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Series method in nonlinear time optimality

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Abstract— **We develop the formal series technique for analysis of the time-optimal control problem. We introduce a concept of equivalence of symmetric control systems in the sense of time optimality. Our approach is based on the consideration of a free algebra of iterated integrals and structures induced in this algebra by control systems. It is close to a homogeneous approximation problem and, in essence, yields that under certain conditions a homogeneous approximation of a system is equivalent to this system in the sense of time optimality.**

I. INTRODUCTION.

In this paper we discuss some advantages of the series method for the analysis of the nonlinear time optimal control problem.

In [1] we considered an approximation of nonlinear affine control systems in a neighborhood of an equilibrium in the sense of the time optimality. Our method was based on the representation of an affine control system in the form of a series of nonlinear power moments [2]. We succeeded in constructing the "algebraic theory" for such systems and used this language, in particular, to describe the homogeneous approximation. Briefly, to any system we associate a right ideal in the free algebra of nonlinear power moments; a homogeneous system corresponding to this ideal is a homogeneous approximation of the initial nonlinear control system. Moreover, it turns out that the homogeneous approximation is closely connected with the approximation in the sense of time optimality.

In this paper we present analogous algebraic constructions for the partial case of symmetric systems. We introduce and discuss algebraic tools and formulate the main theorem on approximation in the sense of time optimality.

II. SERIES METHOD IN A LOCAL BEHAVIOR ANALYSIS OF SYMMETRIC CONTROL SYSTEMS

A. The endpoint map

Consider symmetric control systems of the form

$$\dot{x}=\sum_{i=1}^{m}u_iX_i(x),\quad x\in\mathbb{R}^n,\ u_i\in\mathbb{R},\tag{1}$$

where $X_1(x),\ldots,X_m(x)$ are real analytic vector fields in a neighborhood of the origin in $\mathbb{R}^n$. Below we are interested in the behavior of trajectories starting at the origin,

$$x(0)=0.\tag{2}$$

We admit controls $u(t)=(u_1(t),\ldots,u_m(t))\in L_\infty([0,T];\mathbb{R}^m)$ such that $\|u\|\le 1$ where $\|u\|^2=\sum_{i=1}^{m}(\operatorname{ess\,sup}_{t\in[0,T]}|u_i(t)|)^2$; there exist $T_0>0$ such that for $0\le T\le T_0$ trajectories of (1), (2) corresponding to such controls are well defined.

It is convenient to "stretch" all controls to the same time interval, say, $[0,1]$. Namely, for a control $u(t)$, $t\in[0,1]$, and a number $0<\theta\le T_0$ let us use the notation $u^\theta(t)=u(\frac{t}{\theta})$, $t\in[0,\theta]$. Below we denote by $x(t)=x(t;u^\theta)$, $t\in[0,\theta]$, the solution of the Cauchy problem

$$\dot{x}=\sum_{i=1}^{m}u_i^\theta(t)X_i(x),\quad x(0)=0.\tag{3}$$

Let B_1 be the unit ball in $L_\infty([0,1];\mathbb{R}^m)$.

Definition 1: We say that the mapping $\mathcal{E}_{X_1,\ldots,X_n}:[0,T_0]\times B_1\to\mathbb{R}^n$ defined by

$$\mathcal{E}_{X_1,\ldots,X_n}(\theta,u)=x(\theta;u^\theta)$$

is *the endpoint map* defined by the Cauchy problem (1), (2).

B. Series representation

In order to study the behavior of the map $\mathcal{E}_{X_1,\ldots,X_n}(\theta,u)$, the explicit expression of $\mathcal{E}_{X_1,\ldots,X_n}(\theta,u)$ would be helpful which does not include a trajectory $x(t;u^\theta)$ as in the definition of $\mathcal{E}_{X_1,\ldots,X_n}$, but depends only on θ and u. Such expression (a generalization of the well-known Cauchy formula for linear differential equations) was firstly proposed by M. Fliess. Namely, the following theorem holds [3].

Theorem 1: Consider a system of the form (1) and suppose that the vector fields $X_1,\ldots,X_m$ are real analytic in a neighborhood of the origin. Then there exists $0<T\le T_0$ such that for any $0<\theta\le T$ and any control $u\in B_1$ the endpoint map is represented in the form of an absolutely convergent series

$$\mathcal{E}_{X_1,\ldots,X_n}(\theta,u)=\sum_{k=1}^{\infty}\sum_{1\le i_1,\ldots,i_k\le m}c_{i_1\ldots i_k}\eta_{i_1\ldots i_k}(\theta,u),\tag{4}$$

where

$$\eta_{i_1\ldots i_k}(\theta,u)=\int_0^\theta\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}}\prod_{j=1}^{k}u_{i_j}^\theta(\tau_j)d\tau_k\cdots d\tau_2d\tau_1\tag{5}$$

are "iterated integrals" and

$$c_{i_1\ldots i_k}=X_{i_k}X_{i_{k-1}}\cdots X_{i_1}E(0)\tag{6}$$

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are constant vector coefficients (where we denote $E(x) = x$).

Let us briefly discuss Theorem 1. The right hand side of (4) includes "objects" of two kinds. The objects of the first kind are the constant coefficients – vectors in $\mathbb{R}^n$ – of the form (6). They are determined by the vector fields $X_1, \dots, X_m$ (more precisely, by the values of these vector fields and their derivatives at the origin) and, moreover, they depend on local coordinates. The objects of the second kind are the iterated integrals (5). They are "completely independent" in the sense that they are the same for all systems of the form (1). It turns out that the set of iterated integrals can be regarded as *a free associative algebra* (w.r.t. the certain operation); we introduce it in the next subsection.

