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Further Results on a State Observer for Continuous Oscillating Systems under Intrinsic Pulsatile Feedback

Alexander N. Churilov, Alexander Medvedev and Alexander I. Shepeljavyi

***Abstract*— A static gain observer for linear continuous plants with intrinsic pulse-modulated feedback is analyzed. The purpose of the observer is to drive the state estimation error to zero and asymptotically synchronize the sequence of pulse modulation instants estimated by the observer with that of the plant. Conditions on the observer gain matrix locally stabilizing the observer error along an arbitrary periodic plant solution are derived and illustrated by simulation for the case of pulsatile testosterone regulation.**

I. INTRODUCTION

Continuous dynamics with instant modulated impulses give rise to a broad class of hybrid systems with important applications in electronics and telecommunications. Mathematical tools of impulsive control theory are covered in [1], [2]. Since impulse-modulated signals are typically introduced in technical systems for control or communication, the generated modulated impulse sequence is known there exactly.

In biological systems, the mechanism of impulse modulation constitutes e.g. the basis of pulsatile endocrine feedback regulation and underlies the secretion of important hormones such as testosterone, insulin, cortisol, etc. [3]. In a mathematical model for pulsatile feedback introduced and analyzed in [4], the impulses mark the release instants of certain hormones and communicate the secreted quantities.

The impulse control of the endocrine systems is orchestrated by the brain and cumbersome or impossible to observe in the human due to ethical reasons. This poses an observation problem where the inaccessible for measurement hormone concentrations are reconstructed from the available in blood stream hormone measurements. In endocrinology, this is routinely done by means of deconvolution techniques, [5]. Only recently, an observer structure solving this problem has been suggested and analyzed in [6]. There, for the case of one impulse in the least period of the plant, the conditions under which the observer state estimation error converges to zero and the sequence of pulse modulation instances estimated by the observer asymptotically synchronizes with that of the plant are proved.

The present paper takes further the analysis of [6] by considering a general solution to the plant equations with an arbitrary number of the fired impulses in the least period, i.e. the m-cycle. Specializations to 1-cycles and 2-cycles for an important testosterone regulation case are provided and worked out in detail.

The paper is organized as follows. First the equations for the plant with a pulse-modulated feedback and the static feedback observer are summarized. Then a mapping describing the propagation of the observer state vector from one firing time to another is introduced and its properties are studied. The notion of a synchronous mode is defined describing a situation when the firing instances of the observer coincide with those of the plant. Local stability of synchronous modes is studied and the conditions under which the observer feedback gain guarantees local stability of a m-cycle are derived and illustrated by numerical simulations.

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A. Churilov and A. Shepeljavyi are with the Faculty of Mathematics and Mechanics, St. Petersburg State University, Universitetsky av. 28, Peterhof, 198504, St.Petersburg, Russia, a_churilov@mail.ru, as@as1020.spb.edu

A. Medvedev is with Information Technology, Uppsala University, SE-751 05 Uppsala, Sweden, alexander.medvedev@it.uu.se

II. SYSTEM EQUATIONS

Consider a plant governed by the equations

$$\frac{dx}{dt} = Ax + B\xi(t), \qquad z = Cx, \qquad y = Lx. \tag{1}$$

where A, B, C, L are real constant matrices of sizes $p \times p$, $p \times 1$, $1 \times p$, $q \times p$, respectively, z is the scalar controlled output, y is the vector measurable output, and x is the state vector. The following matrix relationships apply to (1) and are essential for further analysis:

$$CB = 0, \qquad LB = 0.$$

The matrix A is Hurwitz stable and the matrix pair (A, L) is observable. The signal ξ is an intrinsic (non-measurable) pulse-modulated feedback of the controlled output z to the state vector x

$$\xi(t) = \sum_{n=0}^{\infty} \lambda_n \delta(t - t_n), \tag{2}$$

$$t_{n+1} = t_n + T_n, \quad T_n = \Phi(z(t_n)), \quad \lambda_n = F(z(t_n)). \tag{3}$$

Here $\delta(\cdot)$ is a Dirac delta function, the time instant t_n is the firing time of a delta function (frequency modulation) and λ_n represents the corresponding weight (amplitude modulation) [7]. The functions $\Phi(\cdot)$ and $F(\cdot)$ are continuous, strictly monotonous and bounded with strictly positive lower bounds.

Notice that system (1)–(3) is hybrid [8], [9] since it possesses both continuous and discrete dynamics.

The plant is subject to unknown initial conditions $x(0)$ and the first firing instant of the pulsatile feedback occurs after the initial time instant, $t_0 \geqslant 0$.

As demonstrated in [4], the above assumptions imply that all the solutions of system (1)–(3) are bounded and there are no equilibria. This corresponds to the self-sustained oscillations arising in endocrine systems with pulsatile feedback, [3].

