2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Finite-gain L_∞ Stabilization of Satellite Formation Flying with Input Saturation

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***Abstract*— In this paper, we consider a formation keeping problem for a satellite formation flying system. The relative motion dynamics is designed with polytopic uncertainties by considering elliptical reference orbit, noncoplarnar formation and unknown angular rate, angular acceleration within some boundary. We propose a composite nonlinear feedback control law obtained by the solution to an algebraic Riccati inequality (ARI). Then, in the presence of input saturation and disturbances, internal and external stability (Finite-gain L_∞ stability) of the relative motion dynamics are investigated using proposed control law. Finally, numerical examples is presented to demonstrate the validity of the proposed controller.**

I. INTRODUCTION

Satellite formation flying (SFF) has recently attracted a significant amount of interest for many current and future missions. Since SFF has several advantages including cost efficiency, safety, mission flexibility, easy maintenance of space systems, etc, in recent year, many researchers have considered various control problems for SFF. For example, based on the Clohessy-Wiltshire (CW) equation, SFF control problem has been well studied [1]–[3]. However, CW equation is a linear approximation assuming circular or near circular reference orbit without uncertainties and external disturbances. Therefore, many researchers have studied the control problem considering an elliptic reference orbit, uncertainties and disturbances. In the case of the elliptic reference orbit, [4] developed a nonlinear relative motion dynamics and Lyapunov-based, nonlinear, output feedback, robust control law. In [5], a new relative motion dynamics allowing elliptic and noncoplanar formation was derived. In [6], an uncertainty model with eccentricity and semi-major axis variation considerations was derived.

One of the important issues in control engineering is input saturation. Since input saturation affect not only the performance of the systems, but also the stability of the systems, this problem has received the attention of many researchers in the past several decades [8]–[14]. In [8], a low gain feedback control law to achieve semi-global stabilization on the asymptotically null controllable by bounded controls (ANCBC) for linear systems subject to input saturation was derived. A basic concept of the low gain feedback is that the peak value of the control signal can avoid the saturation level by decreasing the low gain parameter.

Based on the low gain feedback law, a low-and-high gain feedback law was introduced in [9], [10]. The high gain feedback law is used to achieve closed-loop performances such as robust stabilization, disturbance rejection, etc (see,e.g., [11], [12], [14], and reference therein).

In this paper, a feedback controller design method for satellites is investigated in the presence of input saturation and disturbances. First, we consider the relative motion dynamics allowing elliptic and noncoplanar formation proposed in [5] and assume that angular velocities and its derivation of leader satellite are unknown parameters within some boundary. Second, based on the low-and-high gain feedback control law, we propose a composite nonlinear feedback (CNF) control law obtained by an algebraic Riccati inequality (ARI). Then finite-gain L_∞ stability of the satellite formation flying system is investigated using the proposed feedback control law.

The outline of this paper is as follows. The relative motion dynamics proposed in [5] is briefly described in Section II. In Section III, we review the condition of finite-gain L_∞ stability and provide problem formulation. In Section IV, the relative motion dynamics with time-varying polytopic uncertainties and the control strategy are provided, and finite-gain L_∞ stability is derived. Furthermore, based on the proposed control law, we present controller design algorithm and investigate some properties of the gain parameters. In Section V, numerical examples are presented, and conclusion are presented in Section VI.

Throughout this paper, we will use the following notations. For any vector $\mathrm{x} \in \mathcal{R}^n$, $\mathrm{x} \geq 0$ means that all the components of x, denoted $x_{(i)}$, are nonnegative. $\|\mathrm{x}\|$ denotes the Euclidean norm of x, the L_∞-norm of x is defined as $\|\mathrm{x}\|_\infty \triangleq ess\ \sup_{t\geq 0}\|\mathrm{x}(t)\|$. $|\mathrm{x}|$ and $|\mathrm{x}|_\infty$ denote the absolute value and the maximum absolute value of each element of x. The elements of the matrix A are denoted by $A_{(i,j)}$,$i = 1,..,m$,$j = 1,..,n$.

