2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

A small-gain result for orthant-monotone systems in feedback: the non sign-definite case

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***Abstract*— This note introduces a small-gain result for interconnected MIMO orthant-monotone systems for which no matching condition is required between the partial orders in input and output spaces of the considered subsystems. Previous results assumed that the partial orders adopted would be induced by positivity cones in input and output spaces and that such positivity cones should fulfill a compatibility rule: namely either be coincident or be opposite. Those two configurations corresponded to positive-feedback or negative feedback cases. We relax those results by allowing arbitrary orthant orders.**

I. INTRODUCTION

Monotone dynamical systems [8] and monotone control systems [4] are an important class of models in several areas of applications, and in particular in the emerging field of systems biology. They are usually defined on partially ordered spaces and enjoy the feature of preserving the partial order along solutions for nonnegative times (precise definitions to be given later).

Such properties have rich consequences in terms of the possible dynamical behaviours that monotone systems may exhibit both in isolation (see for instance the celebrated Hirsch's 'Generic Convergence Theorem', [8]) or when interconnected, (see for instance [3]). In particular, for systems which admit well defined Input-State and Input-Output steady-state characteristics, that is maps associating to each constant input a unique globally asymptotically stable equilibrium (I-S characteristic) and its corresponding output value (I-O characteristic), it is possible to tightly quantify Input-Output gains.

It is therefore meaningful to derive sufficient conditions under which feedback interconnections of monotone systems exhibiting I-O steady-state characteristics are guaranteed to yield globally convergent and stable dynamics. This was done in [4] and [2] for the case of SISO systems in Negative and Positive feedback respectively.

Roughly speaking, if the discrete iteration induced by the composition of I-O characteristics has a unique globally asymptotically stable fixed point, then a unique globally asymptotically stable equilibrium exists for the original continuous time interconnected system. Such results are tight provided one is willing to allow for arbitrary delays in the feedback loop (see below).

These results, for the negative feedback case, were later generalized to MIMO systems (possibly including time-delays) [6], as well as to the case of reaction-diffusion partial differential equations [5]. A generalization to MIMO systems of the result concerning positive feedback interconnections can be found in [7].

In the following we limit our considerations to the most common situation of orthant-monotone systems, that is systems for which the partial orders governing the monotonicity property of individual subsystems are induced by positivity cones which are orthants.

A limitation of the results concerning both the positive and negative feedback cases for MIMO systems is that the number of orthants in Euclidean space of dimension n grows as 2^n. The cases of negative or positive feedback in fact correspond to the situation in which orthants defining partial orders in Input and Output spaces are either matched or 'anti-matched'. This is a sharp limitation to the results as only 2 out of 2^n possible cases are contemplated by current small-gain theorems. The goal of this note is to provide sufficient conditions for global asymptotic stability which can be used in the general case of non sign-definite feedback interconnections of MIMO orthant-monotone systems.

Examples of application of the theorem are also discussed.

II. PROBLEM FORMULATION

Monotone control systems are usually defined on subsets of Euclidean space. We denote the state-space by $X \subseteq \mathbb{R}^n$, input space $U \subseteq \mathbb{R}^m$ and output space $Y \subseteq \mathbb{R}^p$. Input signals are assumed to be locally essentially bounded and measurable functions of time, and we denote by $\mathcal{U}$ the corresponding set of input signals. Furthermore we consider orthants $K_U \subseteq \mathbb{R}^m$ and $K_Y \subseteq \mathbb{R}^p$. Partial orders can be accordingly defined in U and Y by letting for all $u_1, u_2 \in U$ and $y_1, y_2 \in Y$:

$$u_1 \succeq_u u_2 \Leftrightarrow u_1 - u_2 \in K_U$$

$$y_1 \succeq_y y_2 \Leftrightarrow y_1 - y_2 \in K_Y.$$

To simplify notations we will omit the subscript $\succeq_{\{u,y\}}$ when clear from the context.

