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New Approaches for $\mathscr{H}_\infty$ Performance Preserving Controller Reduction

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***Abstract*— This paper proposes several $\mathscr{H}_\infty$ performance preserving controller reduction methods. One of the advantages of the proposed methods is that the weighting functions for controller reduction is easy to compute and is readily available from standard $\mathscr{H}_\infty$ control design software. Numerical simulations show that the proposed methods are at least as effective as the best method available in the literature.**

I. INTRODUCTION

It is well-known that the $\mathscr{H}_\infty$ control theory and μ synthesis can be used to design robust performance controllers for highly complex uncertain systems [2], [5], [23], [24]. However, since a great many physical plants are modeled as high order dynamical systems, the controllers designed with these methodologies typically have very high orders (much higher than the plant orders) because of the performance weighting functions and the model uncertainty weighting functions. It is therefore desirable to find ways to reduce the orders of these controllers without sacrificing much of the performance. Of course, it is critical to reduce the controller order in such a way so that the performance degradation is minimized and it should be clearly noted that the absolute error between the full order controller and the reduced order is not critical. What is the most important is that the error in some critical range should be small [1], [3], [6]–[12], [14]–[19], [21], [22].

Motivated from some of the above recent work on controller reductions, we propose some additional controller reduction methods that can be easily and effectively performed. In addition, we shall propose some algorithms that will simplify some existing controller reduction algorithms.

The paper is organized as follows. In Section 2, we review the standard frequency weighted balanced reduction algorithm and show two seemingly obvious but contradicting results: 1. the frequency weighted balanced realization is independent of the particular realizations of the model and weighting functions; 2. the scalar one-sided weighted balanced realization is independent of the weighting function in input side or output side. Section 3 reviews all $\mathscr{H}_\infty$ controller parametrization. In Section 4, we propose several $\mathscr{H}_\infty$ controller reduction methods that can guarantee the closed-loop stability and performance. In Section 5, we show that the weighted gramians for some controller reduction algorithms can be obtained by solving low order Lyapunov equations which can significantly improve the computational accuracy and efficiency. A numerical example is shown in Section 6 to illustrate and compare different controller reduction methods.

The notations used in this paper is fairly standard as in the book [24].

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II. FREQUENCY-WEIGHTED MODEL REDUCTION

In this section, we briefly review the frequency-weighted balanced model reduction technique proposed by Enns [4], [20]. Given the original full-order model $G \in \mathscr{RH}_\infty$, the input weighting matrix $W_i \in \mathscr{RH}_\infty$, and the output weighting matrix $W_o \in \mathscr{RH}_\infty$, our objective is to find a lower-order model G_r such that

$$\|W_o(G - G_r)W_i\|_\infty$$

is made as small as possible. Assume that G, W_i, and W_o have the following state-space realizations:

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right], W_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], W_o = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right]$$

Define

$$A_{in} = \begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix}, B_{in} = \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix},$$

$$A_{out} = \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}, C_{out} = \begin{bmatrix} D_o C_G & C_o \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}, \tilde{Q} = \begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}$$

Then the input weighted Gramian P and the output weighted Gramian Q satisfy the following equations:

$$A_{in}\tilde{P} + \tilde{P}A_{in}^* + B_{in}B_{in}^* = 0 \quad (1)$$

$$\tilde{Q}A_{out} + A_{out}^*\tilde{Q} + C_{out}^*C_{out} = 0 \quad (2)$$

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Now let T be a nonsingular matrix such that

$$TPT^* = (T^{-1})^* Q T^{-1} = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

(i.e., balanced) with $\Sigma_1 = \text{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r})$ and $\Sigma_2 = \text{diag}(\sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N})$ and partition the system accordingly as

$$\left[\begin{array}{c|c} TA_G T^{-1} & TB_G \\ \hline C_G T^{-1} & D_G \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D_G \end{array}\right]$$

Then a reduced-order model G_r is obtained as

$$G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D_G \end{array}\right]$$

Unfortunately, there is generally no known a priori error bound for the approximation error and the reduced-order model G_r is not guaranteed to be stable either.