C. Iterated integrals and free associative algebras

Suppose $\theta > 0$ is fixed. Let us consider iterated integrals *as functionals* of $u(t)$ defined on the space $L_\infty([0,1];\mathbb{R}^m)$. Then the linear span (over $\mathbb{R}$) of all such iterated integrals is the associative algebra with the product operation defined as

$$\eta_{i_1\dots i_k}(\theta,\cdot) \vee \eta_{j_1\dots j_s}(\theta,\cdot) = \eta_{i_1\dots i_k j_1\dots j_s}(\theta,\cdot). \qquad (7)$$

Notice that one-dimensional integrals are the generators of the algebra, so one can write

$$\eta_{i_1\dots i_k}(\theta,\cdot) = \eta_{i_1}(\theta,\cdot) \vee \dots \vee \eta_{i_k}(\theta,\cdot).$$

Substituting $u^\theta(t) = u(\frac{t}{\theta})$ to (5) we easily get

$$\eta_{i_1\dots i_k}(\theta,u) = \theta^k \eta_{i_1\dots i_k}(1,u).$$

Hence, k equals the asymptotic order of the iterated integral $\eta_{i_1\dots i_k}(\theta,u)$ w.r.t. θ as $\theta \to 0$ for any fixed control $u \in B^1$ such that $\eta_{i_1\dots i_k}(1,u) \neq 0$. This justifies the following

Definition 2: We say that the number k is the order of the iterated integral $\eta_{i_1\dots i_k}(\theta,\cdot)$.

This "length order" naturally generates the filtered structure in the algebra of iterated integrals.

Definition 3: Suppose $\theta > 0$ is fixed. The associative algebra of functionals $\mathcal{F}_\theta = \sum_{k=1}^\infty \mathcal{F}_\theta^k$ (over $\mathbb{R}$) where

$$\mathcal{F}_\theta^k = \mathrm{Lin}\{\eta_{i_1\dots i_k}(\theta,\cdot),\ 1 \le i_1,\dots,i_k \le m\}, \quad k \ge 1,$$

with the product operation (7) is called the algebra of iterated integrals. The natural filtration is given by the set of subspaces $\sum_{k=1}^q \mathcal{F}_\theta^k$, $q \ge 1$.

One can show that this associative algebra is *free* [3]. More specifically, if $\sum_{i_1\dots i_k} \alpha_{i_1\dots i_k}\eta_{i_1\dots i_k}(\theta,u) = 0$ for all $u \in B_1$, where $\alpha_{i_1\dots i_k} \in \mathbb{R}$ and the sum is taken over an arbitrary finite set of indices $i_1,\dots,i_k$ such that $1 \le i_1,\dots,i_k \le m$, then all coefficients $\alpha_{i_1\dots i_k}$ vanish.

This motivates introducing an abstract free associative graded algebra generated by m elements which is isomorphic to any $\mathcal{F}_\theta$.

Namely, let us consider the set of m abstract free elements called *letters*; we denote them by $\eta_1,\dots,\eta_m$. Strings of the letters are called *words*; we denote them by $\eta_{i_1\dots i_k} = \eta_{i_1} \vee \dots \vee \eta_{i_k}$ (we use the same sign $\vee$ as in $\mathcal{F}_\theta$ to denote the concatenation operation). All finite linear combinations of words (over $\mathbb{R}$) form a free associative algebra with the natural gradation $\mathcal{F} = \sum_{k=1}^\infty \mathcal{F}^k$, where the homogeneous subspace $\mathcal{F}^k$ is defined as a linear span of products of k generators

$$\mathcal{F}^k = \mathrm{Lin}\{\eta_{i_1\dots i_k} = \eta_{i_1} \vee \dots \vee \eta_{i_k},\ 1 \le i_1,\dots,i_k \le m\}, \qquad (8)$$

$k \ge 1$. Then $\mathcal{F}$ is naturally isomorphic to $\mathcal{F}_\theta$ for any $\theta > 0$.

Definition 4: The free associative algebra $\mathcal{F}$ over $\mathbb{R}$ with the abstract generators $\eta_1,\dots,\eta_m$, the product operation

$$\eta_{i_1} \vee \dots \vee \eta_{i_k} = \eta_{i_1\dots i_k}, \quad k \ge 2,$$

and the graded structure (8) generated by the "length order"is called *the Fliess algebra.*

Sometimes it is convenient to extend the algebra $\mathcal{F}$ and consider the algebra $\mathcal{F}+\mathbb{R}$ with the unity element (which can be thought of as the empty word) assuming $1 \vee a = a \vee 1 = a$ for any $a \in \mathcal{F} + \mathbb{R}$.

Taking into account the graded structure, we introduce the following convenient notation.

Definition 5: We say that an element $a \in \mathcal{F}$ is of order k and write $\mathrm{ord}(a) = k$ iff $a \in \mathcal{F}^k$. If an element is of some order we say that it is homogeneous.

In the free associative algebra $\mathcal{F}$, the *free* Lie algebra $\mathcal{L}$ is defined which is generated by the same set of generators $\eta_1,\dots,\eta_m$, and the bracket operation is defined in the usual way as $[\ell_1,\ell_2] = \ell_1 \vee \ell_2 - \ell_2 \vee \ell_1$. Then $\mathcal{F}$ is the universal enveloping for $\mathcal{L}$.