Definition 1: A time-invariant input-output orthant-monotone control system is a mapping $\psi : X \times \mathcal{U} \to \mathcal{Y}$, such that for all initial states $x_0 \in X$ and all ordered input pairs $u_1, u_2 \in \mathcal{U}$ (viz. $u_1(t) \succeq u_2(t)$ for all $t \geq 0$), the corresponding output signals $y_1 = \psi(x_0, u_1)$ and $y_2 = \psi(x_0, u_2)$ fulfill $y_1(t) \succeq y_2(t)$ for all $t \geq 0$. Together with ψ we also understand that a state transition map: $\varphi : \mathbb{R}_{\geq 0} \times X \times \mathcal{U}$ exists such that

1) $\varphi(0, x_0, u) = x_0$, for all $x_0 \in X$ and all $u \in \mathcal{U}$;

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978-1-61284-799-3/11/$26.00 ©2011 IEEE

2) $\varphi(t_1, \varphi(t_2, x_0, u), \sigma_{t_2} u) = \varphi(t_1 + t_2, x_0, u)$, for all $t_1, t_2 \geq 0$, all $x_0 \in X$ and all $u \in \mathcal{U}$; the operator σ_t denotes the shift by t units backwards in time;
3) If $y = \psi(x_0, u)$, and $\tilde{y} = \psi(\varphi(t, x_0, u), \sigma_t u)$, then $y(\tau) = \tilde{y}(\tau - t)$ for all $\tau \geq t \geq 0$.

A typical instance of Input-Output orthant-monotone control system arises when systems of ordinary differential equations are considered,

$$\dot{x} = f(x, u) \qquad y = h(x), \tag{1}$$

where $f : \tilde{X} \times U \to \mathbb{R}^n$ is defined in some open neighborhood of $\tilde{X}$ of X and is locally Lipschitz continuous in x and jointly continuous in x and u, while $h : X \to Y$ is a continuous read-out map. By letting $K_X \subset \mathbb{R}^n$ be any convex pointed closed cone, we define the induced partial order

$$x_1 \succeq_x x_2 \Leftrightarrow x_1 - x_2 \in K_X$$

for all $x_1, x_2 \in X$. As pointed out in [4], if h is a monotone map, namely

$$x_1 \succeq_x x_2 \Rightarrow h(x_1) \succeq_y h(x_2) \tag{2}$$

and the map f fulfills suitable "infinitesimal" conditions, namely:

$$\begin{aligned} x_1 \succeq_x x_2, \quad & u_1 \succeq_u u_2 \\ & \Rightarrow \quad f(x_1, u_1) - f(x_2, u_2) \in \mathcal{TC}_{x_1 - x_2} K_X \end{aligned}$$

then the induced flow, $\varphi(t, x_0, u)$ is monotone with respect to inputs and initial conditions, viz. it fulfills:

$$\begin{aligned} x_1 \succeq_x x_2, \quad & u_1(t) \succeq_u u_2(t) \; \forall\, t \geq 0 \\ & \Rightarrow \varphi(t, x_1, u_1) \succeq_x \varphi(t, x_2, u_2), \quad \forall\, t \geq 0 \end{aligned} \tag{3}$$

(the symbol $\mathcal{TC}_x K$ in the previous equation denotes the tangent cone to K at x). Hence, it is straightforward from combining (3) and (2) that:

$$\begin{aligned} x_1 \succeq_x x_2, \quad & u_1(t) \succeq_u u_2(t) \; \forall\, t \geq 0 \\ \Rightarrow \quad & h(\varphi(t, x_1, u_1)) \succeq_y h(\varphi(t, x_2, u_2)), \quad \forall\, t \geq 0. \end{aligned} \tag{4}$$

In particular then, for any $x_0 \in X$, it holds that:

$$\begin{aligned} & u_1(t) \succeq_u u_2(t) \quad \forall\, t \geq 0 \\ & \Rightarrow h(\phi(t, x_0, u_1)) \succeq_y h(\phi(t, x_0, u_2)) \; \forall\, t \geq 0, \end{aligned} \tag{5}$$

which is equivalent to Definition 1 (notice that $\psi(x, u) := h(\varphi(\cdot, x, u))$ for systems given in state-space form). We are now ready to define the notion of input-output characteristic.