Theorem 1: The frequency weighted balanced realization is independent of the particular realizations of G, W_i, and W_o.

Proof: It is known that any two minimal realizations of a transfer matrix can be related by a similarity transformation [24]. Then it is easy to show that the weighted gramians P and Q do not depend on the particular realizations of W_i and W_o. Now let any other realization of G be given by

$$G = \left[\begin{array}{c|c} T_g A_G T_g^{-1} & T_g B_G \\ \hline C_G T_g^{-1} & D_G \end{array}\right]$$

for a nonsingular matrix T_g. Then the input weighted Gramian $\hat{P}$ and the output weighted Gramian $\hat{Q}$ satisfy

$$P = T_g^{-1} \hat{P} (T_g^{-1})^*, \quad Q = T_g^* \hat{Q} T_g$$

and $PQ = T_g^{-1} \hat{P} \hat{Q} T_g$. Hence the weighted balanced realization will not depend on the particular realization of G either. □

Theorem 2: Let W and G be scalar transfer functions. Then the input weighted balanced realization of G with input weighting W is the same as the output weighted balanced realization of G with output weighting W.

Proof: Assume that W and G have the following state space realizations:

$$W(s) = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right], \quad G(s) = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right]$$

Then

$$W^T(s) = \left[\begin{array}{c|c} A_w^T & C_w^T \\ \hline B_w^T & D_w^T \end{array}\right], \quad G^T(s) = \left[\begin{array}{c|c} A_g^T & C_g^T \\ \hline B_g^T & D_g^T \end{array}\right]$$

are also state space realizations of $W(s)$ and $G(s)$.

Note that the input weighted balanced realization of G with input weighting function W is the same as the output weighted balanced realization of G^T with output weighted function W^T since $(GW)^T = W^T G^T$. Hence by Theorem 1, the input weighted balanced realization of G with input weighting W is the same as the input weighted balanced realization of $G^T(s)$ with input weighting function $W^T(s)$. Then the conclusion follows by noting that $G^T(s) W^T(s) = (W(s)G(s))^T = W(s)G(s)$. □

It should be noted that the above conclusion does not hold in general for matrix cases. Furthermore, the weighted balanced realization with two-sided weighting functions can be quite tricky as demonstrated in the following example.

Example 1: Let $G_1 = \frac{2s+7}{(s+2)(s+5)}$. Let W_i and W_o be given by

$$W_i = \frac{s+2}{s+1}, \quad W_o = \frac{1}{s+2}.$$

Then a 1st order weighted balanced approximation with input weighting function W_i and output weighting function W_o is given by $\hat{G}_1 = \frac{1.79}{s+2.5783}$.

Next let $W = W_i W_o = \frac{1}{s+1}$. Then a 1st order weighted balanced approximation with (input or output) one-sided weighting function W is given by $\tilde{G}_1 = \frac{1.82}{s+2.62}$. Moreover,

$$\left\| W_o(G_1 - \hat{G}_1) W_i \right\|_\infty = \left\| W(G_1 - \hat{G}_1) \right\|_\infty = 0.0093$$
$$< \left\| W(G_1 - \tilde{G}_1) \right\|_\infty = 0.011.$$

Next, let $G_2 = \frac{2(s+1)}{(s+2)(s+5)}$. Then the 1st order weighted balanced approximation with input weighting function W_i and output weighting function W_o is given by

$$\hat{G}_2 = \frac{1.5556}{s+5.7037}$$

and the 1st order weighted balanced approximation with (input or output) one-sided weighting function W is given by

$$\tilde{G}_2 = \frac{1.53}{s+6.097}.$$

Moreover,

$$\left\| W_o(G_2 - \hat{G}_2) W_i \right\|_\infty = \left\| W(G_2 - \hat{G}_2) \right\|_\infty = 0.0727$$
$$> \left\| W(G_2 - \tilde{G}_2) \right\|_\infty = 0.0517.$$

This example shows that it is not clear if one-sided weighted method will do better than two-sided weighted method since they produce different results for different problems.