Thus, along with the endpoint map and its series representation (4) we can consider its "abstract analog", the formal series (with coefficients in $\mathbb{R}^n$) of elements of $\mathcal{F}$ of the form

$$\mathcal{E}_{X_1,\dots,X_m} = \sum_{k=1}^\infty \sum_{1 \le i_1,\dots,i_k \le m} c_{i_1\dots i_k}\eta_{i_1\dots i_k}. \qquad (9)$$

D. Changes of variables and shuffles

Notice that a change of variables in system (1) leads to some transformation of the series representation of the endpoint map. More specifically, suppose we know the series representation of the endpoint map $\mathcal{E}_{X_1,\dots,X_m}$, i.e.,

$$\mathcal{E}_{X_1,\dots,X_m}(\theta,u) = \sum_{k=1}^\infty \sum_{1 \le i_1,\dots,i_k \le m} c_{i_1\dots i_k}\eta_{i_1\dots i_k}(\theta,u),$$

where $c_{i_1\dots i_k}$ are *constant vector coefficients*. This representation coincides with (4), however, here we "forget" that the coefficients $c_{i_1\dots i_k}$ can be found via the vector fields $X_1,\dots,X_m$ by formula (6).

Suppose $y = Q(x)$ is a real analytic change of variables defined in a neighborhood of the origin and such that $Q(0) = 0$. Then in the new coordinates the initial system takes the form $\dot{y} = \sum_{i=1}^m u_i Y_i(y)$ where $Y_i(y) = Q'(x)X_i(x)|_{x=Q^{-1}(y)}$, $i = 1,\dots,m$. Then we obviously get

$$\mathcal{E}_{Y_1,\dots,Y_m}(\theta,u) = Q(\mathcal{E}_{X_1,\dots,X_n}(\theta,u))$$

for any rather small $\theta > 0$ and any $u \in B_1$. Since $Q(x)$ is real analytic, the series representation of $\mathcal{E}_{Y_1,\dots,Y_m}$ can be found directly using the series of $\mathcal{E}_{X_1,\dots,X_n}$, *without explicit calculating of vector fields* $Y_i(y)$. Namely, suppose

$Q(x) = \sum_{p=1}^{\infty} \frac{1}{p!} Q^{(p)}(0) x^p$. Hence, in order to express $\mathcal{E}_{Y_1,\dots,Y_m}(\theta, u)$ as a series of iterated integrals of the form (4), we need to find the product of iterated integrals. Calculating the product of two iterated integrals (considered as functionals of u) we get

$$\eta_{m_1\dots m_k}(\theta, u) \cdot \eta_{m_{k+1}\dots m_{k+r}}(\theta, u) = {\sum}' \eta_{m_{j_1}\dots m_{j_{k+r}}}(\theta, u),$$

where the sum $\sum'$ is taken over elements $(j_1, \dots, j_{k+r}) \in S_{k,r}$ of the set of all *shuffle permutations* $S_{k,r} = \{\sigma \in S_{k+r} : \sigma^{-1}(1) < \cdots < \sigma^{-1}(k),\ \sigma^{-1}(k+1) < \cdots < \sigma^{-1}(k+r)\}$ (here S_{k+r} is the set of all permutations of $(1, \dots, k+r)$). In fact, to multiply two integrals over domains $0 \le \tau_1 \le \cdots \le \tau_k \le \theta$ and $0 \le \tau_{k+1} \le \cdots \le \tau_{k+r} \le \theta$ we should "shuffle" two sets of variables $\{\tau_1, \dots, \tau_k\}$ and $\{\tau_{k+1}, \dots, \tau_{k+r}\}$ in all possible ways but keeping their "interior orders".

In the associative algebra, the corresponding operation is called *the shuffle product* [4].

Definition 6: The shuffle product $\sqcup\!\sqcup$ in $\mathcal{F}$ is defined on basis elements by the rule

$$\eta_{m_1\dots m_k} \sqcup\!\sqcup\, \eta_{m_{k+1}\dots m_{k+r}} = \sum_{(j_1,\dots,j_{k+r}) \in S_{k,r}} \eta_{m_{j_1}\dots m_{j_{k+r}}}.$$

Thus, the "usual product" of iterated integrals as functionals corresponds to the shuffle product in the abstract algebra. One can express this statement as follows:

$$\eta_{m_1\dots m_k}(\theta, u) \cdot \eta_{s_1\dots s_r}(\theta, u) = (\eta_{m_1\dots m_k} \sqcup\!\sqcup\, \eta_{s_1\dots s_r})(\theta, u)$$

where in the right hand side we mean that one calculates the shuffle product in $\mathcal{F}$ and *then* substitutes iterated integrals from $\mathcal{F}_\theta$ instead of the corresponding elements of $\mathcal{F}$. This equality also implies that the shuffle product is commutative and associative.

Sometimes it is more convenient to use another definition of the shuffle product. It can be easily shown that Definitions 6 and 7 are equivalent.

Definition 7: The shuffle product in $\mathcal{F}$ is defined on basis elements by the recurrent formula

$$\begin{aligned} \eta_{m_1\dots m_k} \sqcup\!\sqcup\, \eta_{s_1\dots s_r} = {} & (\eta_{m_1\dots m_{k-1}} \sqcup\!\sqcup\, \eta_{s_1\dots s_r}) \vee \eta_{m_k} \\ & + (\eta_{m_1\dots m_k} \sqcup\!\sqcup\, \eta_{s_1\dots s_{r-1}}) \vee \eta_{s_r}, \end{aligned} \tag{10}$$

where $1 \sqcup\!\sqcup\, a = a \sqcup\!\sqcup\, 1 = a$ for any $a \in \mathcal{F} + \mathbb{R}$.