Definition 2: A system admits a steady-state input-output characteristic if, for each constant input signal $u \in \mathcal{U}$, there exists a unique $\bar{y} \in Y$, such that the output $y = \psi(x_0, u)$ satisfies for all $x_0 \in X$:

$$\lim_{t \to +\infty} y(t) = \bar{y}.$$

Moreover the map $\gamma : U \to Y$ which to each input value associates the corresponding asymptotic output value is *continuous* (this is the so called steady-state characteristic).

Notice that $k(\cdot)$ is, for any input-output monotone system, also trivially a monotone map. In the following we consider interconnected input-output orthant-monotone systems of the following form:

$$\begin{gathered} y_1 = \psi_1(x_1, u_1) \qquad y_2 = \psi_2(x_2, u_2) \\ y_1 = u_2, \quad y_2 = u_1. \end{gathered} \tag{6}$$

where $\psi_1 : X_1 \times \mathcal{U}_1 \to \mathcal{Y}_1$ and $\psi_2 : X_2 \times \mathcal{U}_2 \to \mathcal{Y}_2$ are systems defined on state, input and output spaces given by X_1, U_1, Y_1 and X_2, U_2, Y_2 respectively. To make sense of (6) we obviously need to assume that $U_1 \subseteq \mathbb{R}^{m_1}$ and $Y_2 \subseteq \mathbb{R}^{p_2}$ and in particular that $m_1 = p_2$; symmetrically we have $U_2 \subseteq \mathbb{R}^{m_2}$ and $Y_1 \subseteq \mathbb{R}^{p_1}$ with $m_2 = p_1$.

For technical reasons, which will become clear later, we ask that U_1, Y_1, U_2, Y_2 be boxes, that is cartesian products of possibly unbounded real intervals.

We do not assume, however, that $K_{U_1} = \pm K_{Y_2}$ nor that $K_{U_2} = \pm K_{Y_1}$, and that is the main point of departure of the present paper with respect to previous small gain results in the literature. We denote by Λ_{12} and Λ_{21}, diagonal matrices of suitable dimensions with entries in $\{-1, 1\}$ so that

$$K_{U_1} = \Lambda_{21} K_{Y_2}, \qquad K_{U_2} = \Lambda_{12} K_{Y_1}. \tag{7}$$

It is useful in the following developments to introduce the notion of interval for a partially ordered space. Given a partial order $\succeq$, we define the set $[a, b] = \{x : b \succeq x \succeq a\}$, in analogy to intervals of the real line. As we will need more than one partial order even for the same underlying Euclidean space, it is convenient to specify as a subscript the order associated to a particular interval set. Accordingly we let $[a, b]_K$ denote the interval a, b as defined by considering the partial order induced by K.

An easy but essential step in the following developments is to realize that, for the case of orthant-induced orders, intervals are always closed boxes. Moreover, it is possible to express any given box as an interval regardless of the adopted partial order. The following Lemma is not hard to prove:

Lemma 1: Let $K_1, K_2 \subset \mathbb{R}^n$ be orthants such that $K_1 = \Lambda K_2$ for some diagonal matrix with entries in $\{-1, 1\}$. Let $\Lambda_+ = \max\{\Lambda, 0\}$ and $\Lambda_- = -\min\{\Lambda, 0\} = I - \Lambda_+$; then

$$[a, b]_{K_1} = [\Lambda_+ a + \Lambda_- b, \Lambda_+ b + \Lambda_- a]_{K_2}. \tag{8}$$

It is worth pointing out that for symmetric intervals this takes the simpler form:

$$[-a, a]_{K_1} = [-\Lambda a, \Lambda a]_{K_2}. \tag{9}$$

To make notations compact it is useful to define for any given Λ the following block-matrix:

$$\tilde{\Lambda} = \begin{bmatrix} \Lambda_+ & \Lambda_- \\ \Lambda_- & \Lambda_+ \end{bmatrix}. \tag{10}$$

III. Main result

We are now ready to state our main result.