II. SYSTEM MODEL

In this section, we describe the relative motion dynamics of the follower satellite relative to the leader satellite as depicted in Fig.1. We consider the satellites in a noncoplanar and elliptical orbit. The relative position and the relative velocities are represented in the local reference frame attached to the leader satellite (i.e., $\hat{x}$-$\hat{y}$-$\hat{z}$ axes) by $\bar{\rho} = [x, y, z]'$ and $[v_x, v_y, v_z]'$, respectively. The orbital dynamics of the leader and follower satellite in the inertial reference frame is given by

$$\ddot{\mathrm{r}} + \tfrac{\mu}{|\mathrm{r}|^3} r = \mathrm{u}^l + \mathrm{w}^l \tag{1}$$

$$\ddot{\mathrm{r}} + \ddot{\bar{\rho}} + \tfrac{\mu}{|\mathrm{r}+\bar{\rho}|^3}(\mathrm{r} + \bar{\rho}) = \mathrm{u}^f + \mathrm{w}^f \tag{2}$$

where $\mathrm{r} \in \mathbf{R}^3$ is the vector from the Earth's center to the leader satellite, $\mu = 3.986\times 10^{14} m^3/s^2$ is the gravitational constant of the Earth, u^l and u^f are the the control force vector and w^l and w^f are the disturbance vector acting on the leader and follower satellite, respectively. After some algebraic manipulations on (1) and (2), the general nonlinear dynamics for satellite formation flying can be written as [5]

$$\dot{x}_1 = x_1 \tag{3}$$

$$\dot{x}_2 = (\dot{\nu}^2 - \frac{\mu}{\gamma})x_1 + \ddot{\nu}x_3 + 2\dot{\nu}x_4 + u_x + w_x \tag{4}$$

$$\dot{x}_3 = x_4 \tag{5}$$

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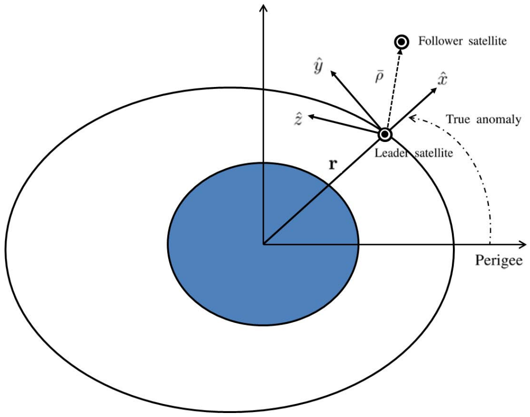

Fig. 1: Relative motion of leader and follower satellites.

$$\dot{x}_4 = -\ddot{\nu}x_1 - 2\dot{\nu}x_2 + (\dot{\nu}^2 - \frac{\mu}{\gamma})x_3 + u_y + w_y \tag{6}$$

$$\dot{x}_5 = x_6 \tag{7}$$

$$\dot{x}_6 = -\frac{\mu}{\gamma}x_5 + u_z + w_z \tag{8}$$

where $[x_1, x_2, x_3, x_4, x_5, x_6]' = [x, v_x, y, v_y, z, v_z]'$, ν, $\dot{\nu}$, and $\ddot{\nu}$ denote the true anomaly of the leader, and its first, second derivations, respectively, $\gamma = [(r_x + x)^2 + y^2 + z^2]^{3/2}$, r_x denotes the distance from the center of the Earth to the leader satellite, $\mathrm{u} = [u_x, u_y, u_z]' = \mathrm{u}^f - \mathrm{u}^l$, and $\mathrm{w} = [w_x, w_y, w_z]' = \mathrm{w}^f - \mathrm{w}^l$.