III. $\mathscr{H}_\infty$ Controller Parameterization

We consider a closed-loop system shown in Figure 1 with the n-th order generalized plant G. Suppose that K is an m-th order controller which stabilizes the closed-loop system. We are interested in investigating controller reduction methods that can preserve the closed-loop stability and minimize the performance degradation of the closed-loop systems with reduced order controllers.

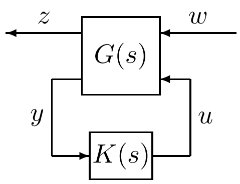


Fig. 1. Closed-loop System Diagram

It is now well known [5], [24] that all stabilizing controllers satisfying $\|T_{zw}\|_\infty < \gamma$ can be parameterized as

$$K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathscr{RH}_\infty, \quad \|Q\|_\infty < \gamma \qquad (3)$$

where M_∞ is of the form

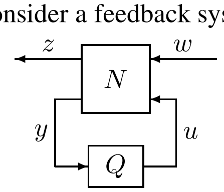
$$M_\infty = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right]$$

such that $\hat{D}_{12}$ and $\hat{D}_{21}$ are invertible and $\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1$ and $\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$ are both stable, i.e., M_{12}^{-1} and M_{21}^{-1} are both stable.

The problem to be considered here is to find a controller $\hat{K}$ with a minimal possible order such that the $\mathscr{H}_\infty$ performance requirement $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ is satisfied. This is clearly equivalent to finding a Q so that it satisfies the above constraint and the order of $\hat{K}$ is minimized. However, directly finding such a Q has proven to be very difficult.

The following lemma is useful in the subsequent development [24].

Lemma 1: Consider a feedback system shown below



where N is a suitably partitioned matrix

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}.$$

Then, the closed-loop transfer matrix from w to z is given by

$$T_{zw} = \mathcal{F}_\ell(N, Q) = N_{11} + N_{12}Q(I - N_{22}Q)^{-1}N_{21}.$$

Assume that the feedback loop is well-posed, i.e., $\det(I - N_{22}Q(\infty)) \neq 0$, and either N_{21} has full row rank or N_{12} has full column rank and $\|N\| \leq 1$ then $\|\mathcal{F}_\ell(N, Q)\|_\infty < 1$ if $\|Q\|_\infty < 1$.

IV. Proposed Controller Reduction Methods

Note that $K_0(s) := M_{11}(s)$ is the central controller that satisfies $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$. Now suppose $\hat{K}$ is a reduced order controller that also satisfies $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$. Then $\hat{K}$ can be represented as

$$\hat{K} = \mathcal{F}_\ell(M_\infty, Q) = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}$$

for some $Q \in \mathscr{H}_\infty$. Let

$$\Delta_K := \hat{K} - K_0. \qquad (4)$$

Then

$$\Delta_K = M_{12}Q(I - M_{22}Q)^{-1}M_{21}$$

and Q can be expressed in Δ_K as

$$Q = (I + M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22})^{-1}M_{12}^{-1}\Delta_K M_{21}^{-1}.$$

Hence finding a reduced order controller $\hat{K}$ such that $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$ is reduced to find a $\hat{K}$ such that $Q \in \mathscr{H}_\infty$ and $\|Q\|_\infty < \gamma$.

Note that $M_{12}^{-1} \in \mathscr{H}_\infty$, $M_{21}^{-1} \in \mathscr{H}_\infty$, and it can also be verified that $M_{21}^{-1}M_{22} \in \mathscr{H}_\infty$ and $M_{22}M_{12}^{-1} \in \mathscr{H}_\infty$.