With this concept in hands, we get the formula for transformation of the endpoint map,

$$\mathcal{E}_{Y_1,\dots,Y_m} = \sum_{p=1}^{\infty} \frac{1}{p!} Q^{(p)}(0) (\mathcal{E})^{\sqcup\!\sqcup p} =$$

$$= \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{j_1+\cdots+j_n=p} \frac{\partial Q^{j_1+\cdots+j_n}(0)}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} (\mathcal{E})_1^{\sqcup\!\sqcup j_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup (\mathcal{E})_n^{\sqcup\!\sqcup j_n} \tag{11}$$

where we used the notation $\mathcal{E} = \mathcal{E}_{X_1,\dots,X_n}$ for brevity. Here the shuffle product of the series is calculated termwise, and $a^{\sqcup\!\sqcup p} = a \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, a$ (p times), $a^{\sqcup\!\sqcup 0} = 1$.

III. STRUCTURES IN THE FREE ALGEBRA INDUCED BY THE CONTROL SYSTEM

A. Lie algebra of vector fields

Consider the (filtered) Lie algebra $\widehat{\mathcal{L}} = \sum_{k=1}^{\infty} \widehat{\mathcal{L}}^k$ generated by the set of vector fields $X_1, \dots, X_m$ as

$$\widehat{\mathcal{L}}^1 = \operatorname{Lin}\{X_1, \dots, X_m\}, \quad \widehat{\mathcal{L}}^{k+1} = [\widehat{\mathcal{L}}^1, \widehat{\mathcal{L}}^k], \quad k \ge 1,$$

where $[\cdot,\cdot]$ means the Lie bracket operation, $[X_i, X_j] = X_i X_j - X_j X_i$. The Lie algebra $\widehat{\mathcal{L}}$ encodes the information on the "small-time" behavior of the system, in particular, on its homogeneous approximation. Let us explain this point more specifically. For convenience, denote

$$\widehat{L}^k = \{V(0) : V \in \widehat{\mathcal{L}}^k\} \subset \mathbb{R}^n, \qquad k \ge 1.$$

In other words, $\widehat{L}^k$ is a subspace (in $\mathbb{R}^n$) of values at the origin of all vector fields from $\widehat{\mathcal{L}}^k$.

Then the subspace $\sum_{k=1}^{\infty} \widehat{L}^k$ defines the dimension of the *orbit* of the system through the origin. In particular, the orbit is of full dimension iff the *Rashevsky-Chow condition*

$$\sum_{k=1}^{\infty} \widehat{L}^k = \mathbb{R}^n \tag{12}$$

holds. For symmetric control systems like (1) this condition also implies local controllability what means that any point from a certain neighborhood of the origin can be reached from any other point from this neighborhood. Throughout of the paper, we suppose this property to be satisfied.

B. Core Lie subalgebra

For a given system of the form (1), consider the linear map $c : \mathcal{F} \to \mathbb{R}^n$ defined as

$$c(\eta_{i_1\dots i_k}) = X_{i_k} \cdots X_{i_1} E(0) = c_{i_1\dots i_k}$$

where $c_{i_1\dots i_k}$ are vector coefficients of series (9). Consider the subspaces of $\mathcal{L}$ of the form

$$\mathcal{P}^k = \{\ell \in \mathcal{L}^k : c(\ell) \in \widehat{L}^1 + \cdots + \widehat{L}^{k-1}\}, \quad k \ge 1,$$

and put

$$\mathcal{L}_{X_1,\dots,X_m} = \sum_{k=1}^{\infty} \mathcal{P}^k.$$

Lemma 1: $\mathcal{L}_{X_1,\dots,X_m}$ is a (graded) Lie subalgebra of $\mathcal{L}$.

Lemma 2: The Lie subalgebra $\mathcal{L}_{X_1,\dots,X_m}$ is invariant w.r.t. nonsingular changes of variables in the system.

Definition 8: We call $\mathcal{L}_{X_1,\dots,X_m}$ *the core Lie subalgebra* corresponding to system (1).

The core Lie subalgebra $\mathcal{L}_{X_1,\dots,X_m}$ is intrinsic, coordinate independent object. Just this subalgebra is responsible for the homogeneous approximation of the system. Let us explain the term "core Lie subalgebra". Let N be a degree of nonholonomy, i.e. the minimal integer such that $\sum_{k=1}^{N} \widehat{L}^k = \mathbb{R}^n$. First, notice that the map $c : \mathcal{L} \to \mathbb{R}^n$ induces the filtration in $\mathbb{R}^n$ defined by $\mathbb{R}^n = \widehat{L}^1 + \cdots + \widehat{L}^N$. Let us introduce the associated graded linear space. Namely, consider factor subspaces $[\mathcal{L}^1] = \widehat{L}^1$ and $[\mathcal{L}^i] = \widehat{L}^i / (\widehat{L}^1 + \cdots + \widehat{L}^{i-1})$, $i = 2, \dots, N$, then the direct sum $[\mathcal{L}^1] \dotplus \cdots \dotplus [\mathcal{L}^N]$ is a graded linear space isomorphic to the initial filtered space $\mathbb{R}^n$. Let us consider the induced graded homomorphism $g : \mathcal{L} \to \mathbb{R}^n$ defined for $\ell \in \mathcal{L}^i$ by $g(\ell) = [c(\ell)]$ if $i = 1, \dots, N$, and $g(\ell) = 0$ if $i \geq N+1$. Then $\mathcal{L}_{X_1,\dots,X_m}$ equals the core of g, i.e. $\mathcal{L}_{X_1,\dots,X_m} = \mathrm{Ker}(g)$. Hence, $\mathrm{Im}(g) = \mathbb{R}^n$ is isomorphic to $\mathcal{L}/\mathrm{Ker}(g)$.

