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# Convex Relaxations for Nonconvex Optimal Control Problems

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***Abstract*— An optimal control problem with a nonlinear control system embedded is considered. Using the endpoint map of the control system, such problems can be written as nonlinear programs on the set of admissible controls. Though necessary optimality conditions exist for such problems, they are often nonconvex and such conditions are not sufficient. A relaxation procedure is outlined which generates a convex program whose solution value is a guaranteed lower bound on the solution value of the original problem. This result is a crucial step towards developing deterministic global optimization techniques for optimal control problems using a branch-and-bound framework. The major contribution is that, unlike other developments along these lines, the convex underestimating program derived here is valid on the original function space; i.e. there is no need to discretize the control.**

## I. INTRODUCTION

We consider the open-loop optimal control problem informally stated as

$$\begin{aligned} &\inf_{\mathbf{u}\in\mathcal{U}} \phi(\mathbf{u}(t_f), \mathbf{x}(t_f, \mathbf{u})) \qquad (1)\\ &\text{s.t.} \quad \mathbf{g}(\mathbf{u}(t_f), \mathbf{x}(t_f, \mathbf{u})) \leq \mathbf{0}\\ &\qquad \mathbf{q}(t, \mathbf{u}(t), \mathbf{x}(t, \mathbf{u})) \leq \mathbf{0}, \quad \text{a.e.} \quad t \in [t_0, t_f], \end{aligned}$$

where $\mathcal{U}$ is a subset of $(L^1([t_0, t_f]))^{n_u}$ and, for each $\mathbf{u} \in \mathcal{U}$, $\mathbf{x}(\cdot, \mathbf{u})$ is an absolutely continuous solution of

$$\begin{aligned} \dot{\mathbf{x}}(t, \mathbf{u}) &= \mathbf{f}(t, \mathbf{u}(t), \mathbf{x}(t, \mathbf{u})), \quad \text{a.e.} \quad t \in [t_0, t_f], \qquad (2)\\ \mathbf{x}(t_0, \mathbf{u}) &= \mathbf{x}_0, \end{aligned}$$

which is assumed unique. This problem is of general interest and has been the subject of intense research for decades [1]. Nonetheless, there is no general purpose algorithm for solving (1) to global optimality. In this article, we begin to address this problem using ideas which originated in the context of branch-and-bound algorithms for the global solution of nonlinear programs on Euclidean spaces.

In order to apply branch-and-bound global optimization to a given problem, there must be procedures available for computing guaranteed upper and lower bounds on the solution value of the original problem when restricted to a given subset of the search space. Upper bounding is typically done by simply evaluating the objective function at any feasible point. Obtaining a lower bound is much more difficult and is typically the key development required to extend branch-and-bound techniques to a new class of problems. In this article, we present a lower bounding procedure for (1). However, at this time we cannot present a complete branch-and-bound global optimization algorithm for (1) due to difficulties that arise in defining an exhaustive search procedure in an infinite dimensional search space.

To compute a guaranteed lower bound on (1), a procedure is presented for constructing an auxiliary optimal control problem, called a relaxation of (1), with the properties (a) the optimal objective value is guaranteed to underestimate the infimum in (1), and (b) it is convex in the sense that the feasible set is a convex subset of $\mathcal{U}$ and the mapping taking $\mathbf{u}$ to the objective value is convex on this set. Because it is convex, this relaxed problem is in principle solvable to global optimality. For example, necessary and sufficient optimality conditions for such programs are derived in [2], and gradient based solution methods are proposed. Supposing that such a solution can be obtained, this procedure generates a guaranteed lower bound on the solution of (1). In [2], conditions were also studied under which (1) can be guaranteed to be convex, based on arguments similar to those presented in §IV. In contrast, the method presented here is not used to verify convexity, but rather to construct a convex optimization problem which underestimates a given instance of (1), even when (1) is nonconvex.

Several approaches for computing lower bounds for (1) have been proposed when the controls $\mathbf{u}$ are approximated by some finite representation [3], [4], [5]. Under a uniqueness assumption on the embedded ODEs, this approximation results in a nonlinear program on a Euclidean space [6]. Thus, the development of lower bounding methods enabled the use of branch-and-bound to solve the approximate optimal control problem to global optimality. However, approximating $\mathbf{u}$ is not only unsatisfying, but potentially requires a large number of variables in order to approximate a single control accurately, which can make global optimization impractical. It is shown here that the relaxation procedure used in [5] can actually provide relaxations of (1) on the original, infinite-dimensional space, which eliminates the need for control parametrization, at least in the lower bounding calculation.

### A. Basic approach

Let $I = [t_0, t_f] \subset \mathbb{R}$, $\bar{U} \subset \mathbb{R}^{n_u}$ compact, and suppose that $U$ maps $I$ into compact, convex subsets of $\bar{U}$, $U(t)$. In the remainder of the article, the set of admissible controls is defined by

$$\mathcal{U} \equiv \{\mathbf{u} \in (L^1(I))^{n_u} : \mathbf{u}(t) \in U(t) \text{ a.e. in } I\} \qquad (3)$$

and is assumed nonempty. It is trivial to verify that $\mathcal{U}$ is a convex subset of the vector space $(L^1(I))^{n_u}$. Finally, let $D \subset \mathbb{R}^{n_x}$ be open and suppose that the mappings in (1) have

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the form $\phi : \bar{U} \times D \to \mathbb{R}$, $\mathbf{g} : \bar{U} \times D \to \mathbb{R}^{n_g}$, and $\mathbf{q} : I \times \bar{U} \times D \to \mathbb{R}^{n_q}$. Assumptions regarding the control systems (2) are discussed in §IV. We note here that the solution has the form $\mathbf{x} : I \times \mathcal{U} \to D$.

