

# Matrix Approach to Boolean Calculus

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**Abstract**—Using semi-tensor product of matrices and the matrix expression of logic, formulas for calculating Boolean derivatives are obtained. Using this form, the solvability of Boolean algebraic equations and Boolean differential equations is considered. Its application to fault detection of combinational circuits is investigated. Then we define the Boolean integrals as the inverse of the Boolean derivative in certain sense. Two kinds of integrals are proposed. The inverse of a partial derivative with respect to  $x_i$  is called the  $i$ th primitive function. The inverse of a differential form is called the indefinite integral. A necessary and sufficient condition for the existence of the indefinite integral is proved.

## I. INTRODUCTION

Right after G. Boole invented an algebra in 1847, which is lately called the Boolean algebra, an effort rose, which attempts to establish Boolean analogues of concepts and results from Calculus. The first version of Boolean Differential Calculus was proposed by Daniell in 1917 [14]. Some forty years later after Shannon proposed the switching algebra in the evaluation of switching circuit designing, it was discovered that the partial derivatives of Boolean functions are particularly useful in switching theory [23], [3]. Since then, the Boolean derivative has been developed quickly, both in view of applications and for its own algebraic interest [22], [5], [34], [27], [30], [15].

There are several definitions on Boolean derivative, we adopt the common definition of Boolean derivative, which can be found, for instance, in [33], [29]. A general definition and basic properties and some applications can be found in [29]. The fundamental requirements and satisfactory of Boolean derivatives are discussed in [25].

Many applications of Boolean derivatives have been reported. The applications include control of Boolean networks [20], synthesis of discrete event systems [26], logical circuit analysis [5], [19], asynchronous circuit design [28], image edge detection [2], selection probabilities of stack filters [16], cellular automata and finite state machine [21], [32], etc. These evidence that Boolean derivative is a useful tool.

Recently, a new matrix product, called the semi-tensor product, has been proposed and it has been successfully applied to the analysis and control of Boolean networks [13]. We also refer to [6], [7], [8], [9], [10], [11], [12] for its applications to the topological structure, controllability, observability, stabilization, disturbance decoupling, etc. of

Boolean (control) networks. The key point of this approach is to convert a logical expression into an algebraic form, and then the known methods for analyzing conventional static and/or dynamic systems are applicable to logical (dynamic) systems. This new technique is the crucial tool for the investigation in this paper.

The first interesting topic is the calculation of Boolean derivatives [35], [18]. Using semi-tensor product, [19] attempts to provide the general formulas for calculating the Boolean derivatives. Following [19], we provide a formula for calculating the structure matrix of a Boolean derivative. It is essentially equivalent to the fast implementation presented by [1], which is concerned with the  $j$ th partial derivative transform, using matrix multiplication [2]. But our vector form is convenient in later use. Then the Boolean algebraic and differential equations are investigated. Algorithms are provided to solve the equations. As an application, the fault detection of combinational circuits is investigated.

Another interesting topic is the counterpart of the Boolean derivative, that is, the Boolean integral. There are much less literatures on Boolean integral. [31] provides some interesting insights for Boolean integral. But it seems to the author that there is no convergent definition yet. Using the formula for calculating Boolean derivatives, we formulate the Boolean integral, as the inverse of the Boolean derivative. Two kinds of the inverses of Boolean derivatives are defined. The inverse of a partial derivative, is called the primitive function; the inverse of a differential form, called the indefinite integral. Using the derivative algorithm, the indefinite integral can be calculated easily. The uniqueness of the indefinite integral (up to a complement equivalence) is proved.

The rest of this paper is organized as follows: Section 2 consists of some preliminaries. Section 3 discusses the calculation of Boolean derivatives. Some easily computable formulas are developed. Section 4 is devoted to solving Boolean algebraic equations and Boolean differential equations. The Boolean integral is proposed in Section 5. Section 6 is a brief conclusion.

## II. PRELIMINARIES

First, we introduce some notations.