Lemma 2: Suppose that $\Delta_K := \hat{K} - K_0$ is stable. Then

$$Q = (I + M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22})^{-1}M_{12}^{-1}\Delta_K M_{21}^{-1} \in \mathscr{H}_\infty$$

if one of the following conditions holds

(a) $\left\|\Delta_K M_{21}^{-1}M_{22}M_{12}^{-1}\right\|_\infty < 1$;

(b) $\left\|M_{21}^{-1}M_{22}M_{12}^{-1}\Delta_K\right\|_\infty < 1$;

(c) $\left\|M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22}\right\|_\infty < 1$;

(d) $\left\|M_{22}M_{12}^{-1}\Delta_K M_{21}^{-1}\right\|_\infty < 1$;

(e) $\left\|LM_{12}^{-1}\Delta_K M_{21}^{-1}M_{22}L^{-1}\right\|_\infty < 1$ for some square L such that $L, L^{-1} \in \mathscr{H}_\infty$;

(f) $\left\|J^{-1}M_{22}M_{12}^{-1}\Delta_K M_{21}^{-1}J\right\|_\infty < 1$ for some square J such that $J, J^{-1} \in \mathscr{H}_\infty$.

Conditions (a) and (b) were used in [19] to obtain reduced order controllers with impressive results.

Lemma 3: Let L and J be square transfer matrices such that $L, L^{-1}, J, J^{-1} \in \mathscr{H}_\infty$. Then

$$\min_{L, L^{-1} \in \mathscr{H}_\infty} \left\|LM_{12}^{-1}\Delta_K M_{21}^{-1}M_{22}L^{-1}\right\|_\infty$$

$$\leq \left\| \Delta_K M_{21}^{-1} M_{22} M_{12}^{-1} \right\|_\infty$$

and

$$\min_{J, J^{-1} \in \mathscr{H}_\infty} \left\| J^{-1} M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} J \right\|_\infty$$

$$\leq \left\| M_{21}^{-1} M_{22} M_{12}^{-1} \Delta_K \right\|_\infty .$$

This lemma shows that the least conservative stability conditions are (e) and (f) with appropriate L and J. Unfortunately, finding the optimal L and J is quite difficult and further research is needed.

However, there is no guarantee that $\|Q\|_\infty < \gamma$ will be satisfied even if any of the above conditions is satisfied. Hence the reduced order controller is not guaranteed to satisfy $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$ and this condition has to be verified for each reduced controller.

Another approach proposed in [10] considers the error $M_{12}^{-1} \Delta_K M_{21}^{-1}$.

Lemma 4: Suppose that $\Delta_K := \hat{K} - K_0$ is stable. Then

$$Q = (I + M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22})^{-1} M_{12}^{-1} \Delta_K M_{21}^{-1}$$

is stable and $\|Q\|_\infty < \gamma$ if

$$\left\| M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty < \frac{\gamma}{1 + \gamma \|M_{22}\|_\infty}.$$

Hence $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$ is guaranteed if the weighted approximation error $\left\| M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty$ is sufficiently small.

Nevertheless, this method may still be conservative. We shall propose some other methods below.

Theorem 3: Let $K_0 = M_{11}$ be a stabilizing controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$ and $\varepsilon > 0$. Then $\hat{K}$ is also a stabilizing controller such that $\left\| \mathcal{F}_\ell(G, \hat{K}) \right\|_\infty < \gamma$ if

$$\left\| M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \begin{bmatrix} \varepsilon\gamma M_{22} & I \end{bmatrix} \right\|_\infty < \frac{\varepsilon\gamma}{\sqrt{1+\varepsilon^2}}.$$

Proof: Let

$$\tilde{\Delta} = \begin{bmatrix} \tilde{\Delta}_1 & \tilde{\Delta}_2 \end{bmatrix} := M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \begin{bmatrix} \varepsilon\gamma M_{22} & I \end{bmatrix}$$

Then

$$\begin{aligned} Q &= (I + \frac{\tilde{\Delta}_1}{\gamma\varepsilon})^{-1} \tilde{\Delta}_2 = \left(I + \frac{\tilde{\Delta}}{\gamma\varepsilon} \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} \tilde{\Delta} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= \mathcal{F}_\ell \left(N, \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} \frac{\tilde{\Delta}}{\gamma} \right) \gamma \end{aligned}$$

where

$$N = \begin{bmatrix} 0 & \begin{bmatrix} \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} I \end{bmatrix} \\ \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} \frac{1}{\sqrt{1+\varepsilon^2}} I \\ 0 \end{bmatrix} \end{bmatrix}$$

and $N'N = I$. By Lemma 1, $\|Q\|_\infty < \gamma$ if

$$\left\| \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} \frac{\tilde{\Delta}}{\gamma} \right\|_\infty < 1$$

or equivalently $\left\| \tilde{\Delta} \right\|_\infty < \frac{\varepsilon\gamma}{\sqrt{1+\varepsilon^2}}$. □

Similarly, we have the following dual result.