In order to estimate the state vector of (1), an observer mimicking the dynamics of the plant is introduced as:

$$\frac{d\hat{x}}{dt} = A\hat{x} + B\hat{\xi}(t) + K(y - \hat{y}), \quad \hat{y} = L\hat{x}, \quad \hat{z} = C\hat{x}, \tag{4}$$

where

$$\hat{\xi}(t) = \sum_{n=0}^{\infty} \hat{\lambda}_n \delta(t - \hat{t}_n), \tag{5}$$

$$\hat{t}_{n+1} = \hat{t}_n + \hat{T}_n, \quad \hat{T}_n = \Phi(\hat{z}(\hat{t}_n)), \quad \hat{\lambda}_n = F(\hat{z}(\hat{t}_n)) \tag{6}$$

and K is a static feedback gain chosen so that the matrix $D = A - KL$ is Hurwitz. Without loss of generality, it is assumed that $\hat{t}_0 \geqslant t_0$.

Summing up, the overall system comprising the plant and the observer is governed by (1)–(3), (4)–(6).

III. POINTWISE MAPPING AND ITS PROPERTIES

In a previous study of the plant dynamics under limit cycles [4], the following pointwise mapping has been treated:

$$x(t_n^-) \mapsto x(t_{n+1}^-).$$

For the dynamics of observer (4), a mapping of the form

$$\hat{x}(\hat{t}_n^-) \mapsto \hat{x}(\hat{t}_{n+1}^-)$$

cannot be obtained since $\hat{x}(\hat{t}_{n+1}^-)$ not only depends on $\hat{x}(\hat{t}_n^-)$ but as well on $\hat{t}_n$, which yields a more complicated functional dependence.

Pick some solution $x(t)$ of plant equations (1)–(3) with the parameters t_i, λ_i, $i = 0, 1, 2, \ldots$. Consider the pointwise mapping:

$$(\hat{x}(\hat{t}_n^-),\ \hat{t}_n) \mapsto (\hat{x}(\hat{t}_{n+1}^-),\ \hat{t}_{n+1}). \tag{7}$$

For some t and x, select integer numbers k and s, $k \leqslant s$, such that

$$t_k \leqslant t < t_{k+1}, \qquad t_s \leqslant t + \Phi(Cx) < t_{s+1}.$$

Define $P(x,t) = P_{k,s}(x,t)$ with

$$\begin{aligned} P_{k,s}(x,t) = {} & \mathrm{e}^{A(t+\Phi(Cx)-t_s)} x(t_s^+) \\ & - \mathrm{e}^{D\Phi(Cx)} \left[\mathrm{e}^{A(t-t_k)} x(t_k^+) - x - F(Cx)B \right] \\ & - \sum_{j=k+1}^{s} \lambda_j e^{D(t+\Phi(Cx)-t_j)} B. \end{aligned}$$

For brevity sake, denote $x_k = x(t_k^-)$ and $\hat{x}_n = \hat{x}(t_n^-)$.

Theorem 1: Pointwise mapping (7) is given by the equations

$$\hat{x}_{n+1} = P(\hat{x}_n, \hat{t}_n), \quad \hat{t}_{n+1} = \hat{t}_n + \Phi(C\hat{x}_n). \tag{8}$$

Proof: See Appendix 1.

Introduce additional notation referring to mapping (7). Define a function

$$Q_{k,s}(q) = \begin{bmatrix} P_{k,s}(x,t) \\ t + \Phi(Cx) \end{bmatrix}, \quad \text{where} \quad q = \begin{bmatrix} x \\ t \end{bmatrix}.$$

Set $Q(q) = Q_{k,s}(q)$ for $t_k \leqslant t < t_{k+1}$, $t_s \leqslant t + \Phi(Cx) < t_{s+1}$. Then

$$\hat{q}_{n+1} = Q(\hat{q}_n) \quad \text{where} \quad \hat{q}_n = \begin{bmatrix} \hat{x}_n \\ \hat{t}_n \end{bmatrix}, \quad Q(q) = \begin{bmatrix} P(x,t) \\ t + \Phi(Cx) \end{bmatrix}.$$

Iterations of the operator Q will be also considered further on

$$Q^{(m)}(q) = \underbrace{Q(Q(\ldots(Q(}_{m} q))\ldots)).$$

Theorem 2: The mapping $P(x,t)$ is continuous. If the scalar functions $F(\cdot)$, $\Phi(\cdot)$ have continuous derivatives then the partial derivatives

$$P'_x = \tfrac{\partial P}{\partial x}, \quad P'_t = \tfrac{\partial P}{\partial t}$$

are continuous everywhere.

Proof: See Appendix 2.

According to the definition, P'_x is a $p \times p$-matrix, and P'_t is a p-dimensional column. Then the Jacobian of $Q(q)$ is

$$Q'(q) = \begin{bmatrix} P'_x(x,t) & P'_t(x,t) \\ \Phi'(Cx)C & 1 \end{bmatrix}.$$

By the chain rule, the Jacobian of the m-th iteration of the mapping is

$$\begin{aligned} \left(Q^{(m)}\right)'(q) = {} & Q'\left(Q^{(m-1)}(q)\right) Q'\left(Q^{(m-2)}(q)\right) \times \ldots \\ & \times Q'\left(Q(q)\right) Q'(q). \end{aligned}$$

IV. SYNCHRONOUS MODE

Let $x(t)$ be some solution of plant equations (1)–(3) with the parameters t_k, λ_k, T_k and $x_k = x(t_k^-)$. Suppose that the plant is already running at the moment when the observer is initiated, i.e. $t_a \leqslant \hat{t}_0 < t_{a+1}$ for some $a \geqslant 1$.