- $\mathcal{M}_{m \times n}$ : the set of  $m \times n$  real matrices.
- $\mathbf{1}_{m \times n}$  ( $\mathbf{0}_{m \times n}$ ): a matrix in  $\mathcal{M}_{m \times n}$  with all entries equal 1 (correspondingly, 0). If no ambiguity is possible, we simply use  $\mathbf{1}_n$  for  $\mathbf{1}_{n \times n}$   $\mathbf{0}$  for  $\mathbf{0}_n$ , or  $\mathbf{0}_n^T$ , or  $\mathbf{0}_{m \times n}$ .
- $\mathcal{D} = \{1, 0\}$ .
- $\delta_n^k$  is the  $k$ -th column of the identity matrix  $I_n$ .
- $\Delta_n := \{\delta_n^1, \dots, \delta_n^n\}$ . For compactness,  $\Delta := \Delta_2$ .

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TABLE I  
STRUCTURE MATRICES OF OPERATORS

Operator	Structure Matrix
$\neg$	$M_{\neg} = \delta_2 [2 \ 1]$
$\wedge$	$M_{\wedge} = \delta_2 [1 \ 2 \ 2 \ 2]$
$\vee$	$M_{\vee} = \delta_2 [1 \ 1 \ 1 \ 2]$
$\rightarrow$	$M_{\rightarrow} = \delta_2 [1 \ 2 \ 1 \ 1]$
$\leftrightarrow$	$M_{\leftrightarrow} = \delta_2 [1 \ 2 \ 2 \ 1]$
$\vee(\text{or } \oplus)$	$M_{\oplus} = \delta_2 [2 \ 1 \ 1 \ 2]$

- $\text{Col}_i(A)$  ( $\text{Row}_i(A)$ ) is the  $i$ -th column ( $i$ -th row) of a matrix  $A$ , the set of all the columns (rows) of  $A$  is denoted by  $\text{Col}(A)$  ( $\text{Row}(A)$ ).
- A matrix  $L \in M_{n \times m}$  is called a logical matrix if its columns,  $\text{Col}(L) \subset \Delta_n$ . The set of  $n \times m$  logical matrices is denoted by  $\mathcal{L}_{n \times m}$ .
- Let  $L \in \mathcal{L}_{n \times m}$ . Then

$$L = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_m}].$$

For the sake of brevity, it is denoted as

$$L = \delta_n [i_1, i_2, \dots, i_m].$$

- A matrix  $A = (a_{i,j}) \in \mathcal{M}_{m \times n}$  is called a Boolean matrix if its entries  $a_{i,j} \in \mathcal{D}$ . The set of  $m \times n$  Boolean matrices is denoted by  $\mathcal{B}_{m \times n}$ . Let  $A = (a_{i,j}), B = (b_{i,j}) \in \mathcal{B}_{m \times n}$ . Then  $\neg A = (-a_{i,j})$ ; and  $A \wedge B = (a_{i,j} \wedge b_{i,j})$ , etc.

*Definition 2.1:* [9], [13] Let  $M \in \mathcal{M}_{m \times n}$  and  $N \in \mathcal{M}_{p \times q}$ . The semi-tensor product of matrices, denoted by  $M \ltimes N$ , is defined as

$$M \ltimes N := (M \otimes I_{s/n}) (N \otimes I_{s/p}), \quad (1)$$

where  $s = \text{lcm}\{n, p\}$  is the least common multiple of  $n$  and  $p$ ,  $\otimes$  is the Kronecher product of matrices.

*Remark 2.2:* Throughout this paper, unless else product symbol is used, the matrix product is assumed to be semi-tensor product, which contains the conventional matrix product as its particular case when  $n = p$ . Hence, the symbol  $\ltimes$  can be omitted.

*Definition 2.3:* A  $k$ -ary logical function (or operator) is a mapping  $f : \mathcal{D}^k \rightarrow \mathcal{D}$ . It is commonly expressed as  $f(x_1, \dots, x_k)$ , where  $x_i \in \mathcal{D}, i = 1, \dots, k$ .

To use the matrix expression of logic, we identify  $1 \sim \delta_2^1$  and  $0 \sim \delta_2^2$ . Under this vector form, a logical function  $f : \mathcal{D}^k \rightarrow \mathcal{D}$  becomes a function  $f : \Delta^k \rightarrow \Delta$ , (or equivalently,  $f : \Delta_{2^k} \rightarrow \Delta$ ).