In order to construct a convex underestimating program for (1), convex underestimating functions are derived for the mappings

$$\mathcal{U} \ni \mathbf{u} \longmapsto \mathcal{F}_\phi(\mathbf{u}) \equiv \phi(\mathbf{u}(t_f), \mathbf{x}(t_f, \mathbf{u})),$$
$$\mathcal{U} \ni \mathbf{u} \longmapsto \mathcal{F}_\mathbf{g}(\mathbf{u}) \equiv \mathbf{g}(\mathbf{u}(t_f), \mathbf{x}(t_f, \mathbf{u})),$$

and the family of mappings

$$\mathcal{U} \ni \mathbf{u} \longmapsto \mathcal{F}_{\mathbf{q},t}(\mathbf{u}) \equiv \mathbf{q}(t, \mathbf{u}(t), \mathbf{x}(t, \mathbf{u})),$$

for a.e. $t \in I$. Defining the relaxed program with these convex underestimators in place of the mappings above, both convexity and the desired underestimation property follow from standard arguments [7], [2].

The development of convex underestimators for the mappings above is based on McCormick's relaxation technique [8]. In §II, McCormick's technique is introduced and it is shown that the desired relaxations of $\mathcal{F}_\phi$, $\mathcal{F}_\mathbf{g}$ and $\mathcal{F}_{\mathbf{q},t}$ can be constructed provided that relaxations of the end-point map of (2) are available. In §III and IV, McCormick's technique is combined with some basic results concerning integrals and control systems to yield procedures for relaxing integral functionals and the end-point maps of control systems.

## II. McCormick's Relaxations

McCormick's relaxation technique is widely used for constructing convex underestimators for nonlinear functions on $\mathbb{R}^n$. The technique is briefly described here. The novelty of this presentation is that some of the key results are shown to hold on an arbitrary vector space. Certainly, the standard definition of convexity (concavity) makes sense in this setting. Relaxations in this context are defined as follows.

*Definition 1:* Let $Q$ denote an arbitrary vector space, $\mathcal{U} \subset Q$ convex, and $h, h^c, h^C : \mathcal{U} \to \mathbb{R}$. The function $h^c$ is called a *convex relaxation* of $h$ on $\mathcal{U}$ if $h^c$ is convex on $\mathcal{U}$ and $h^c(\mathbf{u}) \leq h(\mathbf{u})$, $\forall \mathbf{u} \in \mathcal{U}$. Similarly, $h^C$ is called a *concave relaxation* of $h$ on $\mathcal{U}$ if $h^C$ is concave on $\mathcal{U}$ and $h^C(\mathbf{u}) \geq h(\mathbf{u})$, $\forall \mathbf{u} \in \mathcal{U}$. The terms convex and concave relaxation will also be used for vector functions when these conditions hold elementwise.

We first develop McCormick's relaxations for the case $Q \equiv \mathbb{R}^{n_u}$. McCormick's relaxation technique applies to factorable functions. Roughly speaking, a function is factorable if it can be defined by the finite recursive application of binary additions, binary multiplications and composition with a pre-defined library of univariate functions, typically including exponential and logarithmic functions, square root, odd and even integer powers, trigonometric functions, etc. Letting $\mathcal{E}$ denote this collection of univariate functions, we have the following definition.

*Definition 2:* Let $Q \equiv \mathbb{R}^{n_u}$, $\mathcal{U} \subset Q$ convex. A function $h : \mathcal{U} \to \mathbb{R}$ is *factorable* if it can be expressed in terms of a finite number of factors $v_1, \ldots, v_m$ such that, given $\mathbf{u} \in \mathcal{U}$, $v_i = u_i$ for $i = 1, \ldots, n_u$, and $v_k$ is defined for each $n_u < k \leq m$ as either

(a) $v_k = v_i + v_j$, with $i, j < k$, or
(b) $v_k = v_i v_j$, with $i, j < k$, or
(c) $v_k = w_k(v_i)$, with $i < k$ and $w_k \in \mathcal{E}$,

and $h(\mathbf{u}) = v_m(\mathbf{u})$. A vector function is called factorable if each element is factorable.

Supposing that $\mathcal{U}$ is an $n_u$-dimensional, closed, bounded interval, $[u_1^L, u_1^U] \times \ldots \times [u_{n_u}^L, u_{n_u}^U]$, and $h$ is factorable, a standard application of McCormick's relaxation technique generates relaxations of $h$ on $\mathcal{U}$ by associating with each factor $v_k$ the quantities $(v_k^L, v_k^U, v_k^c, v_k^C)$, which are, respectively, lower and upper bounds for $v_k$ and convex and concave relaxations of $v_k$ on $\mathcal{U}$. The computation is initialized by letting $(v_k^L, v_k^U, v_k^c, v_k^C) = (u_k^L, u_k^U, u_k, u_k)$, for all $k \leq n_u$, and computing these values for the remaining factors recursively based on rules for each basic operation: $+$, $\times$ and composition with an element of $\mathcal{E}$. These propagation steps are based on the following lemma. The lemma is well known for $Q = \mathbb{R}^{n_u}$. Here, it is shown to hold in the case of a general vector space.

*Lemma 1:* Let $Q$ be a vector space and $\mathcal{U} \subset Q$ convex. Let $v_i^L, v_i^U, v_j^L, v_j^U \in \mathbb{R}$, $v_i, v_j, v_i^c, v_j^c, v_i^C, v_j^C : \mathcal{U} \to \mathbb{R}$ and suppose that $v_i(\mathbf{u}) \in [v_i^L, v_i^U]$, $\forall \mathbf{u} \in \mathcal{U}$, $v_i^c$ and $v_i^C$ are, respectively, convex and concave relaxations of $v_i$ on $\mathcal{U}$, and that the analogous conditions hold for $v_j$. Further, let $w, w^c : [v_i^L, v_i^U] \to \mathbb{R}$ and suppose that $w^c$ is a convex relaxation of $w$ on $[v_i^L, v_i^U]$. Finally, let $z^{\min}$ be a minimum of $w^c$ on $[v_i^L, v_i^U]$ and define

$$v_+^c(\mathbf{u}) = v_i^c(\mathbf{u}) + v_j^c(\mathbf{u}),$$
$$v_\times^c(\mathbf{u}) = \max(\alpha_i(\mathbf{u}) + \alpha_j(\mathbf{u}) - v_j^L v_i^L, \beta_i(\mathbf{u}) + \beta_j(\mathbf{u}) - v_j^U v_i^U),$$
$$v_w^c(\mathbf{u}) = w^c(\mathrm{mid}(v_i^c(\mathbf{u}), v_i^C(\mathbf{u}), z^{\min})).$$

where mid returns the middle value of its arguments and

$$\alpha_i(\mathbf{u}) = \min(v_j^L v_i^c(\mathbf{u}), v_j^L v_i^C(\mathbf{u})),$$
$$\alpha_j(\mathbf{u}) = \min(v_i^L v_j^c(\mathbf{u}), v_i^L v_j^C(\mathbf{u})),$$
$$\beta_i(\mathbf{u}) = \min(v_j^U v_i^c(\mathbf{u}), v_j^U v_i^C(\mathbf{u})),$$
$$\beta_j(\mathbf{u}) = \min(v_i^U v_j^c(\mathbf{u}), v_i^U v_j^C(\mathbf{u})).$$