Considering the solution $\hat{x}(t)$ of observer equations (4)–(6) subject to the initial conditions

$$\hat{t}_0 = t_a, \qquad \hat{x}(\hat{t}_0^-) = x(t_a^-),$$

yields

$$\hat{x}_n = x_{n+a}, \quad \hat{t}_n = t_{n+a}, \quad \hat{\lambda}_n = \lambda_{n+a}, \quad n = 0,\, 1,\, 2,\, \ldots,$$

and $\hat{x}(t) = x(t)$ for $t \geqslant t_a$. Such a solution $\hat{x}(t)$ will be called *a synchronous mode* with respect to $x(t)$.

Recall the mapping $P(x,t)$. Since $\hat{t}_n = t_{n+a}$ and $\hat{t}_{n+1} = t_{n+a+1}$ for a synchronous mode, it follows that

$$\hat{x}_{n+1} = P_{n+a,n+a+1}(\hat{x}_n,\, \hat{t}_n)$$

for all $n \geqslant 0$. For brevity, denote $n_a = n+a$, $\Phi'_k = \Phi'(Cx_k)$, $F'_k = F'(Cx_k)$. The synchronous mode with respect to $x(t)$ is characterized by the vector sequence

$$\hat{q}_n^0 = \begin{bmatrix} x_{n_a} \\ t_{n_a} \end{bmatrix}. \tag{9}$$

Then $Q(\hat{q}_n^0) = Q_{n_a,n_a+1}(\hat{q}_n^0)$.

Theorem 3: The Jacobian of $Q(\cdot)$ at $\hat{q}_n^0$ is

$$J_{n_a} \stackrel{def}{=} Q'(\hat{q}_n^0) \tag{10}$$

where the matrix J_k has the following matrix blocks

$$(J_k)_{11} = \Phi_k' A x_{k+1} C + \mathrm{e}^{DT_k} \left(I_p + F_k' BC\right),$$
$$(J_k)_{12} = A x_{k+1} - \mathrm{e}^{DT_k} A(x_k + \lambda_k B),$$
$$(J_k)_{21} = \Phi_k' C, \qquad (J_k)_{22} = 1.$$

Proof: See Appendix 3.

V. LOCAL STABILITY OF A SYNCHRONOUS MODE WITH RESPECT TO A 1-CYCLE

Let $x(t)$ be a 1-cycle. Then $x_n \equiv x_0$, $\lambda_n \equiv \lambda_0$, $T_n \equiv T_0$. Consider a synchronous mode with respect to $x(t)$ and let $\hat{q}_n^0$ be corresponding vector sequence (9) such that $\hat{q}_{n+1}^0 = Q(\hat{q}_n^0)$ is satisfied.

Jacobian (10) does not depend on n, i.e. $J_{n_a} \equiv J_0$, and comprises the following matrix blocks

$$\begin{aligned} (J_0)_{11} &= \Phi_0' A x_0 C + \mathrm{e}^{DT_0} \left(I_p + F_0' BC\right), \\ (J_0)_{12} &= A x_0 - \mathrm{e}^{DT_0} A(x_0 + \lambda_0 B), \\ (J_0)_{21} &= \Phi_0' C, \qquad (J_0)_{22} = 1. \end{aligned} \tag{11}$$

Let $\hat{q}_n$ be a perturbed value of $\hat{q}_n^0$: $\hat{q}_{n+1} = Q(\hat{q}_n)$ and $\|\hat{q}_n - \hat{q}_n^0\|$ be small. Then

$$\hat{q}_{n+1} - \hat{q}_{n+1}^0 = J_0(\hat{q}_n - \hat{q}_n^0) + o(\|\hat{q}_n - \hat{q}_n^0\|).$$

The linear operator J_0 is a contraction mapping iff it is Schur stable, i.e. all its eigenvalues lie inside the unit circle. This is consistent with the previously reported results. In [6] it is shown that Schur stability of J_0 implies orbital stability of the synchronous mode.

VI. LOCAL STABILITY OF A SYNCHRONOUS MODE WITH RESPECT TO A 2-CYCLE

Let $x(t)$ be a 2-cycle. Then $x_{n+2} = x_n$ and

$$x_n = \begin{cases} x_0, & \text{if } n \text{ is even}, \\ x_1, & \text{if } n \text{ is odd}. \end{cases}$$