*Theorem 2.4:* [13] Let  $f : \mathcal{D}^k \rightarrow \mathcal{D}$ . Then there exists a unique logical matrix  $M_f \in \mathcal{L}_{2 \times 2^k}$ , called the structure matrix of  $f$ , such that in vector form we have

$$f(x_1, \dots, x_k) = M_f x, \quad (2)$$

where  $x = \ltimes_{i=1}^k x_i \in \Delta_{2^k}$ .

For convenience, we give the structure matrices of some commonly used logical operators in Table I.

*Remark 2.5:* Let  $M_f \in \mathcal{L}_{2 \times 2^k}$  be the structure matrix of  $f : \mathcal{D}^k \rightarrow \mathcal{D}$ . Then  $\text{Row}_1(M_f)$  is the truth table of  $f$  (in row form) and  $\text{Row}_2(M_f) = \neg \text{Row}_1(M_f)$ . We simply denote

$$m_f := \text{Row}_1^T(M_f).$$

Finally, we need a lemma, which is useful in the sequel.

*Lemma 2.6:* [13]

- 1) Let  $x \in \mathbb{R}^m$  and  $y \in \mathbb{R}^n$  be two column vectors. Then there exist a unique swap matrix  $W_{[n,m]} \in \mathcal{M}_{mn \times mn}$  such that

$$W_{[m,n]} xy = yx. \quad (3)$$

- 2) Let  $x \in \mathbb{R}^t$  and  $A$  is a given matrix. Then

$$xA = (I_t \otimes A)x. \quad (4)$$

- 3) Let  $x = \ltimes_{i=1}^k x_i$ . Then

$$x^2 = M_r^{2^k} x, \quad (5)$$

where

$$M_r^n := \text{diag}[\delta_n^1 \delta_n^2 \dots \delta_n^n].$$

A Boolean algebra on  $\mathcal{D}$  is a quadruple  $(\mathcal{D}, +_{\mathcal{B}}, \times_{\mathcal{B}}, \neg)$ . We refer to [24] for an elementary definition. For our purpose, we only consider Galois algebra in which  $+_{\mathcal{B}} = \oplus$  and  $\times_{\mathcal{B}} = \wedge$ . We firstly define the Boolean product of matrices under such algebra.

*Definition 2.7:* For a Boolean algebra  $\mathcal{B} = (\mathcal{D}, \oplus, \wedge, \neg)$ , we define

- 1) The Boolean product of  $A = (a_{i,j}) \in \mathcal{B}_{m \times n}$  and  $B = (b_{i,j}) \in \mathcal{B}_{n \times s}$  is defined as

$$A \ltimes_{\mathcal{B}} B := (c_{i,j}) \in \mathcal{B}_{m \times s}, \quad (6)$$

where

$$c_{i,j} = a_{i,1} \wedge b_{1,j} \oplus a_{i,2} \wedge b_{2,j} \oplus \dots \oplus a_{i,n} \wedge b_{n,j}.$$

- 2) The Boolean semi-tensor product of  $A = (a_{i,j}) \in \mathcal{B}_{m \times n}$  and  $B = (b_{i,j}) \in \mathcal{B}_{p \times q}$  is defined as

$$A \ltimes_{\mathcal{B}} B = (A \otimes I_{s/n}) \ltimes_{\mathcal{B}} (A \otimes I_{s/p}), \quad (7)$$

where  $s = \text{lcm}(n, p)$ .

Note that (6) is a particular case of (7). Properties of Boolean matrix, like commutative law, associative law and distributive law, etc. are immediate consequences by the definition [13].

### III. BOOLEAN DERIVATIVES

The Boolean derivative in this paper is defined as [4]

*Definition 3.1:* Let  $f(x_1, \dots, x_n) : \mathcal{D}^n \rightarrow \mathcal{D}$  be a logical function.