Lemma 3: The subspace $\mathcal{L}_{X_1,\dots,X_m}$ is of codimension n in the space $\mathcal{L}$. Hence, if elements $\ell_1, \dots, \ell_n$ are such that $\mathcal{L} = \mathrm{Lin}\{\ell_1, \dots, \ell_n\} + \mathcal{L}_{X_1,\dots,X_m}$ then vectors $c(\ell_1), \dots, c(\ell_n)$ are linearly independent.

Some other properties of the core Lie subalgebra can be found in [5].

C. Left ideal generated by the system

The following concept proposed in [1] is closely connected with the core Lie subalgebra.

Definition 9: We say that

$$\mathcal{J}_{X_1,\dots,X_m} = (\mathcal{F} + \mathbb{R}) \vee \mathcal{L}_{X_1,\dots,X_m}$$

is the left ideal generated by the system (1).

Notice that, due to its definition, *the left ideal is graded*,

$$\mathcal{J}_{X_1,\dots,X_m} = \sum_{k=1}^{\infty} (\mathcal{J}_{X_1,\dots,X_m} \cap \mathcal{F}^k). \tag{13}$$

Moreover, it is invariant w.r.t. nonsingular changes of variables in the system what follows directly from Lemma 2. The following property of the left ideal is crucial for further considerations.

Lemma 4: If $a \in \mathcal{J}_{X_1,\dots,X_m} \cap \mathcal{F}^k$ then $c(a) \in c(\mathcal{F}^1 + \dots + \mathcal{F}^{k-1})$.

The next lemma uses the well known concept of the Poincaré-Birkhoff-Witt basis. Suppose $\{\ell_i\}_{i=1}^{\infty}$ is a basis of $\mathcal{L}$ which consists of homogeneous elements. Recall that, due to the Poincaré-Birkhoff-Witt Theorem, the set

$$\{\ell_{j_1} \vee \dots \vee \ell_{j_r} : 1 \leq j_1 \leq \dots \leq j_r,\ r \geq 1\} \tag{14}$$

forms a basis of $\mathcal{F}$. Let us return to our series. Recall that, due to Lemma 3, $\mathcal{L}_{X_1,\dots,X_m}$ is of codimension n in $\mathcal{L}$.

Below we use the following notations. Let $\{\ell_1, \dots, \ell_n\}$ be *any* set of homogeneous elements of $\mathcal{L}$ such that

$$\mathcal{L} = \mathrm{Lin}\{\ell_1, \dots, \ell_n\} + \mathcal{L}_{X_1,\dots,X_m}. \tag{15}$$

Assume that $\mathrm{ord}(\ell_i) \leq \mathrm{ord}(\ell_j)$ if $1 \leq i < j \leq n$. Denote by $\{\ell_j\}_{j=n+1}^{\infty}$ any (homogeneous) basis of $\mathcal{L}_{X_1,\dots,X_m}$.

Lemma 5: The set

$$\{\ell_{j_1} \vee \dots \vee \ell_{j_r} : 1 \leq j_1 \leq \dots \leq j_r,\ r \geq 1,\ j_r \geq n+1\} \tag{16}$$

forms a basis of the left ideal $\mathcal{J}_{X_1,\dots,X_m}$.

Lemma 6: $\mathcal{J}_{X_1,\dots,X_m} \cap \mathcal{L}^k = \mathcal{P}^k$ for any $k \geq 1$ and, therefore,

$$\mathcal{J}_{X_1,\dots,X_m} \cap \mathcal{L} = \mathcal{L}_{X_1,\dots,X_m}.$$

As a corollary, we get that two structures induced by the control system, $\mathcal{L}_{X_1,\dots,X_m}$ and $\mathcal{J}_{X_1,\dots,X_m}$, define each other uniquely.

Below we introduce the inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{F}$ assuming that the basis $\{\eta_{i_1 \dots i_k} : k \geq 1, 1 \leq i_1, \dots, i_k \leq m\}$ is orthonormal. Denote by $\widetilde{\ell}_i$ *the orthogonal projection of* ℓ_i *on the subspace* $\mathcal{J}^{\perp}_{X_1,\dots,X_m}$.

The following lemma follows from the remarkable theorem by R. Ree [6] on a connection of Lie elements and shuffles.

Lemma 7: The set

$$\{\widetilde{\ell}_1^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \dots \sqcup\!\sqcup \widetilde{\ell}_n^{\sqcup\!\sqcup q_n} :\ q_1 + \dots + q_n \geq 1\}$$

forms a basis of $\mathcal{J}^{\perp}_{X_1,\dots,X_m}$.

IV. HOMOGENEOUS APPROXIMATION FROM THE ALGEBRAIC VIEWPOINT

A. Definition of homogeneous approximation

The concept of a homogeneous approximation is one of the central ones in the nonlinear control theory [7], [8], [9], [10], [11], [12]. Let us give a definition of a homogeneous approximation in terms of the endpoint map.