Let $\hat{q}_n^0$ be vector sequence (9) corresponding to a synchronous mode with respect to $x(t)$. From (10), it follows that $J_{n+2} = J_n$ for all n and J_0, J_1 have the following matrix blocks

$$\begin{aligned} (J_0)_{11} &= \Phi_0' A x_1 C + \mathrm{e}^{DT_0} \left(I_p + F_0' BC\right), \\ (J_0)_{12} &= A x_1 - \mathrm{e}^{DT_0} A(x_0 + \lambda_0 B), \\ (J_0)_{21} &= \Phi_0' C, \qquad (J_0)_{22} = 1, \end{aligned} \tag{12}$$

$$\begin{aligned} (J_1)_{11} &= \Phi_1' A x_0 C + \mathrm{e}^{DT_1} \left(I_p + F_1' BC\right), \\ (J_1)_{12} &= A x_0 - \mathrm{e}^{DT_1} A(x_1 + \lambda_1 B), \\ (J_1)_{21} &= \Phi_1' C, \qquad (J_1)_{22} = 1. \end{aligned} \tag{13}$$

For the sequence $\hat{q}_n^0$ characterizing a synchronous mode of the observer with respect to $x(t)$, it applies

$$\hat{q}_{n+2}^0 = Q(Q(\hat{q}_n^0)) = Q^{(2)}(\hat{q}_n^0).$$

A perturbation of $\hat{q}_n^0$ obeys

$$\hat{q}_{n+2} - \hat{q}_{n+2}^0 = J_{n_a+1} J_{n_a} (\hat{q}_n - \hat{q}_n^0) + o(\|\hat{q}_n - \hat{q}_n^0\|). \tag{14}$$

Thus, in (14) , either $J_{n_a+1} J_{n_a} = J_0 J_1$ or $J_{n_a+1} J_{n_a} = J_1 J_0$ can occur. It can be easily shown that the products $J_0 J_1$ and $J_1 J_0$ are Schur stable or unstable simultaneously, see Appendix 4. Hence the matrix $J_{n_a+1} J_{n_a}$ is a contraction iff $J_1 J_0$ is Schur stable. Arguing similarly to [6], it can be verified that such a contraction property implies orbital stability of the synchronous mode.

VII. ON THE EXISTENCE OF THE MATRIX K

Since the pair (A, L) is observable, the matrix K can be chosen in such a way that the matrix $D = A - KL$ has arbitrary pre-defined eigenvalues. The following theorems demonstrate that a proper choice of K can ensure stability of a linearized observer system.

Theorem 4: Let $x(t)$ be a 1-cycle and the notation of Section V apply. Suppose

$$-1 < \Phi_0' C A x_0 + 1 < 1. \tag{15}$$

Then there is always a matrix K such that $D = A - KL$ is Hurwitz and the Jacobi matrix J_0 with the blocks given by (11) is Schur stable.

Theorem 5: Let $x(t)$ be a 2-cycle and notation of Section VI apply. Suppose

$$-1 < (\Phi_0' C A x_0 + 1)(\Phi_1' C A x_1 + 1) < 1. \tag{16}$$

Then there is always a matrix K such that $D = A - KL$ is Hurwitz and the product matrix $J_0 J_1$, where J_0, J_1 are comprised of the blocks given by (12), (13), is Schur stable.

The proofs of Theorems 4, 5 are given in Appendix 5.

Schur stability conditions can be easily generalized to the statement on stability of m-cycle:

$$-1 < \prod_{k=0}^{m-1} (\Phi_k' C A x_{k+2} + 1) < 1.$$

VIII. NUMERICAL EXAMPLE

Assume the following values in model (1)–(3)

$$A = \begin{bmatrix} -0.08 & 0 & 0 \\ 2 & -0.15 & 0 \\ 0 & 0.5 & -0.2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad h = 2.7,$$

$$\Phi(z) = 60 + 40 \, \frac{(z/h)^2}{1 + (z/h)^2}, \quad F(z) = 3 + \frac{2}{1 + (z/h)^2}.$$

The existence and stability conditions of 1-cycles and 2-cycles in system (1), (2) are provided in [4]. From Theorem 3 there, one can deduce the existence of a 2-cycle with the following parameters

$$x_0^\mathsf{T} = \begin{bmatrix} 0.0028 & 0.0799 & 0.3321 \end{bmatrix}^\mathsf{T},$$

$$x_1^\mathsf{T} = \begin{bmatrix} 0.0390 & 1.0989 & 4.4896 \end{bmatrix}^\mathsf{T},$$

where T denotes transpose. The rest of the parameters for the 2-cycle are evaluated as: $T_0 = 60.5960$, $\lambda_0 = 4.9702$, $T_1 = 89.3757$, $\lambda_1 = 3.5312$.

In this case $(\Phi_0' C A x_0 + 1)(\Phi_1' C A x_1 + 1) = -0.1915$, so condition (16) is satisfied and the existence of a stabilizing

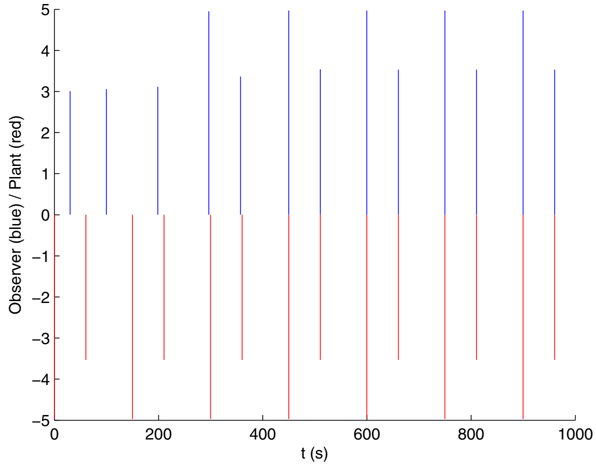

Fig. 1. Initial conditions transients in the firing times and weights of $\hat{\xi}(t)$ relative to $\xi(t)$: Blue lines mark the firing times of the observer $\hat{t}_n$ with hight equal to $\hat{\lambda}_n$. Red lines correspond to the pulse modulation of the plant in 2-cycle with the firing times t_n and the weights $-\lambda_n$.