Then $v_+^c$, $v_\times^c$ and $v_w^c$ are convex relaxation of $v_i + v_j$, $v_i \times v_j$ and $w \circ v_i$, respectively, on $\mathcal{U}$

*Proof:* The case of addition is trivial. Since $v_i^c(\mathbf{u}) \leq v_i^C(\mathbf{u})$ and $v_j^c(\mathbf{u}) \leq v_j^C(\mathbf{u})$, $\forall \mathbf{u} \in \mathcal{U}$, it follows from sign arguments that $\alpha_i$, $\alpha_j$, $\beta_i$ and $\beta_i$ are convex on $\mathcal{U}$. Because the sum of two convex functions is convex and the maximum of two convex functions is convex, it follows that $v_\times$ is convex on $\mathcal{U}$. It is well known [8] that

$$v_i(\mathbf{u}) v_j(\mathbf{u}) \geq \max(v_j^L v_i(\mathbf{u}) + v_i^L v_j(\mathbf{u}) - v_j^L v_i^L, v_j^U v_i(\mathbf{u}) + v_i^U v_j(\mathbf{u}) - v_j^U v_i^U),$$

$\forall \mathbf{u} \in \mathcal{U}$. It follows that $v_\times(\mathbf{u}) \leq v_i(\mathbf{u}) v_j(\mathbf{u})$, $\forall \mathbf{u} \in \mathcal{U}$.

Since $w^c$ is convex, it can be decomposed [8] into a constant part, $A \equiv w^c(z^{\min})$, a convex, non-increasing part, $w_D^c(z) = w^c(\min(z, z^{\min})) - A$, and a convex, non-decreasing part, $w_I^c(z) = w^c(\max(z, z^{\min})) - A$, such that $w^c(v_i(\mathbf{u})) = w_I^c(v_i(\mathbf{u})) + w_D^c(v_i(\mathbf{u})) + A$, $\forall \mathbf{u} \in \mathcal{U}$. By monotonicity arguments,

$$\begin{aligned}
&w_I^c(v_i(\mathbf{u})) + w_D^c(v_i(\mathbf{u})) + A \\
&\geq w_I^c(\max(v_i^c(\mathbf{u}), v_i^L)) + w_D^c(\min(v_i^C(\mathbf{u}), v_i^U)) + A \\
&= w^c(\max(v_i^c(\mathbf{u}), v_i^L, z^{\min})) + \\
&\qquad\qquad w^c(\min(v_i^C(\mathbf{u}), v_i^U, z^{\min})) - A \\
&= w^c(\max(v_i^c(\mathbf{u}), z^{\min})) + w^c(\min(v_i^C(\mathbf{u}), z^{\min})) - A \\
&= w^c(\text{mid}(v_i^c(\mathbf{u}), v_i^C(\mathbf{u}), z^{\min})), \quad \forall \mathbf{u} \in \mathcal{U}.
\end{aligned}$$

Then,

$$w(v_i(\mathbf{u})) \geq w^c(v_i(\mathbf{u})) \geq w^c(\text{mid}(v_i^c(\mathbf{u}), v_i^C(\mathbf{u}), z^{\min})),$$

for all $\mathbf{u} \in \mathcal{U}$. Convexity of the last term follows from the first equality above, since the composition of a convex non-increasing function with a concave functions is convex, the composition of a convex non-decreasing function with a convex function is convex, the maximum of two convex functions is convex, and the minimum of two concave functions is concave [2]. ■

*Remark 1:* The previous lemma enables the computation of $v_k^c$ from knowledge of $(v_i^L, v_i^U, v_i^c, v_i^C)$ for all $i < k$. An analogous set of rules are available for $v_k^C$ [8], and $v_k^L$ and $v_k^U$ can be computed using standard interval arithmetic [9]. According to this procedure, the requirement for functions in $\mathcal{E}$ is that interval extensions, as well as convex and concave relaxations are available. These are provided for many common univariate functions in [10].

In the case where $Q = \mathbb{R}^{n_u}$, $\mathcal{U}$ is an $n_u$-dimensional, closed, bounded interval, and $h$ is factorable, convex and concave relaxations of $h$ on $\mathcal{U}$ can now be obtained directly by recursive application of the rules in Lemma 1 and Remark 1 to the factors of $h$. It should be noted that the recursive nature of McCormick's relaxations facilitate an automatic computer implementation, e.g. [11], [12].

### A. Relaxations of composite functions

In [10], it was observed that Lemma 1 and Remark 1 can also be used to construct relaxations of composite functions. Since the arguments in [10] are based on application of Lemma 1, the method can easily be extended to composite functions on an arbitrary vector space. To present these results, let $\bar{U}$, $U(t)$, $\mathcal{U}$ be defined as in §I-A. Relaxations of the mapping $\mathcal{F}_{\mathbf{q},t}$ are derived for fixed $t \in I$, under the assumption that convex and concave relaxations of $\mathbf{x}(t, \cdot)$ are available (see §IV). The following assumption is required.

*Assumption 1:* Functions $\mathbf{x}^L, \mathbf{x}^U : I \to \mathbb{R}^{n_x}$ are available such that $\mathbf{x}(t, \mathbf{u}) \in X(t) \equiv [\mathbf{x}^L(t), \mathbf{x}^U(t)]$, $\forall (t, \mathbf{u}) \in I \times \mathcal{U}$, and $X(t) \subset D$, $\forall t \in I$.