Definition 10: Suppose a bracket generating affine system of the form (1) is given. The (bracket generating) system

$$\dot{z} = \sum_{i=1}^{m} u_i Z_i(z), \quad z \in \mathbb{R}^n,\ u_i \in \mathbb{R}, \tag{17}$$

with real analytic $Z_1(z), \dots, Z_m(z)$ is called a homogeneous approximation for system (1) if

(i) its endpoint map $\mathcal{E}_{Z_1,\dots,Z_m}$ is homogeneous,

$$\mathcal{E}_{Z_1,\dots,Z_m}(\theta, u) = H_\theta(\mathcal{E}_{Z_1,\dots,Z_m}(1, u)), \quad \theta > 0,\ u \in B_1,$$

where $H_\theta(z) = (\theta^{w_1} z_1, \dots, \theta^{w_n} z_n)$ is a dilation and $1 \leq w_1 \leq \dots \leq w_n$ are some integers;

(ii) there exists a real analytic change of variables $y = Q(x)$ in the initial system ($Q(0) = 0$, $\det Q'(0) \neq 0$) such that $\mathcal{E}_{Z_1,\dots,Z_m}$ approximates the endpoint map of the initial system in the new coordinates; namely, for any $u \in B_1$

$$H_\theta^{-1}(Q(\mathcal{E}_{X_1,\dots,X_m}(\theta, u)) - \mathcal{E}_{Z_1,\dots,Z_m}(\theta, u)) \to 0$$

as $\theta \to 0$.

B. A principal part of the series

For the further arguments, it is convenient to introduce the dual basis for (14). Let us re-write the basis (14) as

$$\{\ell_{j_1}^{p_1} \vee \dots \vee \ell_{j_s}^{p_s} : s \geq 1,\ j_1 < \dots < j_s,\ p_i \in \mathbb{N}\} \tag{18}$$

where $\ell^p = \ell \vee \dots \vee \ell$ (p times). Suppose

$$\{d_{i_1 \dots i_r}^{q_1 \dots q_r} : r \geq 1,\ i_1 < \dots < i_r,\ q_i \in \mathbb{N}\}$$

is the *dual basis* for (18), i.e. $\langle \ell_{j_1}^{p_1} \vee \dots \vee \ell_{j_s}^{p_s}, d_{i_1 \dots i_r}^{q_1 \dots q_r} \rangle = 1$ if $s = r$, $j_k = i_k$, $p_k = q_k$, $k = 1, \dots, s$, and $= 0$ otherwise. Then, as it was proven in [13], $d_{i_1 \dots i_r}^{q_1 \dots q_r} = \frac{1}{q_1! \dots q_r!} d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \dots \sqcup\!\sqcup d_{i_r}^{\sqcup\!\sqcup q_r}$ where $d_i = d_i^1$. This dual basis gives us another basis of $\mathcal{J}^{\perp}_{X_1,\dots,X_m}$.

Lemma 8: The set

$$\{d_1^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \dots \sqcup\!\sqcup d_n^{\sqcup\!\sqcup q_n} :\ q_1 + \dots + q_n \geq 1\} \tag{19}$$

forms a basis of $\mathcal{J}^{\perp}_{X_1,\dots,X_m}$.

Now let us re-expand the series $\mathcal{E}_{X_1,\dots,X_m}$ w.r.t. the dual basis. We get

$$\mathcal{E}_{X_1,\dots,X_m} = \mathcal{S} + \mathcal{T}, \tag{20}$$

where

$$\mathcal{S} = \sum_{1\le i_1<\cdots<i_r\le n} \frac{1}{q_1!\dots q_r!}\, c(\ell_{i_1}^{q_1}\vee\cdots\vee\ell_{i_r}^{q_r})\, d_{i_1}^{\sqcup q_1}\sqcup\cdots\sqcup d_{i_r}^{\sqcup q_r},$$

$$\mathcal{T} = \sum_{\substack{1\le i_1<\cdots<i_r\\ i_r\ge n+1}} \frac{1}{q_1!\dots q_r!}\, c(\ell_{i_1}^{q_1}\vee\cdots\vee\ell_{i_r}^{q_r})\, d_{i_1}^{\sqcup q_1}\sqcup\cdots\sqcup d_{i_r}^{\sqcup q_r}.$$

Notice that $\mathcal{T}$ is a sum of elements of $\mathcal{J}_{X_1,\dots,X_m}$.

Let us return to the transformation of the endpoint map. Suppose $y = Q(x)$ is a nonsingular change of variables, then

$$\mathcal{E}_{Y_1,\dots,Y_m} = Q(\mathcal{E}_{X_1,\dots,X_m}) = \sum_{p=1}^{\infty} \frac{1}{p!} Q^{(p)}(0)(\mathcal{S}+\mathcal{T})^{\sqcup p} = \\ = Q(\mathcal{S}) + \mathcal{T}',$$

where

$$\mathcal{T}' = \sum_{j\ge 1} \frac{1}{p!j!} Q^{(p+j)}(0)(\mathcal{S})^{\sqcup p}\sqcup(\mathcal{T})^{\sqcup j}$$

is a sum of elements of $\mathcal{J}_{X_1,\dots,X_m}$. Due to Lemma 4, this means that the "main part" of the series $\mathcal{E}_{Y_1,\dots,Y_m}$ (determining the homogeneous approximation) is contained in the series $Q(\mathcal{S})$. Hence, it is sufficient to find a homogeneous approximation for the series $\mathcal{S}$. It can be shown that there exists a change of variables $y = Q(x)$ such that

$$(Q(\mathcal{S}))_i = d_i + \text{"elements of order} > w_i\text{"}$$

where $w_i = \operatorname{ord}(d_i) = \operatorname{ord}(\ell_i)$, $i = 1,\dots,n$. This means that d_i are *principal terms* for $(Q(\mathcal{S}))_i$ and, therefore, for $(\mathcal{E}_{Y_1,\dots,Y_m})_i$. So, we get the following theorem.