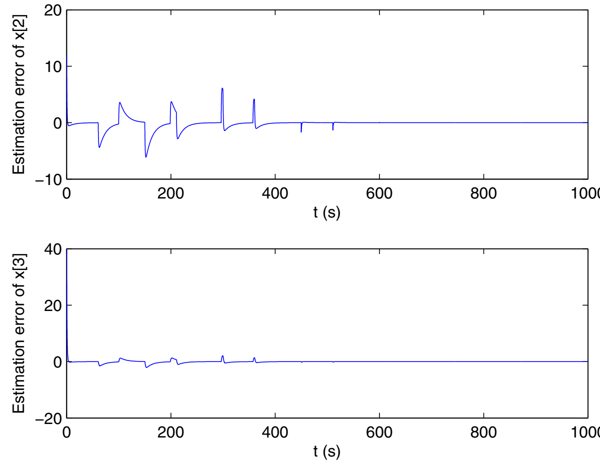

Fig. 2. Transients in the continuous (2nd and 3rd) components of the observer residual $r(t) = x(t) - \hat{x}(t)$.

observer gain K is guaranteed. Choose the observer feedback gain as

$$K = \begin{bmatrix} 0 & 0 \\ 1.2 & 0 \\ 0 & 1.2 \end{bmatrix}.$$

Then $A - KL$ is Hurwitz stable and the eigenvalues of $J_1 J_0$ amount to $\{-0.1844,\ -0.0000,\ -0.0000,\ 0.0000\}$, proving $J_1 J_0$ to be Schur stable. Therefore, a stable synchronous mode is imposed on the observer under the 2-cycle by the choice of the observer gain.

Fig. 1 illustrates the transients in the sequence $\hat{\xi}(t)$ relative to $\xi(t)$, caused by a mismatch between the initial conditions of the plant and those of the observer. The initial values of the observer states are selected so that $\hat{t}_0 - t_0 = T_0/2$ and $\hat{x}(\hat{t}_0^-) - x(\hat{t}_0) = [1\ 1\ 1]^\mathsf{T}$. Since pulse modulation generates unbounded Dirac δ-functions, the sequences $\xi(t), \hat{\xi}(t)$ are depicted in terms of the firing times t_n, $\hat{t}_n$ and the weights λ_n, $\hat{\lambda}_n$.

The blue vertical lines of hight $\hat{\lambda}_n$ positioned at $\hat{t}_n$ correspond to the observer firing sequence $\hat{\xi}(t)$. The pulse modulation of the plant is shown by red lines of hight $-\lambda_n$ positioned at t_n (with $t_0 = 0$). It can be seen that the firing instants of $\hat{\xi}(t)$ become synchronized with those of $\xi(t)$ and the impulse weights $\hat{\lambda}_n$ asymptotically converge to λ_n.

The convergence of the observer residual is illustrated in Fig. 2. Notice that the signals in question are continuous and measurable at the system output.

IX. CONCLUSIONS

Analytical tools for state estimation in linear continuous time-invariant systems under inaccessible for measurement pulse-modulated feedback are developed in this paper. The states of the considered system undergo jumps at certain times modulated by other states. Such a pulsatile feedback occurs in neural as well as in endocrine systems and gives rise to homeostatic biological regulation. By local stability analysis of the observer, it is shown that with a proper choice of the observer gain, one can obtain an asymptotically converging estimate of the system states and a synchronization between of the periodic impulse sequences of the plant and that of the observer.

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APPENDIX 1. PROOF OF THEOREM 1

The state estimation error of the observer $r(t) = x(t) - \hat{x}(t)$ obeys

$$\frac{dr}{dt} = Dr + B(\xi(t) - \hat{\xi}(t)), \qquad \hat{z} = z - Cr. \tag{17}$$

Pick an arbitrary integer $n \geqslant 0$ and suppose that

$$t_k \leqslant \hat{t}_n < t_{k+1}, \qquad t_s \leqslant \hat{t}_n + \Phi(C\hat{x}_n) < t_{s+1}$$

for some k and s. Introduce a number $m \geqslant 0$ such that $s = k + m$. Now the following steps have to be considered to arrive to the result of the Theorem.