*Remark 2:* Sufficient conditions for functions $\mathbf{x}^L$ and $\mathbf{x}^U$ to bound $\mathbf{x}$ in the sense of Assumption 1, as well as efficient methods for computing them, can be found in [13].

Now suppose that, for each fixed $t \in I$, $U(t) = [\mathbf{u}^L(t), \mathbf{u}^U(t)]$ is a closed, bounded $n_u$-dimensional interval and $\mathbf{q}(t, \cdot, \cdot)$ is factorable on $U(t) \times X(t)$. The method presented in [10] provides functions

$$\mathbf{q}_t^c, \mathbf{q}_t^C : U(t) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_q}$$

with the following property: If $\boldsymbol{\psi}_t^c, \boldsymbol{\psi}_t^C : \mathcal{U} \to \mathbb{R}^{n_x}$ are, respectively, convex and concave relaxations of $\mathbf{x}(t, \cdot)$ on $\mathcal{U}$, then the composite mappings

$$\begin{aligned}
U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) &\longmapsto \mathbf{q}_t^c(\mathbf{p}, \boldsymbol{\psi}_t^c(\mathbf{u}), \boldsymbol{\psi}_t^C(\mathbf{u})), \\
U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) &\longmapsto \mathbf{q}_t^C(\mathbf{p}, \boldsymbol{\psi}_t^c(\mathbf{u}), \boldsymbol{\psi}_t^C(\mathbf{u})),
\end{aligned}$$

are, respectively, convex and concave relaxations of

$$U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) \longmapsto \mathbf{q}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{u}))$$

on $U(t) \times \mathcal{U}$. Using these properties, it is easily verified that the mapping

$$\mathcal{U} \ni \mathbf{u} \longmapsto \mathcal{F}_{\mathbf{q},t}^c(\mathbf{u}) \equiv \mathbf{q}_t^c(\mathbf{u}(t), \boldsymbol{\psi}_t^c(\mathbf{u}), \boldsymbol{\psi}_t^C(\mathbf{u}))$$

is a convex relaxation of $\mathcal{F}_{\mathbf{q},t}$ on $\mathcal{U}$.

The construction of the functions $\mathbf{q}_t^c$ and $\mathbf{q}_t^C$ is discussed in detail in [10]. Informally, these functions are simply the results of recursively applying Lemma 1 (and Remark 1) to the factors of $\mathbf{q}(t, \cdot, \cdot)$, noting that bounds and relaxations on the first and second arguments are given by $(\mathbf{u}^L(t), \mathbf{u}^U(t), \mathbf{u}(t), \mathbf{u}(t))$ and $(\mathbf{x}^L(t), \mathbf{x}^U(t), \boldsymbol{\psi}_t^c(\mathbf{u}), \boldsymbol{\psi}_t^C(\mathbf{u}))$, respectively.

It is not difficult to see that relaxations of $\mathcal{F}_\phi$ and $\mathcal{F}_\mathbf{g}$ can also be constructed by analogous procedures. Thus, the task of deriving a convex underestimating program for (1) has been reduced to that of deriving convex and concave relaxations for the end-point map of the control system (2). That development occupies the remainder of the article.

## III. RELAXING INTEGRAL FUNCTIONALS

Let $\bar{U}$, $U(t)$ and $\mathcal{U}$ be defined as in §I-A. In this section, relaxations of the functional

$$\mathcal{U} \ni \mathbf{u} \longmapsto \mathcal{H}(\mathbf{u}) \equiv \int_{t_0}^{t} \mathbf{h}(s, \mathbf{u}(s)) ds, \tag{4}$$

are considered, where $\mathbf{h} : I \times \bar{U} \to \mathbb{R}^n$. Though no integral functionals appear in (1), the development in this section is required for relaxing the end-point maps of control systems in §IV. Indeed, integral functionals have not been included in (1) because they can be treated by augmenting quadrature variables to the control system (2). For the benefit of §IV, the following lemma is stated for a more general functionals than above.

*Lemma 2:* Let $\mathbf{h} : I \times \bar{U} \times \mathcal{U} \to \mathbb{R}^n$ and suppose that the mapping $t \longmapsto \mathbf{h}(t, \mathbf{u}(t), \mathbf{u})$ is in $(L^1(I))^n$ for every $\mathbf{u} \in \mathcal{U}$. If, for a.e. $t \in I$, the mapping

$$U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) \longmapsto \mathbf{h}(t, \mathbf{p}, \mathbf{u})$$

is convex on $U(t) \times \mathcal{U}$, then the mapping

$$\mathcal{U} \ni \mathbf{u} \longmapsto \mathcal{H}(\mathbf{u}) \equiv \int_{t_0}^{t} \mathbf{h}(s, \mathbf{u}(s), \mathbf{u}) ds$$

is convex on $\mathcal{U}$, for every $t \in I$.

*Proof:* Choose any $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$ and any $\lambda \in (0,1)$. For a.e. $s \in I$, the hypothesis on $\mathbf{h}$ and the fact that $\mathbf{u}_1(s), \mathbf{u}_2(s) \in U(s)$ imply that

$$\begin{aligned}&\mathbf{h}(s, \lambda\mathbf{u}_1(s) + (1-\lambda)\mathbf{u}_2(s), \lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2)\\ &\quad \leq \lambda\mathbf{h}(s, \mathbf{u}_1(s), \mathbf{u}_1) + (1-\lambda)\mathbf{h}(s, \mathbf{u}_2(s), \mathbf{u}_2).\end{aligned}$$

Since this holds for any $s \in I$, linearity and monotonicity of the integral imply that, for any $t \in I$,

$$\begin{aligned}&\int_{t_0}^{t} \mathbf{h}(s, \lambda\mathbf{u}_1(s) + (1-\lambda)\mathbf{u}_2(s), \lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2)ds\\ &\leq \lambda \int_{t_0}^{t} \mathbf{h}(s, \mathbf{u}_1(s), \mathbf{u}_1)ds + (1-\lambda)\int_{t_0}^{t} \mathbf{h}(s, \mathbf{u}_2(s), \mathbf{u}_2)ds.\end{aligned}$$