Theorem 4: Let $K_0 = M_{11}$ be a stabilizing controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$ and $\varepsilon > 0$. Then $\hat{K}$ is also a stabilizing controller such that $\left\| \mathcal{F}_\ell(G, \hat{K}) \right\|_\infty < \gamma$ if

$$\left\| \begin{bmatrix} \varepsilon\gamma M_{22} \\ I \end{bmatrix} M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \right\|_\infty < \frac{\varepsilon\gamma}{\sqrt{1+\varepsilon^2}}.$$

Remark 1: Note that $\varepsilon > 0$ should be used as a design parameter. One may start from $\varepsilon = 0$ and in this case the above controller reduction methods are reduced to

$$\left\| M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \right\|_\infty .$$

In this case, the $\mathscr{H}_\infty$ performance is satisfied if the above error is sufficiently small by Lemma 4. However, it should be noted that the $\mathscr{H}_\infty$ performance may still be satisfied even if the inequality in Lemma 4 is not satisfied. Hence it is necessary to verify the exact $\mathscr{H}_\infty$ performance for each reduced order controller. On the other hand, when ε is very large, the method is equivalent to

$$\left\| M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} M_{22} \right\|_\infty$$

or

$$\left\| M_{22} M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \right\|_\infty .$$

Again the exact $\mathscr{H}_\infty$ performance has to be verified for each reduced order controller.

$\mathscr{H}_\infty$ Controller Reduction KZ Algorithm 1

- Find a reduced order controller $\hat{K}$ using the following criterion

$$\left\| M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \begin{bmatrix} \varepsilon\gamma M_{22} & I \end{bmatrix} \right\|_\infty .$$

$\mathscr{H}_\infty$ Controller Reduction KZ Algorithm 2

- Find a reduced order controller $\hat{K}$ using the following criterion

$$\left\| \begin{bmatrix} \varepsilon\gamma M_{22} \\ I \end{bmatrix} M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \right\|_\infty .$$

The related state space realizations of the relevant transfer matrices are given by

$$M_{21}^{-1} \begin{bmatrix} \varepsilon\gamma M_{22} & I \end{bmatrix} =$$

$$\left[\begin{array}{c|cc} \hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 & \varepsilon\gamma \left(\hat{B}_2 - \hat{B}_1 \hat{D}_{21}^{-1} \hat{D}_{22} \right) & -\hat{B}_1 \hat{D}_{21}^{-1} \\ \hline \hat{D}_{21}^{-1} \hat{C}_2 & \varepsilon\gamma \hat{D}_{21}^{-1} \hat{D}_{22} & \hat{D}_{21}^{-1} \end{array} \right]$$

$$\begin{bmatrix} \varepsilon\gamma M_{22} \\ I \end{bmatrix} M_{12}^{-1}$$

$$= \left[\begin{array}{c|c} \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_2\hat{D}_{12}^{-1} \\ \hline \varepsilon\gamma\left(\hat{C}_2 - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1\right) & \varepsilon\gamma\hat{D}_{22}\hat{D}_{12}^{-1} \\ -\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{12}^{-1} \end{array}\right].$$

V. Computational Issues in Weighted Balanced Controller Reduction

In this section, we shall look at how the frequency-weighted balanced model reduction method in Section 2 can be used to solve the controller reductions in the last section. We shall start with the simple case

$$HY\,Algorithm: \quad \left\| M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \right\|_\infty.$$

Define

$$G = K_0 = M_{11} = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_{11} \end{array}\right]$$

$$W_i = M_{21}^{-1} = \left[\begin{array}{c|c} \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{B}_1\hat{D}_{21}^{-1} \\ \hline \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1} \end{array}\right]$$

$$W_o = M_{12}^{-1} = \left[\begin{array}{c|c} \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & -\hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{12}^{-1} \end{array}\right].$$