(i) Since $t_k \leqslant \hat{t}_n < t_{k+1}$, one has $x(\hat{t}_n^+) = e^{A(\hat{t}_n - t_k)} x(t_k^+)$. Hence

$$r(\hat{t}_n^+) = x(\hat{t}_n^+) - \hat{x}(\hat{t}_n^+) = e^{A(\hat{t}_n - t_k)} x(t_k^+) - \hat{x}(\hat{t}_n^-) - \hat{\lambda}_n B.$$

(ii) If $\hat{t}_n < t < t_{k+1}$ and $m \geqslant 1$ then $r(t) = e^{D(t-\hat{t}_n)} r(\hat{t}_n^+)$, hence

$$r(t_{k+1}^+) = e^{D(t_{k+1}-\hat{t}_n)} r(\hat{t}_n^+) + \lambda_{k+1} B.$$

(iii) If $t_{k+1} < t < t_{k+2}$ and $m \geqslant 2$ then $r(t) = e^{D(t-\hat{t}_{k+1})} r(\hat{t}_{k+1}^+)$, hence

$$r(t_{k+2}^+) = e^{D(t_{k+2}-\hat{t}_n)} r(\hat{t}_n^+) + \lambda_{k+1} e^{D(t_{k+2}-t_{k+1})} B + \lambda_{k+2} B.$$

(iv) If $t_{k+2} < t < t_{k+3}$ and $m \geqslant 3$ then

$$\begin{aligned} r(t_{k+3}^+) = {} & e^{D(t_{k+3}-\hat{t}_n)} r(\hat{t}_n^+) + \lambda_{k+1} e^{D(t_{k+3}-t_{k+1})} B \\ & + \lambda_{k+2} e^{D(t_{k+3}-t_{k+2})} B + \lambda_{k+3} B. \end{aligned}$$

(v) In general,

$$\begin{aligned} r(t_{k+m}^+) = {} & e^{D(t_{k+m}-\hat{t}_n)} r(\hat{t}_n^+) \\ & + \lambda_{k+1} e^{D(t_{k+m}-t_{k+1})} B \\ & + \lambda_{k+2} e^{D(t_{k+m}-t_{k+2})} B \\ & \vdots \\ & + \lambda_{k+m-1} e^{D(t_{k+m}-t_{k+m-1})} B \\ & + \lambda_{k+m} B. \end{aligned}$$

Recalling that $k + m = s$, one can write

$$r(t_s^+) = e^{D(t_s-\hat{t}_n)} r(\hat{t}_n^+) + \sum_{j=k+1}^{s} \lambda_j e^{D(t_s-t_j)} B.$$

(vi) Finally,

$$\begin{aligned} \hat{x}(\hat{t}_{n+1}^-) &= x(\hat{t}_{n+1}^-) - r(\hat{t}_{n+1}^-) \\ &= e^{A(\hat{t}_{n+1}-t_{k+m})} x(t_{k+m}^+) \\ &\quad - e^{D(\hat{t}_{n+1}-t_{k+m})} r(t_{k+m}^+) \end{aligned}$$

and $\hat{t}_{n+1} = \hat{t}_n + \Phi(C\hat{x}_n)$. This can be rewritten as

$$\hat{x}_{n+1} = e^{A(\hat{t}_n+\Phi(C\hat{x}_n)-t_s)} x(t_s^+) - e^{D(\hat{t}_n+\Phi(C\hat{x}_n)-t_s)} r(t_s^+).$$

Then the statement of Theorem 1 is obtained by substituting formulas (i) into (v) and (v) into (vi).

Appendix 2. Proof of Theorem 2

Lemma 1: The function $P_{k,s}(x,t)$ can be represented as

$$P_{k,s}(x,t) = u_k(x,t) + v_s(x,t) + w(x,t),$$

where

$$u_k(x,t) = e^{D\Phi(Cx)} \left[-e^{A(t-t_k)} x(t_k^+) + \sum_{j=1}^{k} \lambda_j e^{D(t-t_j)} B \right],$$

$$v_s(x,t) = e^{A(t+\Phi(Cx)-t_s)} x(t_s^+) - \sum_{j=1}^{s} \lambda_j e^{D(t+\Phi(Cx)-t_j)} B,$$

$$w(x,t) = e^{D\Phi(Cx)} \left[x + F(Cx) B \right].$$

Moreover, the following recursions hold

$$\begin{aligned} & u_k(x,t) - u_{k-1}(x,t) = \\ & \quad - \lambda_k e^{D\Phi(Cx)} \left[e^{A(t-t_k)} - e^{D(t-t_k)} \right] B \end{aligned} \tag{18}$$

and

$$\begin{aligned} & v_s(x,t) - v_{s-1}(x,t) = \\ & \quad \lambda_s \left[e^{A(\Phi(Cx)+t-t_s)} - e^{D(\Phi(Cx)+t-t_s)} \right] B. \end{aligned} \tag{19}$$

Proof of Lemma. Recursion (18) follows from the relationships

$$\begin{aligned} e^{A(t-t_k)} x(t_k^+) &= e^{A(t-t_k)} \left[e^{A(t_k-t_{k-1})} x(t_{k-1}^+) + \lambda_k B \right] \\ &= e^{A(t-t_{k-1})} x(t_{k-1}^+) + \lambda_k e^{A(t-t_k)} B. \end{aligned}$$

Substituting in the above formula k instead for s, one obtains (19).