The result follows since $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$ and $\lambda \in (0,1)$ are arbitrary. ■

Lemma 2 will be used in its full generality in §IV. For the moment, consider the functional $\mathcal{H}$ as defined in (4), with $\mathbf{h} : I \times \bar{U} \to \mathbb{R}^n$. If, for each $t \in I$, $U(t)$ is interval-valued and $\mathbf{h}(t, \cdot)$ is nonconvex but factorable on $U(t)$, then McCormick's technique can be used to compute a convex relaxation of $\mathbf{h}(t, \cdot)$ on $U(t)$, $\mathbf{h}^c_t : U(t) \to \mathbb{R}^n$. Let $\mathbf{h}^c(t, \mathbf{p}) = \mathbf{h}^c_t(\mathbf{p})$ for all $(t, \mathbf{p})$ such that $t \in I$ and $\mathbf{p} \in U(t)$ and define $\mathcal{H}^c(\mathbf{u}) = \int_{t_0}^{t} \mathbf{h}^c(s, \mathbf{u}(s))ds$. Then Lemma 2 shows that $\mathcal{H}^c$ is a convex relaxation of $\mathcal{H}$ on $\mathcal{U}$, provided that the integral exists. We do not elaborate on this technical detail here, but note that in the case where the map $t \longmapsto U(t)$ is continuous, continuity of $\mathbf{h}^c$ follows from the analysis in [10].

## IV. RELAXING END-POINT MAPS OF CONTROL SYSTEMS

Let $\bar{U}$, $U(t)$, $\mathcal{U}$ and $D$ be defined as in §I-A, and consider the control system (2), where $\mathbf{f} : I \times \bar{U} \times D \to \mathbb{R}^{n_x}$.

*Assumption 2:* $\mathbf{f}$ is continuous on $I \times \bar{U} \times D$ and, for every compact $K \subset D$, $\exists L_K \in \mathbb{R}_+$ such that

$$\|\mathbf{f}(t, \mathbf{p}, \mathbf{z}) - \mathbf{f}(t, \mathbf{p}, \hat{\mathbf{z}})\|_1 \leq L_K \|\mathbf{z} - \hat{\mathbf{z}}\|_1,$$

for every $(t, \mathbf{p}, \mathbf{z}, \hat{\mathbf{z}}) \in I \times \bar{U} \times K \times K$.

Under Assumption 2, it can be shown by standard methods that there exists a closed interval $I' \subset I$ such that, corresponding to each $\mathbf{u} \in \mathcal{U}$ there exists a unique, absolutely continuous solution of (2) on $I'$. It is assumed that such a solution exists on all of $I$; that is, there exists a unique mapping $\mathbf{x} : I \times \mathcal{U} \to D$ satisfying (2) a.e. in $I$ for every $\mathbf{u} \in \mathcal{U}$. The objective of this section is to derive relaxations for the family of mappings $\mathcal{X}_t(\mathbf{u}) \equiv \mathbf{x}(t, \mathbf{u})$ on $\mathcal{U}$, for each $t \in I$. It will be shown that these relaxations are given by the solutions of a suitable auxiliary control system which can be generated using McCormick's relaxation technique.

Let $\mathbf{f}^c, \mathbf{f}^C : I \times \bar{U} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and consider the auxiliary control system

$$\begin{aligned}\dot{\mathbf{c}}(t, \mathbf{u}) &= \mathbf{f}^c(t, \mathbf{u}(t), \mathbf{c}(t, \mathbf{u}), \mathbf{C}(t, \mathbf{u})), \quad \text{a.e.} \quad t \in I, \quad (5)\\ \dot{\mathbf{C}}(t, \mathbf{u}) &= \mathbf{f}^C(t, \mathbf{u}(t), \mathbf{c}(t, \mathbf{u}), \mathbf{C}(t, \mathbf{u})), \quad \text{a.e.} \quad t \in I,\\ \mathbf{c}(t_0, \mathbf{u}) &= \mathbf{x}_0,\\ \mathbf{C}(t_0, \mathbf{u}) &= \mathbf{x}_0,\end{aligned}$$

for every $\mathbf{u} \in \mathcal{U}$.

*Assumption 3:* $\mathbf{f}^c$ and $\mathbf{f}^C$ are continuous on $I \times \bar{U} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$, and $\exists L \in \mathbb{R}_+$ such that

$$\begin{aligned}&\|\mathbf{f}^c(t, \mathbf{p}, \mathbf{z}, \mathbf{y}) - \mathbf{f}^c(t, \mathbf{p}, \hat{\mathbf{z}}, \hat{\mathbf{y}})\|_1\\ &\qquad + \|\mathbf{f}^C(t, \mathbf{p}, \mathbf{z}, \mathbf{y}) - \mathbf{f}^C(t, \mathbf{p}, \hat{\mathbf{z}}, \hat{\mathbf{y}})\|_1\\ &\leq L(\|\mathbf{z} - \hat{\mathbf{z}}\|_1 + \|\mathbf{y} - \hat{\mathbf{y}}\|_1)\end{aligned}$$

for all $(t, \mathbf{p}, \mathbf{z}, \mathbf{y}, \hat{\mathbf{z}}, \hat{\mathbf{y}}) \in I \times \bar{U} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$.

The following definition gives sufficient conditions for the solutions of (5) to describe convex and concave relaxations of $\mathbf{x}$ (see Theorem 1).