Theorem 1: Consider the interconnection of Input-Output monotone systems as in (6), and assume that the closed-loop system $\psi : (X_1 \times X_2) \to \mathcal{Y}_1 \times \mathcal{Y}_2$ is well-posed and admits uniformly bounded output solutions. If ψ_1 and ψ_2 admit input-output steady-state characteristics, $\gamma_1 : U_1 \to Y_1$ and $\gamma_2 : U_2 \to Y_2$, denote with a slight abuse of notation $\gamma_i([a,b]) = [\gamma_i(a), \gamma_i(b)]$ respectively. Provided for all intervals $[u_1^{\min}, u_1^{\max}] \subset U_1$, the discrete iteration

$$\begin{aligned} &[u_{\min}(k+1), u_{\max}(k+1)] \\ &\quad = \gamma_2(\gamma_1([u_{min}(k), u_{\max}(k)])\tilde{\Lambda}_{12})\tilde{\Lambda}_{21} \\ &[u_{\min}(0), u_{\max}(0)] = [u_1^{\min}, u_1^{\max}] \end{aligned} \tag{11}$$

converges to a unique fixed point $[\bar{u}, \bar{u}]$, then the closed-loop system is globally convergent. Namely, for all $(x_1, x_2) \in X_1 \times X_2$, the output $y = \psi(x)$ fulfills:

$$\lim_{t \to +\infty} y(t) = [\gamma_1(\bar{u}), \bar{u}]'.$$

Notice that for the iteration to be well defined we need to have $\gamma_1(U_1) \subseteq U_2$ and $\gamma_2(U_2) \subseteq U_1$.

The proof of Theorem 1 will rely on the following Lemma:

Lemma 2: Suppose given a system with input-output characteristic γ. Let $u \in \mathcal{U}$ be a bounded signal, such that for all $\varepsilon > 0$, $d(u(t), [u_1, u_2]_U) \leq \varepsilon$ for sufficiently large times. Let $y = \varphi(x, u)$, where $x \in X$ is an arbitrary initial condition. Then, for all $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$d(y(t), [\gamma(u_1), \gamma(u_2)]\,) \leq \varepsilon \qquad \forall\, t \geq T_\varepsilon. \tag{12}$$

Proof. Let $\{u_1^n\}_{n=1}^{+\infty}$ and $\{u_2^n\}_{n=1}^{+\infty}$ be two sequences converging to u_1 and u_2 with the property that $u(t) \in [u_1^n, u_2^n]_U$ for all sufficiently large t. Fix n arbitrary, and assume without loss of generality that $u(t) \in [u_1^n, u_2^n]$ for all $t \geq 0$ (if not, just pick as initial condition the state reached at some sufficiently large time so that $u(t)$ is contained in $[u_1^n, u_2^n]$ thereafter). Let $y = \varphi(x, u)$, $y_1 = \varphi(x, u_1^n)$ and $y_2 = \varphi(x, u_2^n)$, where with a slight abuse of notation we have identified the input values u_1^n and u_2^n with the corresponding constant signals. By monotonicity:

$$y_1(t) \leq y(t) \leq y_2(t) \qquad \forall\, t \geq 0. \tag{13}$$

By definition of steady-state characteristic $y_1(t) \to \gamma(u_1^n)$ and $y_2(t) \to \gamma(u_2^n)$ as $t \to +\infty$, hence, by continuity of γ, for all $\varepsilon > 0$ there exists T_ε as in equation (12).

We are now ready to prove our Main Result.

Pick $x = (x_1, x_2) \in X_1 \times X_2$, and let $y = [y_1, y_2] = \varphi(x)$ be the corresponding output response, with $y_1 \in \mathcal{Y}_1$ and $y_2 \in \mathcal{Y}_2$. By assumption y_1 and y_2 are defined for all $t \geq 0$ and bounded. Given the interconnection rules so are also u_1 and u_2. Hence, exploiting the fact that U_1 is a box, there exist $\underline{u}_1$, $\bar{u}_1$ such that $u_1(t) \in [\underline{u}_1, \bar{u}_1]_{U_1}$ for all $t \geq 0$.