Proof of Theorem 2. It is demonstrated below that Theorem 2 is a direct consequence of Lemma 1. The function $P(x,t)$ can have gaps either on the surfaces $M_k = \{(x,t) : t = t_k\}$ or on the surfaces $N_s = \{(x,t) : t + \Phi(x) = t_s\}$.

Let $(x,t) \in M_k$ for some k. From (18)

$$u_k(x,t_k) = u_{k-1}(x,t_k),$$

Hence $P(x,t)$ has no gaps on this surface. This statement remains true for partial derivatives of $P(x,t)$, because $(A - D)B = 0$.

Let $(x,t) \in N_s$ for some s. From (19)

$$v_s(x, t_s - \Phi(Cx)) = v_{s-1}(x, t_s - \Phi(Cx)),$$

so neither $P(x,t)$ has gaps on this surface. The same is true for its partial derivatives.

Clearly, if $(x,t) \in M_k \cap N_s$, for some s and k, then $t = t_k = t_s - \Phi(Cx)$ and from (18), (19), it follows that

$$P_{k,s}(x,t_k) = P_{k-1,s-1}(x,t_k).$$

Combined with the previous two conclusions, this means that $P(x,t)$ has no gaps on $M_k \cap N_s$. The same is true for its partial derivatives.

Appendix 3. Proof of Theorem 3

Two lemmas are needed in order to prove the result of the theorem.

Lemma 2: The following relationship holds:

$$\begin{aligned} P_{k,k+1}(x,t) = {} & \left[e^{A\Phi(Cx)} - e^{D\Phi(Cx)} \right] e^{A(t-t_k)} x(t_k^+) \\ & + \lambda_{k+1} \left[e^{A(t+\Phi(Cx)-t_{k+1})} - e^{D(t+\Phi(Cx)-t_{k+1})} \right] B \\ & + e^{D\Phi(Cx)} \left[x + F(Cx) B \right]. \end{aligned}$$

Proof: The lemma follows from Theorem 1 and the evident formula

$$x(t_{k+1}^+) = e^{A(t_{k+1}-t_k)} x(t_k^+) + \lambda_{k+1} B.$$

Further, the following facts are needed to carry on with the proof.

Lemma 3: The partial derivatives of $P(x_k, t_k)$ with respect to its arguments can be calculated as follows:

$$P'_x(x_k, t_k) = \Phi'_k A x_{k+1} C + \mathrm{e}^{DT_k}\left[I_p + F'_k BC\right],$$
$$P'_t(x_k, t_k) = A x_{k+1} - \mathrm{e}^{DT_k} A(x_k + \lambda_k B).$$

Proof: Obviously $P(x_k, t_k) = P_{k,k+1}(x_k, t_k)$. As it was stipulated by Theorem 2, the partial derivatives of $P(x, t)$ are continuous, so that

$$\frac{\partial}{\partial x} P(x_k, t_k) = \frac{\partial}{\partial x} P_{k,k+1}(x_k, t_k),$$
$$\frac{\partial}{\partial t} P(x_k, t_k) = \frac{\partial}{\partial t} P_{k,k+1}(x_k, t_k).$$

By direct calculation

$$\begin{aligned}\frac{\partial}{\partial x} P_{k,k+1}(x, t) &= \Phi'(Cx)\big[A\mathrm{e}^{A\Phi(Cx)} - D\mathrm{e}^{D\Phi(Cx)}\big] \\ &\times \mathrm{e}^{A(t-t_k)} x(t_k^+) C + \lambda_{k+1}\Phi'(Cx) \\ &\times \big[A\mathrm{e}^{A(t+\Phi(Cx)-t_{k+1})} - D\mathrm{e}^{D(t+\Phi(Cx)-t_{k+1})}\big]BC \\ &+ \Phi'(Cx) D\mathrm{e}^{D\Phi(Cx)}\big[x + F(Cx)B\big]C \\ &+ \mathrm{e}^{D\Phi(Cx)}\big[I_p + F'(Cx)BC\big],\\ \frac{\partial}{\partial t} P_{k,k+1}(x, t) &= \big[\mathrm{e}^{A\Phi(Cx)} - \mathrm{e}^{D\Phi(Cx)}\big]A\mathrm{e}^{A(t-t_k)} x(t_k^+) \\ &+ \lambda_{k+1}\big[A\mathrm{e}^{A(t+\Phi(Cx)-t_{k+1})} - D\mathrm{e}^{D(t+\Phi(Cx)-t_{k+1})}\big]B.\end{aligned}$$

Since $t_k + \Phi(Cx_k) - t_{k+1} = 0$ and $(A - D)B = 0$, one obtains

$$\begin{aligned}\frac{\partial}{\partial x} P_{k,k+1}(x_k, t_k) &= \Phi'(Cx_k)\big[A\mathrm{e}^{A\Phi(Cx_k)} - D\mathrm{e}^{D\Phi(Cx_k)}\big] \\ &\times x(t_k^+) C + \Phi'(Cx_k) D\mathrm{e}^{D\Phi(Cx_k)}\left[x_k + \lambda_k B\right]C \\ &+ \mathrm{e}^{D\Phi(Cx_k)}\big[I_p + F'(Cx_k)BC\big],\\ \frac{\partial}{\partial t} P_{k,k+1}(x_k, t_k) &= \big[\mathrm{e}^{A\Phi(Cx_k)} - \mathrm{e}^{D\Phi(Cx_k)}\big]Ax(t_k^+).\end{aligned}$$

Now the statement of Lemma 3 can be derived by taking into account $x(t_k^+) = x_k + \lambda_k B$ and $\mathrm{e}^{A\Phi(Cx_k)} x(t_k^+) = x_{k+1}$.