*Definition 3:* The auxiliary system of ODEs (5) is called a *C-system* of (2) on $\mathcal{U}$ if, in addition to satisfying Assumption 3, the following condition holds: For any mappings $\boldsymbol{\psi}^c, \boldsymbol{\psi}^C : I \times \mathcal{U} \to \mathbb{R}^{n_x}$ and a.e. $t \in I$, the functions

$$\begin{aligned}U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) &\longmapsto \mathbf{f}^c(t, \mathbf{p}, \boldsymbol{\psi}^c(t, \mathbf{u}), \boldsymbol{\psi}^C(t, \mathbf{u}))\\ U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) &\longmapsto \mathbf{f}^C(t, \mathbf{p}, \boldsymbol{\psi}^c(t, \mathbf{u}), \boldsymbol{\psi}^C(t, \mathbf{u}))\end{aligned}$$

are, respectively, convex and concave relaxations of

$$U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) \longmapsto \mathbf{f}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{u}))$$

on $U(t) \times \mathcal{U}$, provided that $\boldsymbol{\psi}^c(t, \cdot)$ and $\boldsymbol{\psi}^C(t, \cdot)$ are, respectively, convex and concave relaxations of $\mathbf{x}(t, \cdot)$ on $\mathcal{U}$.

If, for each fixed $t \in I$, $U(t)$ is a closed, bounded $n_u$-dimensional interval and $\mathbf{f}(t, \cdot, \cdot)$ is factorable on $U(t) \times X(t)$, then functions $\mathbf{f}^c$ and $\mathbf{f}^C$ satisfying Definition 3 can be readily derived by applying the methods of §II-A. In that case, Assumption 3 follows from the analysis in [10].

The following theorem shows that if (5) is a C-system of (2) on $\mathcal{U}$, then the unique solution of (5) provides relaxations of $\mathbf{x}(t, \cdot)$ on $\mathcal{U}$, for each $t \in I$. The proof uses a standard construction in ODE theory known as successive approximations (or Picard iterates) [14]. In particular, Theorem 2 in the Appendix is required.

*Theorem 1:* Suppose that the auxiliary system of ODEs (5) is a C-system of (2) on $\mathcal{U}$. Then $\mathbf{c}(t, \cdot)$ and $\mathbf{C}(t, \cdot)$ are, respectively, convex and concave relaxations of $\mathbf{x}(t, \cdot)$ on $\mathcal{U}$, for each fixed $t \in I$.

*Proof:* Choose any vectors $\mathbf{x}^L, \mathbf{x}^U \in \mathbb{R}^{n_x}$, such that $\mathbf{x}^L \leq \mathbf{x}(t, \mathbf{u}) \leq \mathbf{x}^U$, $\forall (t, \mathbf{u}) \in I \times \mathcal{U}$. Under Assumption 1, such vectors certainly exist. Let $\mathbf{c}^0(t, \mathbf{u}) = \mathbf{x}^L$ and $\mathbf{C}^0(t, \mathbf{u}) = \mathbf{x}^U$, $\forall (t, \mathbf{u}) \in I \times \mathcal{U}$, and consider the successive approximations defined recursively by

$$\begin{aligned}\mathbf{c}^{k+1}(t, \mathbf{u}) &= \mathbf{x}_0 \qquad (6)\\ &\qquad + \int_{t_0}^{t} \mathbf{f}^c(s, \mathbf{u}(s), \mathbf{c}^k(s, \mathbf{u}), \mathbf{C}^k(s, \mathbf{u}))ds,\\ \mathbf{C}^{k+1}(t, \mathbf{u}) &= \mathbf{x}_0\\ &\qquad + \int_{t_0}^{t} \mathbf{f}^C(s, \mathbf{u}(s), \mathbf{c}^k(s, \mathbf{u}), \mathbf{C}^k(s, \mathbf{u}))ds.\end{aligned}$$

Note that $\mathbf{f}^c$ and $\mathbf{f}^C$ are defined on $I \times \bar{U} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ and Lipschitz on all of $\mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ uniformly on $I \times \bar{U}$ by Assumption 3. Thus, Theorem 2 may be applied to (5), which shows that the successive approximations $\mathbf{c}^k$ and $\mathbf{C}^k$ in (6) exist and, for each fixed $\mathbf{u} \in \mathcal{U}$, converge uniformly on $I$ to the unique solutions of (5), $\mathbf{c}(\cdot, \mathbf{u})$ and $\mathbf{C}(\cdot, \mathbf{u})$.

Next, note that $\mathbf{c}^0(t, \cdot)$ and $\mathbf{C}^0(t, \cdot)$ are trivially convex and concave relaxations of $\mathbf{x}(t, \cdot)$ on $\mathcal{U}$, respectively, for each fixed $t \in I$. Suppose that the same is true of $\mathbf{c}^k$ and $\mathbf{C}^k$. Then, by Definition 3,

$$U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) \longmapsto \mathbf{f}^c(t, \mathbf{p}, \mathbf{c}^k(t, \mathbf{u}), \mathbf{C}^k(t, \mathbf{u}))$$
$$U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) \longmapsto \mathbf{f}^C(t, \mathbf{p}, \mathbf{c}^k(t, \mathbf{u}), \mathbf{C}^k(t, \mathbf{u}))$$

are, respectively, convex and concave relaxations of

$$U(t) \times \mathcal{U} \ni (\mathbf{p}, \mathbf{u}) \longmapsto \mathbf{f}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{u}))$$

on $U(t) \times \mathcal{U}$, for a.e. $t \in I$. Lemma 2 shows that

$$\mathcal{U} \ni \mathbf{u} \longmapsto \int_{t_0}^{t} \mathbf{f}^c(s, \mathbf{u}(s), \mathbf{c}^k(s, \mathbf{u}), \mathbf{C}^k(s, \mathbf{u}))ds$$
$$\mathcal{U} \ni \mathbf{u} \longmapsto \int_{t_0}^{t} \mathbf{f}^C(s, \mathbf{u}(s), \mathbf{c}^k(s, \mathbf{u}), \mathbf{C}^k(s, \mathbf{u}))ds$$

are, respectively, convex and concave on $\mathcal{U}$, for every fixed $t \in I$. Then, (6) shows that $\mathbf{c}^{k+1}(t, \cdot)$ and $\mathbf{C}^{k+1}(t, \cdot)$ are, respectively, convex and concave on $\mathcal{U}$ for every fixed $t \in I$.