Theorem 5: Suppose $\hat{A}$ is stable. Then the input weighted gramian P and the output weighted gramian Q in *HY Algorithm* can be computed from the following Lyapunov equations

$$(\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2)P + P(\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2)' + \hat{B}_1\hat{D}_{21}^{-1}(\hat{B}_1\hat{D}_{21}^{-1})' = 0 \qquad (5)$$

$$Q(\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1) + (\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1)'Q + (\hat{D}_{12}^{-1}\hat{C}_1)'\hat{D}_{12}^{-1}\hat{C}_1 = 0 \qquad (6)$$

Similarly, we have the following results.

Theorem 6: Suppose $\hat{A}$ is stable.

(a) The output weighted gramian in KZ Algorithm 1 is the Q obtained in equation (5).
(b) The input weighted gramian in KZ Algorithm 2 is the P obtained in equation (6).

These results show that the controller reduction algorithms HY, KZ1, and KZ2 can be performed by solving some lower order Lyapunov equations.

Note that if $\hat{A}$ is not stable, then we need to write

$$K_0 = K_{0s} + K_{0u}$$

such that K_{0s} is stable and K_{0u} is antistable. Now let the reduced order controller be

$$\hat{K} = \hat{K}_{0s} + K_{0u}$$

such that $\hat{K}_{0s}$ is a stable approximation of K_{0s} obtained using any algorithm proposed above.

VI. An Example

We consider a four-disk control system studied by Enns [1984]. We shall set up the dynamical system in the standard linear fractional transformation form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= \begin{bmatrix} \sqrt{q_1}H \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\ y &= C_2 x + \begin{bmatrix} 0 & I \end{bmatrix} w \end{aligned}$$

where $q_1 = 1 \times 10^{-6}, q_2 = 1$ and

$$A = \begin{bmatrix} a & 0 \\ I_7 & 0_{7\times1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0_{7\times1} \end{bmatrix}$$

$$a = [\ -0.161 \quad -6.004 \quad -0.58215 \quad -9.9835 \quad -0.40727 \quad -3.982 \quad 0\ ]$$

$$B_1 = [\ \sqrt{q_2}B_2 \quad 0\ ],$$

$$H = [\ 0 \quad 0 \quad 0 \quad 0 \quad 0.55 \quad 11 \quad 1.32 \quad 18\ ]$$

$$C_2 = [\ 0 \quad 0 \quad 6.4432\times10^{-3} \quad 2.3196\times10^{-3} \quad 7.1252\times10^{-2} \quad 1.0002 \quad 0.10455 \quad 0.99551\ ].$$

The optimal $\mathscr{H}_\infty$ norm for T_{zw} is $\gamma_{opt} = 1.1272$. We choose $\gamma = 1.2$ to compute an 8th order suboptimal controller K_o. The controller is reduced using several methods as described in this paper and in the book [24]. The results are listed in Table I where the following abbreviations are used to represent the model reduction methods in addition to those used in [24].

- NU1: $\left\| M_{21}^{-1}M_{22}M_{12}^{-1}\Delta_K \right\|_\infty$
- NU2: $\left\| \Delta_K M_{21}^{-1}M_{22}M_{12}^{-1} \right\|_\infty$
- KZ3: $\left\| M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22} \right\|_\infty$
- KZ4: $\left\| M_{22}M_{12}^{-1}\Delta_K M_{21}^{-1} \right\|_\infty$
- YH: $\left\| M_{12}^{-1}\Delta_K M_{21}^{-1} \right\|_\infty$
- YHx: $\left\| M_{21}^{-1}M_{12}^{-1}\Delta_K \right\|_\infty = \left\| \Delta_K M_{21}^{-1}M_{12}^{-1} \right\|_\infty$

Table I shows that the performance weighted controller reduction methods, PWA, PWRCF, PWLCF, NU1, NU2, KZ1 , KZ2, KZ3, KZ4, YH, YHx, all work very well. It also shows that the unweighted reduction method, UWRCF, also works well. However, it is believed that this is true mostly because of the particular structure of the system matrices [24]. Note that Algorithms KZ1 and KZ2 are reduced to YH when $\varepsilon = 0$. On the other hand, the Algorithms KZ1 and KZ2 are reduced to KZ3 and KZ4 respectively as $\varepsilon \to \infty$.