- 1) The Boolean derivative of  $f$  with respect to  $x_i$  is defined as

$$\frac{\partial f}{\partial x_i} = f(x_1, \dots, x_i, \dots, x_n) \oplus f(x_1, \dots, \neg x_i, \dots, x_n). \quad (8)$$

- 2) The higher order derivative of  $f$  with respect to  $x_{i_1}, \dots, x_{i_k}$  is defined recursively as

$$\frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}} = \frac{\partial}{\partial x_{i_1}} \left( \frac{\partial}{\partial x_{i_2}} \left( \dots \left( \frac{\partial f}{\partial x_{i_k}} \right) \right) \right). \quad (9)$$

We cite some basic properties in the following.

*Proposition 3.2:* [33]

1)  $\frac{\partial f}{\partial x_i}$  is independent of  $x_i$ , and hence

$$\frac{\partial^2 f}{\partial x_i^2} = 0.$$

2)

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}. \quad (10)$$

3)

$$\frac{\partial(f_1 \oplus f_2)}{\partial x_i} = \frac{\partial f_1}{\partial x_i} \oplus \frac{\partial f_2}{\partial x_i}. \quad (11)$$

4)

$$\frac{\partial(f_1 f_2)}{\partial x_i} = \frac{\partial f_1}{\partial x_i} f_2 \oplus f_1 \frac{\partial f_2}{\partial x_i} \oplus \frac{\partial f_1}{\partial x_i} \frac{\partial f_2}{\partial x_i}. \quad (12)$$

5) Denote  $\bar{f} := \neg f$ , and  $\bar{x} := \neg x$ , then

$$\frac{\partial \bar{f}}{\partial x_i} = \frac{\partial f}{\partial x_i}, \quad \text{and} \quad \frac{\partial f}{\partial \bar{x}_i} = \frac{\partial f}{\partial x_i}. \quad (13)$$

Using the vector form of logical expression as introduced in Section 2, we can easily obtain the matrix expression of  $\frac{\partial f}{\partial x_i}$ , denoted by  $M_{\partial_i f}$ . Denote by  $M_f$  the structure matrix of  $f$  and  $x := \times_{i=1}^n x_i$ . Using (8), we have

$$\frac{\partial f}{\partial x_i} = M_{\partial_i f} x = M_f x \oplus M_f x_1 \cdots \bar{x}_i \cdots x_n. \quad (14)$$

Then using Lemma 2.6 to simplify the right hand side of (14), it is easy to have that [19]

$$M_{\partial_i f} = M_{\oplus} M_f [I_{2^n} \otimes M_f (I_{2^{i-1}} \otimes M_{\neg})] M_r^{2^n}. \quad (15)$$

Then the higher order derivatives can also be calculated recursively [19].

In the following we shall give an explicit form of the structure matrices of the derivatives. Consider the structure matrix of  $g(x_1, \dots, x_n) := f(x_1, \dots, \bar{x}_i, \dots, x_n)$ . Assume the structure matrices of  $f$  and  $g$  are  $M_f$  and  $M_g$  respectively. Using (4), we have

$$\begin{aligned} M_g x &= M_f x_1 \cdots M_{\neg} x_i \cdots x_n \\ &= M_f (I_{2^{i-1}} \otimes M_{\neg}) x. \end{aligned}$$

That is,

$$M_g = M_f (I_{2^{i-1}} \otimes M_{\neg}). \quad (16)$$

The following proposition is obvious.

*Proposition 3.3:* Assume  $f(x_1, \dots, x_n)$  and  $g(x_1, \dots, x_n)$  have their truth tables as  $m_f, m_g \in \mathcal{B}_{2^n}$  respectively, and  $\sigma$  is a binary logical operator. Then

$$m_{f \sigma g} = m_f \sigma m_g. \quad (17)$$

Using Remark 2.5 and Proposition 3.3, we have

$$m_{\partial_i f}^T = m_f^T \oplus m_f^T (I_{2^{i-1}} \otimes M_{\neg}). \quad (18)$$