Theorem 2: For any (bracket generating) system (1) there exist a nonsingular real analytic change of variables $y = Q(x)$ such that series (11) (which describes the endpoint map of the system in the new coordinates) has the form

$$(\mathcal{E}_{Y_1,\dots,Y_m})_i = (\sum_{p=1}^{\infty} \tfrac{1}{p!} Q^{(p)}(0)(\mathcal{E}_{X_1,\dots,X_n})^{\sqcup p})_i = d_i + \rho_i, \tag{21}$$

where $\rho_i \in \sum_{j=w_i+1}^{\infty} \mathcal{F}^j$, $w_i = \operatorname{ord}(\ell_i)$, $i = 1,\dots,n$.

Now recall that elements d_i belong to $\mathcal{J}^{\perp}_{X_1,\dots,X_m}$, hence, can be expressed as polynomials of elements of basis (19). For practical purposes, it is more convenient to use this form of a principal part.

Theorem 3: There exist such coordinates $y = Q(x)$ that

$$(\mathcal{E}_{Y_1,\dots,Y_n})_i = \widetilde{\ell}_i + \rho_i, \quad i = 1,\dots,n, \tag{22}$$

where $\rho_i \in \sum_{j=w_i+1}^{\infty} \mathcal{F}^j$, $w_i = \operatorname{ord}(\ell_i)$, $i = 1,\dots,n$.

A complete description of all such coordinates $y = Q(x)$ (so-called "privileged coordinates" [10]) can be found in [14]. In the very partial case of free systems a close approach was proposed in [15].

Theorem 3 means that the principal part of the series $\mathcal{E}_{X_1,\dots,X_n}$ can be constructed in a purely algebraic way by the "standard" procedure of finding of the orthogonal projection of elements $\ell_1,\dots,\ell_n$.

C. Algebraic definition of homogeneous approximation

Now we are ready to give an algebraic analog of Definition 10.

Lemma 9: System (17) is a homogeneous approximation for system (1) in the sense of Definition 10 if and only if its series is of the form $(\mathcal{E}_{Z_1,\dots,Z_m})_i = P_i(d_1,\dots,d_n)$, $i = 1,\dots,n$, where P is a polynomial vector function with nonsingular linear part and P_i are such that $P_i(d_1,\dots,d_n) \in \mathcal{F}^{w_i}$ (where $w_i = \operatorname{ord}(\ell_i)$).

Thus, the series of a system which is a homogeneous approximation is defined, in essence, uniquely (up to a polynomial change of variables preserving homogeneity). One can show that series $\mathcal{E}_{Z_1,\dots,Z_m}$ described in Lemma 9 is realizable [16], [17], so, the approximating system exists and, in essence, is defined uniquely.

It can be shown that $\mathcal{L}_{Z_1,\dots,Z_m} = \mathcal{L}_{X_1,\dots,X_m}$. Moreover, Definition 10 is equivalent to the following "algebraic" coordinate-free definition of a homogeneous approximation.

Definition 11: Suppose two bracket generating symmetric systems (1) and (17) are given. System (17) is called a homogeneous approximation for (1) if

(i) $c_{Z_1,\dots,Z_m}(\mathcal{L}_{Z_1,\dots,Z_m}) = 0$;
(ii) $\mathcal{L}_{X_1,\dots,X_m} = \mathcal{L}_{Z_1,\dots,Z_m}$.

Emphasize that (i) and (ii) mean the homogeneity and the approximation properties respectively.

Taking into account the connection between the core Lie subalgebra $\mathcal{L}_{Z_1,\dots,Z_m}$ and the left ideal $\mathcal{J}_{Z_1,\dots,Z_m}$ we get that conditions (i) and (ii) can be substituted by the equivalent conditions

(i′) $c_{Z_1,\dots,Z_m}(\mathcal{J}_{Z_1,\dots,Z_m}) = 0$;
(ii′) $\mathcal{J}_{X_1,\dots,X_m} = \mathcal{J}_{Z_1,\dots,Z_m}$.

V. TIME OPTIMALITY

A. Time-optimal controls

From now on, we consider the time-optimal control problem for system (1) of the form

$$\begin{gathered} \dot{x} = \sum_{i=1}^{m} u_i X_i(x),\ x(0) = 0,\ x(\theta) = x^0, \\ \sum_{i=1}^{m} u_i^2(t) \le 1,\ t\in[0,\theta] \text{ a.e.}, \quad \theta \to \min. \end{gathered} \tag{23}$$

Definition 12: We say that a pair $(\theta_{x^0}, u_{x^0}) \in \mathbb{R}^+ \times B_1$ is a solution of time-optimal control problem (23) if θ_{x^0} is the optimal time and $v(t) = u_{x^0}\left(\frac{t}{\theta_{x^0}}\right)$, $t \in [0,\theta_{x^0}]$ is a time-optimal control for (23).

Our first observation concerns the character of the optimal control.

Lemma 10: Consider a time-optimal control problem of the form (23) where vector fields $X_1,\dots,X_m$ are real analytic in a neighborhood of the origin, and suppose that the Rashevsky-Chow condition (12) holds. Then there exists a neighborhood of the origin $U(0)$ such that for any $x^0 \in U(0)$ any optimal control $u(t) = u_{x^0}(t)$ satisfies the condition

$$\sum_{i=1}^{m} u_i^2(t) = 1 \ \text{ a.e.},\ t \in [0,1]. \tag{24}$$

Let (θ_{x^0}, u_{x^0}) be a solution of problem (23). Notice that the function $u(t) = \theta_{x^0} u_{x^0}(t\theta_{x^0})$, $t \in [0,1]$, minimizes also the "length functional" $\ell(u) = \int_0^1 \sqrt{\sum_{i=1}^m u_i^2(t)}dt$ and the "energy functional" $J(u) = \int_0^1 \sum_{i=1}^m u_i^2(t)dt$. Recall that the length functional is closely connected with a concept of the sub-Riemannian metrics [10].