Now, considering the under and overestimating properties of the functions $\mathbf{f}^c$ and $\mathbf{f}^C$ described above, for any $\mathbf{u} \in \mathcal{U}$ and a.e. $s \in I$, we have

$$\begin{aligned} &\mathbf{f}^c(s, \mathbf{u}(s), \mathbf{c}^k(s, \mathbf{u}), \mathbf{C}^k(s, \mathbf{u})) \\ &\leq \mathbf{f}(s, \mathbf{u}(s), \mathbf{x}(s, \mathbf{u})), \\ &\leq \mathbf{f}^C(s, \mathbf{u}(s), \mathbf{c}^k(s, \mathbf{u}), \mathbf{C}^k(s, \mathbf{u})). \end{aligned}$$

Combining this with integral monotonicity,

$$\begin{aligned} &\int_{t_0}^{t} \mathbf{f}^c(s, \mathbf{u}(s), \mathbf{c}^k(s, \mathbf{u}), \mathbf{C}^k(s, \mathbf{u}))ds \\ &\leq \int_{t_0}^{t} \mathbf{f}(s, \mathbf{u}(s), \mathbf{x}(s, \mathbf{u}))ds, \\ &\leq \int_{t_0}^{t} \mathbf{f}^C(s, \mathbf{u}(s), \mathbf{c}^k(s, \mathbf{u}), \mathbf{C}^k(s, \mathbf{u}))ds, \end{aligned}$$

for all $(t, \mathbf{u}) \in I \times \mathcal{U}$. Then, (6) shows that

$$\begin{aligned} \mathbf{c}^{k+1}(t, \mathbf{u}) &\leq \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{u}(s), \mathbf{x}(s, \mathbf{u}))ds \\ &\leq \mathbf{C}^{k+1}(t, \mathbf{u}), \quad \forall (t, \mathbf{u}) \in I \times \mathcal{U}, \end{aligned}$$

which, by the integral form of (2), gives

$$\mathbf{c}^{k+1}(t, \mathbf{u}) \leq \mathbf{x}(t, \mathbf{u}) \leq \mathbf{C}^{k+1}(t, \mathbf{u}), \quad \forall (t, \mathbf{u}) \in I \times \mathcal{U}.$$

Therefore, by induction, $\mathbf{c}^k(t, \cdot)$ and $\mathbf{C}^k(t, \cdot)$ are, respectively, convex and concave relaxations of $\mathbf{x}(t, \cdot)$ on $\mathcal{U}$, for each fixed $t \in I$ and every $k \in \mathbb{N}$.

It was shown above that, as $k \to \infty$, $\mathbf{c}^k$ and $\mathbf{C}^k$ converge pointwise to the unique solutions of (5) on $I \times \mathcal{U}$. Then, taking limits, it is clear that $\mathbf{c}(t, \cdot)$ and $\mathbf{C}(t, \cdot)$ are, respectively, convex and concave relaxations of $\mathbf{x}(t, \cdot)$ on $\mathcal{U}$, for each fixed $t \in I$. ∎

According to the previous theorem, the desired relaxations of the endpoint map $\mathcal{X}_t$ are given by $\mathcal{X}_t^c(\mathbf{u}) \equiv \mathbf{c}(t, \mathbf{u})$ and $\mathcal{X}_t^C(\mathbf{u}) \equiv \mathbf{C}(t, \mathbf{u})$, $\forall (t, \mathbf{u}) \in I \times \mathcal{U}$. Combining these relaxations with the analysis in §I-A and II-A, the desired relaxation of (1) can be derived.

## V. CONCLUSIONS AND FUTURE WORK

### A. Conclusions

A method has been presented for computing a rigorous lower bound for the nonconvex optimal control problem (1). In particular, a constructive procedure was described, based on McCormick's relaxation technique, which produces a convex optimization problem whose solution is guaranteed to underestimate the infimum in (1). Supposing that this convex program can be solved to global optimality, using for example the methods described in [2], a guaranteed lower bound on the infimum in (1) is obtained. Computing guaranteed lower bounds is a crucial step required by branch-and-bound global optimization algorithms. Thus, the method developed here provides a key ingredient required for branch-and-bound global optimization of nonconvex optimal control problems. Finally, the proposed lower bounding technique is distinguished from previous work in that it does not require control parametrization. The derived relaxations are valid in the original space of admissible control functions.

### B. Future work

Future work aims to incorporate the relaxations developed here into a general purpose global optimization algorithm for nonconvex optimal control problems.

## APPENDIX

*Theorem 2:* Consider the ODEs (2) and suppose that $\mathbf{f}$ is continuous on $I \times \bar{U} \times \mathbb{R}^{n_x}$ and $\exists L \in \mathbb{R}_+$ such that

$$\|\mathbf{f}(t, \mathbf{p}, \mathbf{z}) - \mathbf{f}(t, \mathbf{p}, \hat{\mathbf{z}})\|_1 \leq L\|\mathbf{z} - \hat{\mathbf{z}}\|_1,$$

for every $(t, \mathbf{p}, \mathbf{z}, \hat{\mathbf{z}}) \in I \times \bar{U} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$. Given any $\mathbf{x}^0 : I \times \mathcal{U} \to \mathbb{R}^{n_x}$ such that $\mathbf{x}^0(\cdot, \mathbf{u})$ is absolutely continuous on $I$ for any $\mathbf{u} \in \mathcal{U}$, the sequence of successive approximations defined recursively by

$$\mathbf{x}^{k+1}(t, \mathbf{u}) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{u}(s), \mathbf{x}^k(s, \mathbf{u}))ds \qquad (7)$$

satisfies the following conditions:

1) For each $\mathbf{u} \in \mathcal{U}$, each $\mathbf{x}^k$ exists and is absolutely continuous on $I$,
2) For each $\mathbf{u} \in \mathcal{U}$, the sequence $\{\mathbf{x}^k(\cdot, \mathbf{u})\}$ converges uniformly on $I$ to an absolutely continuous limit function $\mathbf{x}(\cdot, \mathbf{u})$ satisfying (2) uniquely.