Using the distributive law, we can calculate that

$$\begin{aligned} m_{\partial_i f}^T &= m_f^T \oplus m_f^T (I_{2^{i-1}} \otimes M_{\neg}) \\ &= m_f^T \times_{\mathcal{B}} I_{2^i} \oplus m_f^T \times_{\oplus} (I_{2^{i-1}} \otimes M_{\neg}) \\ &= m_f^T \times_{\mathcal{B}} (I_{2^i} \oplus (I_{2^{i-1}} \otimes M_{\neg})) \\ &= m_f^T \times_{\mathcal{B}} (I_{2^{i-1}} \otimes (I_2 \oplus M_{\neg})) \\ &= m_f^T \times_{\mathcal{B}} (I_{2^{i-1}} \otimes \mathbf{1}_{2 \times 2}). \end{aligned}$$

We conclude that

*Theorem 3.4:* Let  $f(x_1, \dots, x_n)$  be a Boolean function with structure matrix  $M_f$ . Then The structure matrix of  $\frac{\partial f}{\partial x_i}$ , denoted by  $M_{\partial_i f}$ , is

$$M_{\partial_i f} = \begin{bmatrix} \text{Row}_1(M_f) \times_{\mathcal{B}} \Xi_n^i \\ \neg \text{Row}_1(M_f) \times_{\mathcal{B}} \Xi_n^i \end{bmatrix} \quad (19)$$

where

$$\Xi_n^i = I_{2^{i-1}} \otimes \mathbf{1}_{2 \times 2}.$$

Hence, in vector form,

$$\frac{\partial f}{\partial x_i} = M_{\partial_i f} x, \quad (20)$$

where  $x = \times_{i=1}^n x_i$ . Moreover,

$$m_{\partial_i f} = [\Xi_n^i]^T m_f. \quad (21)$$

As we know that  $\frac{\partial f}{\partial x_i}$  is independent of  $x_i$ , so one may be interested in an alternative expression as

$$\frac{\partial f}{\partial x_i} = M_{\partial_{[i]} f} x_1 \cdots x_{i-1} \hat{x}_i x_{i+1} \cdots x_n, \quad (22)$$

where notation " $\hat{x}_i$ " means  $x_i$  is omitted.

To calculate  $M_{\partial_{[i]} f}$ , we dividing  $M_{\partial_i f}$  into  $2^i$  equal blocks as

$$M_{\partial_i f} = [C_1 \ C_2 \ \cdots \ C_{2^i}].$$

One sees easily that to get  $M_{\partial_{[i]} f}$  from  $M_{\partial_i f}$ , we need only to pick out all odd (or even) blocks. It can be done by right-multiply

$$\left( I_{2^{i-1}} \otimes \begin{bmatrix} I_{2^{n-i}} \\ \mathbf{0}_{2^{n-i} \times 2^{n-i}} \end{bmatrix} \right).$$

That is,

$$M_{\partial_{[i]} f} = \begin{bmatrix} \text{Row}_1(M_f) \times_{\mathcal{B}} [\Psi_n^i]^T \\ \neg \text{Row}_1(M_f) \times_{\mathcal{B}} [\Psi_n^i]^T \end{bmatrix} \quad (23)$$

where

$$\begin{aligned} \Psi_n^i &= (I_{2^{i-1}} \otimes [I_{2^{n-i}} \ \mathbf{0}_{2^{n-i} \times 2^{n-i}}]) \times_{\mathcal{B}} (I_{2^{i-1}} \otimes \mathbf{1}_{2 \times 2}) \\ &= I_{i-1} \otimes \mathbf{1}_2^T \otimes I_{n-i}. \end{aligned}$$

Note that the transpose of the first row  $\text{Row}_1^T(M_f)$  is the truth table of  $f$ . We have the following

*Corollary 3.5:* Assume the truth table of a logical function  $f(x_1, \dots, x_n)$  is  $m_f$ . Then the truth table of  $\frac{\partial f}{\partial x_i}$ , in condensed form, is

$$m_{\partial_{[i]} f} = \Psi_n^i m_f. \quad (24)$$

Corollary 3.5 coincides with the result in [1], [2].

The following Corollaries 3.6 and 3.7, which are convenient in numerical computation, are obvious.

