$. If C_0 has constant rank k in a neighbourhood of $p \in M$, then we can find local coordinates $x = (x^1, \dots, x^k, x^{k+1}, \dots, x^n) = (y^1, y^2)$ around p such that $C = \operatorname{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}\right\}$ and, because C is invariant for f, we have the local decomposition [26]:

$$\begin{aligned} \dot{y}^1 &= f^1\left(y^1, y^2, \omega\right), \\ \dot{y}^2 &= f_0^2\left(y^2\right). \end{aligned} \tag{2}$$

Corollary 12: If the subsystem f^1 is locally controllable from p and the vector field f_0^2 has p as a locally asymptotically stable equilibrium, then the control system f is asymptotically stabilisable at p.

Obviously, we can obtain variations of Corollary 12 depending on the type of stability we assume for the uncontrolled dynamics f_0^2.

IV. EXAMPLES

A. First example

Consider the question of whether some kind of stabilising feedback exists for the class of control-affine systems described by

$$\begin{aligned} \dot{x} &= u, \\ \dot{y} &= Q(x), \end{aligned} \tag{3}$$

where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, Q is a quadratic form (see Problem 10.4 in [6, p. 315] and the references therein for the long history of vector-valued quadratic forms and their relevance to control theory), and $u \in U$, with U being a compact convex subset of $\mathbb{R}^m$ containing a neighbourhood of the origin. One way to address this question is to use centre manifold theory, however, if $n \geq 2$, the centre manifold is of dimension at least two and there is no general method for proving or disproving stability.

On the other hand, it has been proven that control systems of the form (3) are locally controllable from the origin $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ if and only if Q is indefinite [16]. In that case, Theorem 10 implies that there exists a piecewise analytic feedback that asymptotically stabilises (3) to the origin.

B. Second example

The control system

$$\begin{aligned} \dot{x} &= u, \\ \dot{y} &= x^2(y-z), \\ \dot{z} &= x^2(z-x), \end{aligned} \tag{4}$$

with $(x,y,z) \in \mathbb{R}^3$ and $u \in [a,b] \subset \mathbb{R}$, is locally controllable from the origin, asymptotically controllable to the origin, and it satisfies Coron's condition for stabilisation [11]. However, as is shown in [11], there does not exist a continuous feedback that asymptotically stabilises (4) to the origin. It is also remarked in [11] that it is not known whether (4) can be stabilised using dynamic feedback. Since (4) is locally controllable we know from Theorem 10 that it can be stabilised to the origin using a piecewise analytic feedback.

C. Third example

Our last example is meant to illustrate the behaviour of control systems when the rank condition of Corollary 12 does not hold. It is an instance of how the notion of singularities, if properly understood, will clarify further the relation between controllability and stabilisability for nonlinear control systems, and indeed the properties of these systems in general. It is worth noting the simplicity of the example.

The control system

$$\dot{x} = (1-u)x,$$

where $x, u \in \mathbb{R}$, becomes asymptotically stable at $x=0$ for any choice of a constant control with value greater than one. At the same time, the system is clearly not locally controllable at $x=0$.

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