*Proof:* Fix any $\mathbf{u} \in \mathcal{U}$. By hypothesis, $\mathbf{x}^0(\cdot, \mathbf{u})$ is absolutely continuous on $I$. Suppose this is true of $\mathbf{x}^k$. Continuity of $\mathbf{f}$ and measurability of $\mathbf{u}$ and $\mathbf{x}^k(\cdot, \mathbf{u})$ imply that $\mathbf{f}(\cdot, \mathbf{u}(\cdot), \mathbf{x}^k(\cdot, \mathbf{u}))$ is measurable (see [15]). Since this function is also bounded a.e. on $I$, it is integrable and hence (7) defines $\mathbf{x}^{k+1}(\cdot, \mathbf{u})$ as an absolutely continuous function on $I$. Induction shows that this property holds for all $k \in \mathbb{N}$.

Define

$$\gamma(t) \equiv \|\mathbf{f}(t, \mathbf{u}(t), \mathbf{x}^1(t, \mathbf{u})) - \mathbf{f}(t, \mathbf{u}(t), \mathbf{x}^0(t, \mathbf{u}))\|_1$$

and let $\bar{\gamma} = \text{ess sup}_{t\in I}\gamma(t)$. The assumption that $U(t) \subset \bar{U}$ for all $t \in I$, with $\bar{U}$ compact, along with the continuity of $\mathbf{f}$, $\mathbf{x}^1$ and $\mathbf{x}^0$, ensures that $\bar{\gamma}$ is finite. It will be shown that

$$\|\mathbf{x}^{k+1}(t,\mathbf{u}) - \mathbf{x}^k(t,\mathbf{u})\|_1 \leq \frac{\bar{\gamma}L^k(t-t_0)^k}{Lk!}, \qquad (8)$$

for all $t \in I$ and every $k \in \mathbb{N}$. For $k = 1$, (7) directly gives

$$\begin{aligned}
&\|\mathbf{x}^2(t,\mathbf{u}) - \mathbf{x}^1(t,\mathbf{u})\|_1 \\
&\leq \int_{t_0}^{t} \|\mathbf{f}(s,\mathbf{u}(s),\mathbf{x}^1(s,\mathbf{u})) - \mathbf{f}(s,\mathbf{u}(s),\mathbf{x}^0(s,\mathbf{u}))\|_1 ds \\
&\leq \bar{\gamma}(t-t_0), \quad \forall t \in I.
\end{aligned}$$

Supposing that (8) holds for some arbitrary $k$, the Lipschitz condition on $\mathbf{f}$ gives

$$\begin{aligned}
&\|\mathbf{x}^{k+2}(t,\mathbf{u}) - \mathbf{x}^{k+1}(t,\mathbf{u})\|_1 \\
&\leq L\int_{t_0}^{t} \|\mathbf{x}^{k+1}(s,\mathbf{u}) - \mathbf{x}^k(s,\mathbf{u})\|_1 ds, \\
&\leq \frac{\bar{\gamma}L^{k+1}}{Lk!}\int_{t_0}^{t}(s-t_0)^k ds, \\
&\leq \frac{\bar{\gamma}L^{k+1}(t-t_0)^{k+1}}{L(k+1)!}, \quad \forall t \in I.
\end{aligned}$$

Thus, induction proves (8). Now, for any $n, m \in \mathbb{N}$ with $m > n$, expansion by the triangle inequality and application of Equation (8) gives

$$\|\mathbf{x}^m(t,\mathbf{u}) - \mathbf{x}^n(t,\mathbf{u})\|_1 \leq \sum_{k=n}^{m-1}\frac{\bar{\gamma}L^k(t_f-t_0)^k}{Lk!}, \qquad (9)$$

for all $t \in I$. But

$$\sum_{k=0}^{\infty}\frac{\bar{\gamma}L^k(t_f-t_0)^k}{Lk!} = \frac{\bar{\gamma}}{L}e^{L(t_f-t_0)} < +\infty,$$

and hence $\lim_{n\to\infty}\sum_{k=n}^{\infty}\frac{\bar{\gamma}L^k(t_f-t_0)^k}{Lk!} = 0$, which implies by (9) that the sequence $\{\mathbf{x}^k(\cdot,\mathbf{u})\}$ is uniformly Cauchy on $I$. Continuity implies that this sequence converges uniformly to a continuous limit function $\mathbf{x}(\cdot,\mathbf{u})$ on $I$.

Next, it is shown that $\mathbf{x}$ is a solution of (2) on $I \times \mathcal{U}$. For any $\mathbf{u} \in \mathcal{U}$, the Lipschitz condition on $\mathbf{f}$ gives,

$$\begin{aligned}
&\|\int_{t_0}^{t}\mathbf{f}(s,\mathbf{u}(s),\mathbf{x}^k(s,\mathbf{u}))ds - \int_{t_0}^{t}\mathbf{f}(s,\mathbf{u}(s),\mathbf{x}(s,\mathbf{u}))ds\|_1 \\
&\leq L\int_{t_0}^{t}\|\mathbf{x}^k(s,\mathbf{u}) - \mathbf{x}(s,\mathbf{u})\|_1 ds, \quad \forall t \in I,
\end{aligned}$$

so uniform convergence of $\{\mathbf{x}^k(\cdot,\mathbf{u})\}$ to $\mathbf{x}(\cdot,\mathbf{u})$ on $I$ implies that $\lim_{k\to\infty}\int_{t_0}^{t}\mathbf{f}(s,\mathbf{u}(s),\mathbf{x}^k(s,\mathbf{u}))ds = \int_{t_0}^{t}\mathbf{f}(s,\mathbf{u}(s),\mathbf{x}(s,\mathbf{u}))ds$, $\forall t \in I$. Then, taking limits on both sides of (7) gives

$$\mathbf{x}(t,\mathbf{u}) = \mathbf{x}_0 + \int_{t_0}^{t}\mathbf{f}(s,\mathbf{u}(s),\mathbf{x}(s,\mathbf{u}))ds, \quad \forall t \in I,$$

which implies that $\mathbf{x}(\cdot,\mathbf{u})$ is absolutely continuous and solves (2). Uniqueness of $\mathbf{x}$ now follows (for each fixed $\mathbf{u} \in \mathcal{U}$) by a standard application of Gronwall's inequality (Proposition 1, Ch. 2, Sec. 4, [16]). ■

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