*Corollary 3.6:* Divide  $m_f^T$  into  $2^i$  blocks

$$m_f^T = (c_{1,1} \ c_{1,2} \ c_{2,1} \ c_{2,2} \ \cdots \ c_{2^{i-1},1} \ c_{2^{i-1},2}).$$

Then  $m_{\partial_{[i]} f}^T$  can be calculated directly by

$$m_{\partial_{[i]} f}^T = (c_{1,1} \oplus c_{1,2} \ c_{2,1} \oplus c_{2,2} \ \cdots \ c_{2^{i-1},1} \oplus c_{2^{i-1},2}). \quad (25)$$



## V. BOOLEAN INTEGRAL

As we mentioned in the introduction, there is no commonly used definition for Boolean integral. [31] provides a framework for Boolean integral. Unfortunately, the Boolean derivative used in [31] is different from the standard one, and hence the integral is in-consistent with the aforementioned Boolean derivative. Moreover, the computation problem has not been solved yet there.

In the following we define the Boolean integrals in the sense that they are precisely the inverse of the Boolean derivatives.

### A. Primitive Function

First, we define primitive function.

*Definition 5.1:* Given a Boolean function  $f(x_1, \dots, x_n)$ .  $F(x_1, \dots, x_{i-1}, z, x_i, \dots, x_n)$  is called the  $i$ th primitive function of  $f(x)$  (or the  $i$ th partial integral of  $f(x)$ ), denoted by

$$\int f(x_1, \dots, x_n) d[i] = F(x_1, \dots, x_{i-1}, z, x_i, \dots, x_n), \quad (35)$$

if

$$\frac{\partial F}{\partial z} = f(x_1, \dots, x_n). \quad (36)$$

In the light of Corollary 3.5, the problem becomes solving the equation

$$\Psi_{n+1}^i m_F = m_f. \quad (37)$$

### B. Indefinite Integral

*Definition 5.2:* Given a logical function  $F(x_1, \dots, x_n)$ . Its differential form, denoted by  $dF$ , is defined as

$$dF := \frac{\partial F}{\partial x_1} dx_1 + \dots + \frac{\partial F}{\partial x_n} dx_n. \quad (38)$$

Note that in (38) the symbol “+” is considered as only an adjacent notation, but not an operator.

*Definition 5.3:* Given a set of functions

$$f_i(x_1, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_n), \quad i = 1, \dots, n.$$

A function  $F(x_1, \dots, x_n)$  is called the indefinite integral of the differential form

$$dh = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$$

(or simply, integral of  $\{f_1, \dots, f_n\}$ ), if

$$\frac{\partial F}{\partial x_i} = f_i, \quad i = 1, \dots, n. \quad (39)$$

Note that according to equation (13) one sees that if  $F$  is an indefinite integral of  $dh$ , then so is  $\bar{F}$ .

Next, we can prove a necessary and sufficient condition of the existence of indefinite integrals.

*Theorem 5.4:* Consider a differential form

$$dh = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n.$$

There exists at least a pair of complemented indefinite integrals, if and only if

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad 1 \leq i < j \leq n. \quad (40)$$

*Proof.* Necessity is trivial. We prove the sufficiency. If (40) is satisfied, we can have third order cross derivatives as

$$\frac{\partial^2 f_i}{\partial x_j \partial x_k} = \frac{\partial^2 f_j}{\partial x_i \partial x_k} = \frac{\partial^2 f_k}{\partial x_i \partial x_j},$$

and even higher order cross derivatives. Using the obtained partial derivatives and following the form of MacLaurin expansion [3] we can construct

$$\begin{aligned} F(x_1, \dots, x_n) = & c \oplus \bigoplus_{i=1}^n f_i|_{\mathbf{0}} \wedge x_i \\ & \oplus \bigoplus_{1 \leq i_1 < i_2 \leq n} \frac{\partial f_{i_1}}{\partial x_{i_2}}|_{\mathbf{0}} \wedge x_{i_1} \wedge x_{i_2} \oplus \dots \\ & \oplus \frac{\partial^{n-1} f_1}{\partial x_2 \dots \partial x_n}|_{\mathbf{0}} \wedge x_1 \wedge x_2 \wedge \dots \wedge x_n. \end{aligned}$$

Then it is ready to verify that

$$F(x) = \int dh.$$

□

*Corollary 5.5:* If  $\int dh$  exists, then it is unique (up to a complement equivalence).