By Lemma 2, for any $\varepsilon > 0$, $d(y_1(t), [\gamma_1(\underline{u}_1), \gamma_1(\bar{u}_1)]_{Y_1}) \leq \varepsilon$ for all sufficiently large t. Hence, by Lemma 1, $d(u_2(t), [\gamma_1([\underline{u}_1, \bar{u}_1])\tilde{\Lambda}_{12}]_{U_2}) \leq \varepsilon$ for sufficiently large t. By applying Lemma 2 once more we get $d(y_2(t), [\gamma_2(\gamma_1([\underline{u}_1, \bar{u}_1])\tilde{\Lambda}_{12})]_{Y_2} \leq \varepsilon$ for sufficiently large t and by Lemma 2 this is equivalent to:

$$d(u_1(t), [\gamma_2(\gamma_1([\underline{u}_1, \bar{u}_1])\tilde{\Lambda}_{12})\tilde{\Lambda}_{21}]_{U_1}) \leq \varepsilon$$

for all sufficiently large t. Let $\gamma_2(\gamma_1([a,b])\tilde{\Lambda}_{12})\tilde{\Lambda}_{21}$ be denoted by $K([a,b])$. By induction we can show that for any k,

$$d(u_1(t), K^k([\underline{u}_1, \bar{u}_1])) \leq \varepsilon$$

for all sufficiently large t. As by assumption the discrete iteration K^k converges to a fixed point $[\bar{u}, \bar{u}]$, we have that for any $\varepsilon > 0$ it is possible to choose k large enough, so that $|u_1(t) - \bar{u}| \leq 2\varepsilon$ for all sufficiently large t. As $\varepsilon > 0$ is arbitrary, $u_1(t) \to \bar{u}$. This shows that $y_2(t) \to \bar{u}$. A similar argument can be employed to show that $y_1(t) \to \gamma_1(\bar{u})$.

It is worth pointing out that, unlike classical small-gain theorems such as [9], boundedness of solutions is assumed rather than being a consequence of the small-gain condition. This was remarked also in [4], where additional technical assumptions are provided for the case of monotone systems of differential equations in feedback.

Let us mention that the positive feedback case corresponds here to $\Lambda_{12} = \Lambda_{21} = I_{m_1}$. As $\tilde{\Lambda}_{12} = \tilde{\Lambda}_{21} = I_{2m_1}$, the iteration (11) decouples into two identical and non-interacting subsystems. Instead, the negative feedback case, amounts to $\Lambda_{12} = I_{m_1}$ and $\Lambda_{21} = -I_{m_1}$. In this case, iteration (11) looks coupled and seems to depart from the original criterion proposed in [4]. However, even iterates of (11) exhibit the desirable decoupled structure of two identical non-interacting subsystems. This allows one to reduce dimension from $2m_1$ to just m_1 and restate the results in terms of the iteration originally proposed in [4].

Remark 1: It is worth pointing out that an even more general class of systems fulfilling the small-gain theorem are those for which the property expressed in Lemma 2 holds, regardless of any monotonicity assumptions.

IV. Linear systems

The case of linear systems deserves special attention, as Theorem 1 is original, to the best of our knowledge, even in the case of finite dimensional monotone linear systems:

$$\dot{x} = Ax + Bu \qquad y = Cx. \tag{14}$$

The steady-state Input-Output characteristic $\gamma : U \to Y$ is trivially the map $\gamma(u) = \Gamma u$ with

$$\Gamma = -CA^{-1}B.$$

As a consequence of monotonicity $\Gamma K_U \subseteq K_Y$. It is clear from the proof of Theorem 1 that in the case of linear systems since U and Y are Euclidean spaces, it is possible to define iteration (11) by only considering symmetric intervals $[-a, a]$; in fact the iteration maps (for systems with odd characteristics) preserves symmetric intervals. Hence, exploiting the simpler formula (9) one may recast (11) as follows:

$$u(k+1) = \Lambda_{21}\Gamma_2\Lambda_{12}\Gamma_1 u(k),$$

where, for the sake of simplicity, we did not explicitly write the iteration for the interval $[-u(k), u(k)]$, but only for one of its extremes. The condition that the latter be a converging iteration amounts to:

$$\rho(\Lambda_{21}\Gamma_2\Lambda_{12}\Gamma_1) < 1. \tag{15}$$

The following result holds for linear systems:

Theorem 2: Consider the following interconnected systems:

$$\begin{array}{rlr} \dot{x}_1 = & A_1x_1 + B_1u_1 & u_2 = y_1 = C_1x_1 \\ \dot{x}_2 = & A_2x_2 + B_2u_2 & u_1 = y_2 = C_2x_2 \end{array} \tag{16}$$

with A_1 and A_2 Hurwitz matrices, whose exponentials preserve the cone K_{X_1} and K_{X_2}, respectively. Moreover, we assume that $B_1K_{U_1} \subset K_{X_1}$, $C_1K_{X_1} \subset K_{Y_1}$ and $B_2K_{U_2} \subset K_{X_2}$, $C_2K_{X_2} \subset K_{Y_2}$. Under such assumptions $\Gamma_1 = -C_1A_1^{-1}B_1$ and $\Gamma_2 = -C_2A_2^{-1}B_2$ define monotone maps, fulfilling $\Gamma_1K_{U_1} \subset K_{Y_1}$ and $\Gamma_2K_{U_2} \subset K_{Y_2}$. Then, provided condition (15) holds, the system (16) is asymptotically stable.

Let now define diagonal matrices Δ_1, Δ_2, Θ_1, Θ_2 with entries in $\{-1,+1\}$ such that $K_{U_1} = \Delta_1[0,+\infty)^{m_1}$, $K_{U_2} = \Delta_2[0,+\infty)^{m_2}$, $K_{Y_1} = \Theta_1[0,+\infty)^{p_1}$ and $K_{Y_2} = \Theta_2[0,+\infty)^{p_2}$. With the above notation $\Lambda_{12} = \Delta_2\Theta_1$ and $\Lambda_{21} = \Delta_1\Theta_2$. Notice moreover that $\Gamma_1K_{U_1} \subset K_{Y_1}$ implies $\Theta_1\Gamma_1K_{U_1} \subset [0,+\infty)^{p_1}$. Hence $\Theta_1\Gamma_1\Delta_1[0,+\infty)^{m_1} \subset [0,+\infty)^{p_1}$. This means that $\Theta_1\Gamma_1\Delta_1$ is a non-negative matrix, and in particular $\Theta_1\Gamma_1\Delta_1 = |\Gamma_1|$ (where $|\cdot|$ denotes componentwise absolute value). Similar considerations apply to $\Theta_2\Gamma_2\Delta_2 = |\Gamma_2|$. The small gain condition (15) can be equivalently written as:

$$\begin{aligned} 1 > \rho(\Lambda_{21}\Gamma_2\Lambda_{12}\Gamma_1) &= \rho(\Delta_1\Theta_2\Gamma_2\Delta_2\Theta_1\Gamma_1) \\ &= \rho(\Theta_2\Gamma_2\Delta_2\Theta_1\Gamma_1\Delta_1) \\ &= \rho(|\Gamma_2||\Gamma_1|) \end{aligned} \tag{17}$$

Since for linear SISO monotone system the DC-gain equals the $\mathcal{L}_2$ (as well as the $\mathcal{L}_\infty$) induced gain, the stability condition that we derived is equivalent to classical linear small-gain results (such as for instance Theorem 7 in [10]).

V. An Example

We show below by means of an example how the result can be used. We also point out that in general the absolute values $|\Gamma_2|$ and $|\Gamma_1|$ cannot be avoided, namely the condition that $\rho(\Gamma_2\Gamma_1) < 1$ is *not* enough to guarantee stability for the case of monotone systems in feedback. This may be counter-intuitive as for the positive feedback case (as well as for the negative feedback one) there is no need to introduce absolute values (indeed for positive feedback Γ_2 and Γ_1 can be taken to be both positive without loss of generality, whereas in the case of negative feedback Γ_1 and Γ_2 can be taken to be of opposite sign, but this is no concern as spectral radius is invariant with respect to sign inversions). Consider the following matrices:

$$A_1 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -10 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 10 & 10 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 5 & 5 \\ -1 & -1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad C_2 = \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

for some $\gamma \geq 0$. Notice that these matrices define monotone systems with respect to the partial orders induced by the following orthants: $K_{X_1} = K_{X_2} = [0,+\infty)^4$, $K_{U_1} = K_{Y_1} = K_{U_2} = [0,+\infty)^2$, while $K_{Y_2} = [0,+\infty) \times (-\infty, 0]$. The DC-gain matrices are given by:

$$\Gamma_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \Gamma_2 = \gamma \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Computing the DC loop gain yields:

$$\Gamma_2\Gamma_1 = \gamma \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$$

Notice that $\Gamma_2\Gamma_1$ is a nilpotent matrix regardless of γ, so that $\rho(\Gamma_2\Gamma_1) = 0$. If one could avoid using absolute values in expressing the small-gain condition (17), this would mean asymptotic stability of the interconnected system regardless of γ. The characteristic polynomial of the closed-loop system reads:

$$\begin{aligned} \chi(s) = {} & s^8 + 21\, s^7 + 155\, s^6 + 545\, s^5 + \\ & (1065 - 44\,\gamma)\, s^4 + (1231 - 168\,\gamma)\, s^3 \\ & + (841 - 204\,\gamma)\, s^2 + (315 - 80\,\gamma)\, s + 50 \end{aligned}$$

which according to the Routh-Hurwitz criterion is asymptotically stable (for non-negative γ) if and only if $\gamma \in [0, \gamma^\star)$ with $\gamma^\star \approx 1.9662$. According to criterion (17), instead:

$$|\Gamma_2| \cdot |\Gamma_1| = \gamma \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},$$

which yields $\rho(|\Gamma_2| \cdot |\Gamma_1|) = 4\gamma$. Then, according to Theorem 2 asymptotic stability of (16) holds provided $|\gamma| < 1/4$. There is a significant gap between the value $1/4$ (provided by the small gain theorem) and the true value $\gamma^\star$ which renders the interconnected system (16) unstable. How to improve the quality of bounds provided by small-gain results is still an interesting open question for future research.

VI. Necessity of the Linear Small-Gain Condition

As highlighted by means of the previous example, stability intervals assessed by means of the small gain criterion can be rather conservative. It is to be emphasized however, that similar arguments apply also when arbitrary time delays are considered in at the system interconnections; namely if the following interconnected system is considered:

$$\begin{array}{cc} y_1 = \psi_1(x_1, u_1) & y_2 = \psi_2(x_2, u_2) \\ y_1^k(t) = u_2^k(t - \tau_1), & y_2^i(t) = u_1^i(t - \tau_2). \end{array} \quad (18)$$

for some nonnegative τ_1, τ_2. We argue next, that if one is willing to allow for arbitrary time-delays as in (18) in the loop, then the small gain condition (15) is also necessary for stability. We prove the following Proposition.

Proposition 1: Consider the linear system:

$$\begin{array}{rll} \dot{x}_1(t) = & A_1x_1(t) + B_1u_1(t), & y_1(t) = C_1x_1(t) \\ \dot{x}_2(t) = & A_2x_2(t) + B_2u_2(t), & y_2(t) = C_2x_2(t). \end{array} \quad (19)$$

where A_1, B_1, C_1 and A_2, B_2, C_2 are as in Theorem 2. We adopt the following delayed interconnections:

$$u_1^i(t) = y_2^i(t - \tau_1^i) \qquad u_2^k(t) = y_1^k(t - \tau_2^k) \quad (20)$$

with $i \in \{1, \ldots, m_1\}$ and $k \in \{1, \ldots, m_2\}$ and τ_1^i, τ_2^k nonnegative reals. Assume that $\rho(\Lambda_{21}\Gamma_2\Lambda_{12}\Gamma_1) > 1$. Then there exists values of $\tau_1^i \geq 0$ and $\tau_2^k \geq 0$, and $T > 0$ such that the system (19) admits a periodic solution of period T.