The case YHx is created only for this example because $M_{21}^{-1}M_{12}^{-1}$ may not make sense in general if the number of inputs is not the same as the number of outputs. Another reason for listing this case is to show that even though $\left\| M_{12}^{-1}\Delta_K M_{21}^{-1} \right\|_\infty = \left\| M_{21}^{-1}M_{12}^{-1}\Delta_K \right\|_\infty = \left\| \Delta_K M_{21}^{-1}M_{12}^{-1} \right\|_\infty$ for a fixed Δ_K, Δ_K will be different when the weighted model reduction method is applied to these different criteria to obtain the reduced order controllers.

Order of $\hat{K}$	7	6	5	4	3	2
PWA	1.196	1.196	1.199	1.197	U	4.99
PWRCF	1.2	1.196	1.207	1.195	2.98	1.67
PWLCF	1.197	1.196	U	1.197	U	U
UWA	U	1.321	U	U	U	U
UWRCF	1.198	1.196	1.199	1.196	U	U
UWLCF	1.985	1.258	27.04	5.059	U	U
SWA	1.327	1.199	2.27	1.47	23.5	U
SWRCF	1.236	1.197	1.251	1.201	13.9	1.42
SWLCF	1.417	1.217	48.04	3.031	U	U
NU1	1.197	1.196	1.199	1.196	U	2.98
NU2	1.197	1.196	1.199	1.196	U	2.98
KZ3	U	1.196	U	1.197	U	U
KZ4	U	1.196	U	1.197	U	U
YH	U	1.196	U	1.197	U	U
YHx	1.197	1.196	1.199	1.196	U	3.11
KZ1 $\varepsilon = 0.1$	U	1.196	U	1.197	U	U
KZ1 $\varepsilon = 1$	U	1.196	U	1.197	U	U
KZ1 $\varepsilon = \infty$	U	1.196	U	1.197	U	U
KZ2 $\varepsilon = 0.1$	U	1.196	U	1.197	U	U
KZ2 $\varepsilon = 1$	U	1.196	U	1.197	U	U
KZ2 $\varepsilon = \infty$	U	1.196	U	1.197	U	U

TABLE I

$\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty$ WITH REDUCED ORDER CONTROLLER: U–CLOSED-LOOP SYSTEM IS UNSTABLE

VII. CONCLUSIONS

In this paper, we first show that the weighted balanced realization (or model reduction) does not depend on the particular realizations of the weighting functions and the system. This is not surprising and seems to be well accepted. However, our past limited numerical experience shows that this result may not be so obvious from numerical simulations with high order weighting functions and systems that are not well-conditioned. Indeed, numerical simulations often seem to suggest that results are realization dependent. The details are not given here due to space limitation and will be reported in [13]. We have also shown that the one-sided scalar weighted balanced realization is not relevant if the weighting function is in the input or in the output. This in turn implies that for our SISO example, the results from NU1 and NU2 are the same which is confirmed from our numerical simulations. However, the results from NU1 and NU2 with one-sided weighting functions are in general not the same as the results from KZ3 and KZ4 with two-sided weighting functions.

To reduce the computational burden in the controller reduction algorithms, we have shown that YH controller reduction method can be obtained by solving two n-th order Lyapunov equations. Furthermore, the two n-th order Lyapunov equations are also used in the KZ1-KZ4 algorithms.

Although PWA, PWRCF, PWLCF, NU1, NU2, KZ1, KZ2, KZ3, KZ4, and YH all work very well, one of the advantages of the proposed methods and NU1, NU2, and YH methods is that the $\mathscr{H}_\infty$ controller reduction weighting functions can be obtained easily from the controller parameterization matrix M_∞.

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