In the following when we consider the integral of a differential form, we assume

**A1** integrable condition (40) holds.

Hence as long as  $dh$  is integrable, we can write an indefinite integral as

$$\int dh = F(x) + C,$$

where  $C \in \mathcal{D}$ .

In later use, we would like to specify  $F$ . So we also use the following notation:

$$\int dh = F(x), \quad F(\mathbf{0}) = 0;$$

$$\int \bar{d}h = \bar{F}(x), \quad \bar{F}(\mathbf{0}) = 1.$$

Next, we consider how to calculate the indefinite integral. The following result is an immediate consequence of Corollary 3.5.

*Theorem 5.6:* Each indefinite integral of a differential form  $dh = f_1 dx_1 + \dots + f_n dx_n$  has a solution  $z$  of the following linear Galois algebraic system as its truth table.

$$\Psi_n \times_{\mathcal{B}} z = b, \quad (41)$$

where

$$\Psi_n = \begin{bmatrix} \Psi_n^1 \\ \vdots \\ \Psi_n^n \end{bmatrix} \in \mathcal{B}_{n \cdot 2^{n-1} \times 2^n}; \quad \text{and} \quad b = \begin{bmatrix} m_{f_{[1]}}^T \\ \vdots \\ m_{f_{[n]}}^T \end{bmatrix} \in \mathcal{B}_{n \cdot 2^{n-1}}.$$

It is worth noting that to get the default solution, we need to set  $z_{2^n} = 0$ .

Example 5.7: Assume  $n = 2$ . Then we have

$$\Psi_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Case 1: Assume  $f_1 = x_2$ ,  $f_2 = \neg x_1$ . Then we have  $m_{\partial_{[1]}f} = [1 \ 0]^T$ ,  $m_{\partial_{[2]}f} = [0 \ 1]^T$ . The equation (41) becomes

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \times_B \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Equivalently, we have

$$z_1 \oplus z_3 = 1, \quad z_2 \oplus z_4 = 0, \quad z_1 \oplus z_2 = 0, \quad z_3 \oplus z_4 = 1.$$

Setting  $z_4 = 0$ , which corresponds to  $F(\mathbf{0}) = 0$ , we have

$$z_1 = 0, \quad z_2 = 0, \quad z_3 = 1, \quad z_4 = 0.$$

That is,  $m_F = [0 \ 0 \ 1 \ 0]^T$ . Hence,  $F = (\neg x_1 \wedge x_2)$ . Writing it into integral form, we have

$$\int x_2 dx_1 + \neg x_1 dx_2 = (\neg x_1) \wedge x_2. \quad (42)$$

We also have

$$\int x_2 \bar{d}x_1 + \neg x_1 \bar{d}x_2 = (\neg x_1) \wedge x_2 \oplus 1. \quad (43)$$

Case 2: Assume  $f_1 = x_2$ ,  $f_2 = 1$ . Then we have  $m_{\partial_{[1]}f} = [1 \ 0]^T$  and  $m_{\partial_{[2]}f} = [1 \ 1]^T$ . It is easy to check that there is no solution. Hence the integral does not exist.

## VI. CONCLUSION

Using semi-tensor product of matrices and the matrix expression of logic, the calculation of Boolean derivative was investigated. A very simple formula was obtained, which converts the calculation of derivative into a modulo-2 matrix-vector product. Using it, three kinds of Boolean integrals were proposed. First, find  $f$  from  $\frac{\partial f}{\partial x_i}$  is called its  $i$ th primitive function. Second, find  $f$  from a differential form  $dh = h_1(\hat{x}_1, x_2, \dots, x_n)dx_1 + \dots + h_n(x_1, \dots, x_{n-1}, \hat{x}_n)dx_n$  is called the indefinite integral of  $dh$ . A necessary and sufficient condition was obtained for the existence of indefinite integral.

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