B. Approximation in the sense of time optimality

In this subsection we introduce the concept of approximation in the sense of time optimality following the ideas of [1] and show the connection with the homogeneous approximation considered above.

Definition 13: Let vector fields $X_1(x), \ldots, X_m(x)$ and $Y_1(x), \ldots, Y_m(x)$ be real analytic in a neighborhood of the origin. Suppose there exists an open domain $\Omega \subset \mathbb{R}^n \backslash \{0\}$, $0 \in \overline{\Omega}$, such that the time-optimal control problem

$$\begin{aligned} &\dot{x} = \sum_{i=1}^m u_i Y_i(x),\ x(0) = 0,\ x(\theta) = x^0, \\ &\sum_{i=1}^m u_i^2(t) \le 1,\ t \in [0,\theta] \text{ a.e.}, \quad \theta \to \min \end{aligned} \tag{25}$$

has the unique solution $(\theta^*_{x^0}, u^*_{x^0})$ for any $x^0 \in \Omega$. Denote by $\{(\theta_{x^0}, u_{x^0}) : u_{x^0} \in U_{x^0}\}$, the set of solutions of time-optimal control problem (23). We say that *the time-optimal control problem (25) approximates the time-optimal control problem (23) (in the domain* Ω*)* if the exists a nonsingular transformation Φ of a neighborhood of the origin of $\mathbb{R}^n$, $\Phi(0) = 0$, such that for any $u_{\Phi(x^0)} \in U_{\Phi(x^0)}$

$$\frac{\theta_{\Phi(x^0)}}{\theta^*_{x^0}} \to 1,$$

$$\frac{1}{\theta}\int_0^\theta |v_{\Phi(x^0)}(t) - v^*_{x^0}(t)|dt \to 0 \quad \text{as } x^0 \to 0,\ x^0 \in \Omega,$$

where $v_{\Phi(x^0)}(t) = u_{\Phi(x^0)}(\frac{t}{\theta_{\Phi(x^0)}})$, $v^*_{x^0}(t) = u^*_{x^0}(\frac{t}{\theta^*_{x^0}})$ are optimal controls for (23) and (25) and $\theta = \min\{\theta^*_{x^0}, \theta_{\Phi(x^0)}\}$.

Thus, the definition means that after a certain change of variables in system (23) the optimal times and optimal controls of problems (23) and (25) become asymptotically equivalent as functions of the end point. Notice that the equivalence of optimal times for a system and its homogeneous approximation was, in essence, studied in [10].

Let us consider the time-optimal control problem for system (17) which is a homogeneous approximation for a system of the form (1) in the sense of Definition 10. Without loss of generality suppose $(Z_i(z))_j$ are polynomials, namely, $(Z_i(z))_j = \sum \alpha^{ij}_{s_1\ldots s_n} z_1^{s_1} \cdots z_n^{s_n}$ where sum is taken over such $(s_1, \ldots, s_n)$ that $s_1 w_1 + \cdots + s_n w_n = w_j - 1$. As above, let H_ε be a dilation, $H_\varepsilon(x) = (\varepsilon^{w_1} x_1, \ldots, \varepsilon^{w_n} x_n)$. Denote by $(\theta^*_{z^0}, u^*_{z^0})$ a solution of the time-optimal control problem

$$\begin{aligned} &\dot{z} = \sum_{i=1}^m u_i Z_i(z),\ z(0) = z^0,\ z(\theta) = 0, \\ &\sum_{i=1}^m u_i^2(t) \le 1,\ t \in [0,\theta] \text{ a.e.},\ \theta \to \min. \end{aligned} \tag{26}$$

Then, due to homogeneity, $\theta^*_{H_\varepsilon(x)} = \varepsilon\theta^*_x$ and $u^*_{H_\varepsilon(x)}(t) = u^*_x(t)$, $t \in [0,1]$. Hence, if some properties concerning the optimal time and control (such that existence, uniqueness etc.) are satisfied in some domain Ω then they are also true in the domain $H_\varepsilon(\Omega)$. Thus, without loss of generality we assume the domain Ω is "pseudo-conic", i.e. if $x \in \Omega$ then $H_\varepsilon(x) \in \Omega$ for any $\varepsilon > 0$.

The main result of this paper is the following theorem on approximation which states that the concept of the homogeneous approximation is closely connected with the approximation of time-optimal control problems.

Theorem 4: Let system (17) be a homogeneous approximation for system (1). Let there exist a (pseudo-conic) open domain $\Omega \subset \mathbb{R}^n \backslash \{0\}$ such that $0 \in \overline{\Omega}$ and for any $z^0 \in \Omega$ the solution $(\theta^*_{z^0}, u^*_{z^0})$ of the time-optimal control problem (26) is unique. Then there exists the set of embedded domains $\Omega(\delta)$, $\delta > 0$, such that $\Omega(\delta_1) \subset \Omega(\delta_2)$ if $\delta_1 > \delta_2 > 0$ and $\Omega = \cup_{\delta > 0}\Omega(\delta)$, in each of which the time optimal control problem (26) approximates the time optimal control problem (23).

Thus, the homogeneous approximation of a symmetric control system also approximates it in the sense of time optimality.

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