Theorem 3 follows directly from Lemma 3.

APPENDIX 4. A REMARK ON SCHUR STABILITY

Here, a proof of a statement utilized in Section VI is provided.

Lemma 4: Let J_0, J_1 be square matrices of the same sizes. Then the products $J_0 J_1$ and $J_1 J_0$ are Schur stable or unstable simultaneously.

Proof. It is known that a square matrix J is Schur stable iff all the solutions of the discrete-time equation $x_{k+1} = Jx_k$ vanish as k increases. For definiteness, let the product $J_0 J_1$ be Schur stable. Then all the solutions of the system

$$x_{k+1} = J_0 J_1 x_k \tag{20}$$

approach zero as k tends to infinity.

Consider the system

$$y_{k+1} = J_1 J_0 y_k. \tag{21}$$

To verify that all the solutions of (21) vanish as time goes to infinity, denote $z_k = J_0 y_k$. Then z_k satisfies the equation

$$z_{k+1} = J_0 J_1 z_k \tag{22}$$

of type (20), and hence $z_k \to 0$ as $k \to \infty$ for any solution of (22). It means that $J_0 y_k \to 0$ as $k \to \infty$ for any solution of (21). Then $J_1 J_0 y_k \to 0$ and so (21) implies $y_{k+1} \to 0$ as $k \to \infty$, whence $y_k \to 0$ as $k \to \infty$. Consequently, $J_1 J_0$ is Schur stable. The lemma is proved.

Remark. The proposition can be easily generalized: if $J_0 J_1 \ldots J_{m-1}$ is Schur stable this property preserves for any cyclic permutation of the multipliers.

APPENDIX 5. PROOFS OF THEOREM 4 AND THEOREM 5

Proof of Theorem 4. The matrix J_0 can be decomposed as $J_0 = \tilde{J}_0 + W(D)$, where

$$\tilde{J}_0 = \begin{bmatrix} \Phi'_0 A x_0 C & A x_0 \\ \Phi'_0 C & 1 \end{bmatrix} = \begin{bmatrix} A x_0 \\ 1 \end{bmatrix}\begin{bmatrix} \Phi'_0 C & 1 \end{bmatrix}$$

and

$$W(D) = \begin{bmatrix} \mathrm{e}^{DT_0}\left(I_p + F'_0 BC\right) & -\mathrm{e}^{DT_0} A(x_0 + \lambda_0 B) \\ 0 & 0 \end{bmatrix}.$$

The elements of the matrix e^{DT_0}, and hence those of $W(D)$, can be made arbitrarily small by choice of K. As for the matrix $\tilde{J}_0$, it is evident that

$$\operatorname{rank} \tilde{J}_0 = 1, \qquad \operatorname{tr} \tilde{J}_0 = \Phi'_0 C A x_0 + 1.$$

Thus the matrix $\tilde{J}_0$ has only one eigenvalue that may be nonzero, it is equal to $\Phi'_0 C A x_0 + 1$. So, if (15) is satisfied, then $\tilde{J}_0$ is Schur stable.

Proof of Theorem 5.

Similarly, decompose

$$J_0 = \tilde{J}_0 + W_0(D), \quad J_1 = \tilde{J}_1 + W_1(D),$$

where $W_0(D)$, $W_1(D)$ contain e^{DT_0}, e^{DT_1} as multipliers. The matrices e^{DT_0}, e^{DT_1} can simultaneously be made arbitrarily small by the choice of K. At the same time,

$$\tilde{J}_0 = \begin{bmatrix} A x_1 \\ 1 \end{bmatrix}\begin{bmatrix} \Phi'_0 C & 1 \end{bmatrix}, \quad \tilde{J}_1 = \begin{bmatrix} A x_0 \\ 1 \end{bmatrix}\begin{bmatrix} \Phi'_1 C & 1 \end{bmatrix}.$$

Hence

$$\tilde{J}_1 \tilde{J}_0 = (\Phi'_1 C A x_1 + 1)\begin{bmatrix} A x_0 \\ 1 \end{bmatrix}\begin{bmatrix} \Phi'_0 C & 1 \end{bmatrix}$$

Thus the matrix $\operatorname{rank} \tilde{J}_1 \tilde{J}_0 = 1$ and the matrix product has only one eigenvalue that may be non-zero, that is the one equal to

$$(\Phi'_0 C A x_0 + 1)(\Phi'_1 C A x_1 + 1).$$

If (16) is satisfied, the matrix $\tilde{J}_1 \tilde{J}_0$ is Schur stable.