Proof. Define $G_1(\omega) = C_1(j\omega - A_1)^{-1}B_1$ and $G_2(\omega) = C_2(j\omega - A_2)^{-1}B_2$. Notice that $G_1(\omega) \to 0$ as $\omega \to +\infty$, and the same applies to $G_2(\omega)$. Clearly $G_1(0) = \Gamma_1$ and $G_2(0) = \Gamma_2$. By continuity of $G_i, i = 1, 2$ and of the spectral radius $\rho(\cdot)$, there exists $\bar{\omega} > 0$ such that:

$$\rho(\Lambda_{21}G_2(\bar{\omega})\Lambda_{12}G_1(\bar{\omega})) = 1. \quad (21)$$

Let $\lambda_{21}(\omega) = \text{diag}[\ldots e^{-j\omega\tau_1^i} \ldots]$ for $i = 1, 2, \ldots, m_1$ and $\lambda_{12}(\omega) = \text{diag}[\ldots e^{-j\omega\tau_2^k} \ldots]$ for $k = 1, 2, \ldots, m_2$. Clearly there exists nonnegative τ_1^is and τ_2^ks so that $\lambda_{12}(\bar{\omega}) = \Lambda_{12}$ and $\lambda_{21}(\bar{\omega}) = \Lambda_{21}$. By definition of $\bar{\omega}$ in (21) we have:

$$\rho(\lambda_{21}(\bar{\omega})G_2(\bar{\omega})\lambda_{12}(\bar{\omega})G_1(\bar{\omega})) = 1.$$

Notice that $\lambda_{21}(\bar{\omega})G_2(\bar{\omega})\lambda_{12}(\bar{\omega})G_1(\bar{\omega})$ can be interpreted as the loop-gain transfer function of (19) with the delayed interconnection (20) evaluated for $s = j\bar{\omega}$. Hence, by standard arguments, the linear system (19), (20) admits a sinusoidal solution of period $1/(2\pi\bar{\omega})$.

As already remarked, Theorem 2 is true for general linear systems provided the matrices $[\Gamma_1]_{ij}$ and $[\Gamma_2]_{ji}$ in condition (17) represent the $\mathcal{L}_2$ induced gains from input j to output i and from output i to input j respectively $j = 1, \ldots, m_1$, $i = 1, \ldots, m_2$. The necessity result stated in Proposition 1 is however new and in fact false for general linear systems even in the SISO case. To see this, consider the simple example described below.

Example 1: Consider the transfer functions given below:

$$G_1(s) = \frac{\gamma}{1+s} \qquad G_2(s) = \frac{1+s}{(1+0.1s)^2}$$

where γ is a positive parameter. We want to study stability of the following closed-loop transfer function:

$$\frac{1}{1 + G_1(s)G_2(s)e^{-s\tau}}$$

corresponding to an interconnection of G_1 and G_2 in closed loop, where τ indicates the sum of the delays at the loop interconnections. Notice that:

$$G_1(s)G_2(s) = \frac{\gamma}{(1+0.1s)^2},$$

This is a low-pass filter, hence the $\mathcal{L}_2$ induced gain equals the DC gain γ. Asymptotic stability for arbitrary delays holds provided $\gamma < 1$. Let us now compute the stability estimates provided by the small-gain theorem. For $G_1(s)$ the $\mathcal{L}_2$-induced gain equals the DC gain γ. However, for the second transfer function, the maximum of $|G_2(j\omega)|$ is achieved at $\omega = 2\sqrt{2}$ and equals $|G_2(j2\sqrt{2})| = 5/3$. This means that the small gain theorem only predicts stability up to $\gamma < 3/5$, giving a conservative estimate of the stability region under arbitrary delays. Of course there is no state-space realization of G_2 that satisfies the monotonicity conditions in Theorem 2.

VII. Conclusion

We generalized existing small-gain theorems for orthant-monotone MIMO systems connected in feedback. The results improve on existing literature as they do not assume any compatibility between the orthant-induced orders pertaining to input and output spaces of interconnected terminals. Though the methods are new also for linear systems, in that they arise from a different point of view, the conditions achieved in this case boil down to classical L_2 or L_∞ small gain results.

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