Let us consider the error vector $\mathrm{e} = \mathrm{x} - \mathrm{x}^t$, where x^t denotes the target vector, i.e., $\mathrm{x}^t = [x_1^t, x_2^t, x_3^t, x_4^t, x_5^t, x_6^t]'$, and $r_x \gg x, y, z$. Furthermore, by assuming that the leader satellite is perfectly controlled, the control force vector of the leader satellite equals to the disturbance vector, i.e., $\mathrm{u}^l = \mathrm{w}^l$. Then, using Eqs.(14)-(16) of [5], the relative motion error dynamics of the leader and follower satellites can be written as

$$\dot{\mathrm{e}} = A_1\mathrm{e} + A_2\mathrm{x}^t + f(\mathrm{x}) + B\mathrm{u} + D\mathrm{w} \tag{9}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \dot{\nu}^2 & 0 & \ddot{\nu} & 2\dot{\nu} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\ddot{\nu} & -2\dot{\nu} & \dot{\nu}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \dot{\nu}^2 & 0 & \ddot{\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\ddot{\nu} & 0 & \dot{\nu}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

$$f(\mathrm{x}) = [0, -\frac{\mu}{\gamma}x_1, 0, -\frac{\mu}{\gamma}x_3, 0, -\frac{\mu}{\gamma}x_5]' \tag{12}$$

$$B = D = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}' \tag{13}$$

where $\mathrm{u} = [u_x, u_y, u_z]'$ and $\mathrm{w} = [w_x, w_y, w_z]'$ denote the control force vector and disturbance vector acting on the follower satellite, respectively.

III. Preliminaries and Problem Formulation

In this section, for a given dynamics (9), we formulate a control design problem under the following assumptions.

Assumption 3.1: $\dot{\nu}$ and $\ddot{\nu}$ are not measured. However, when the leader is perfectly controlled, the uncertain parameters are bounded as $0 < \underline{a} \le \dot{\nu} \le \bar{a}$ and $\underline{b} \le \ddot{\nu} \le \bar{b}$.

Assumption 3.2: The satellite system is subject to input saturation as $-u_{0(i)} \le u_{(i)} \le u_{0(i)}, i = 1, 2, 3$.

Then the relative motion error dynamics (9) can be rewritten as

$$\dot{\mathrm{e}} = A_1\mathrm{e} + A_2\mathrm{x}^t + f(\mathrm{x}) + Bsat(\mathrm{u}) + D\mathrm{w} \tag{14}$$

The saturation function $sat(\mathrm{u})$ is defined as

$$sat(u_{(i)}) = sign(u_{(i)})min(u_{o(i)}, |u_{(i)}|), \ \ i = 1, 2, 3 \tag{15}$$

We next pose the problem to be solved in this paper.

Problem 3.1: For the relative motion dynamics (14), determine a state feedback controller such that

(a) Internal stability : when $\mathrm{w} = 0$, the closed-loop system trajectories converge asymptotically to the origin.

(b) External stability (Finite-gain L_∞ stability) : when $\mathrm{w} \ne 0$, there exist finite scalars $\kappa, \eta > 0$ such that

$$\|\mathrm{e}(t)\|_\infty \le \kappa\|\mathrm{e}(0)\| + \eta\|\mathrm{w}(t)\|_\infty, \ \ \forall t > 0 \tag{16}$$

Before we begin the controller design which solves *Problem 3.1*, we consider some useful lemmas.

Lemma 3.1: [16] If a real scale function $w(t)$ satisfies the following differential inequality

$$\dot{w}(t) \le -\gamma w(t) + cv(t) \tag{17}$$

where $\gamma > 0$ and $c > 0$ then

$$w(t) \le e^{-\gamma t}w(0) + c\int_0^t e^{-\gamma\tau}v(t - \tau)\,d\tau \tag{18}$$

Next, we consider finite-gain L_∞ stability. The following lemma presents the condition that the linear system is finite-gain L_∞ stable.

Lemma 3.2: Consider the linear system

$$\dot{x} = Ax + Bu, \ \ x(0) = x_0 \tag{19}$$

where $x \in \mathbf{R}^n, u \in \mathbf{R}^m$. Let $\gamma, \zeta > 0$ and suppose there is a positive definite $P \in \mathbf{R}^{n\times n}$ that satisfies the ARI

$$A'P + PA - \frac{1}{\gamma^2}PBB'P + \zeta P \le 0 \tag{20}$$

Then, for all $x_0 \in \mathbf{R}^n$, the system is finite-gain L_∞ stable and its L_∞ gains κ and η are less than or equal to $\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$ and $\sqrt{\frac{\gamma^2}{\zeta\lambda_{min}(P)}}$, respectively.

Proof: Let us consider such a Lyapunov function $V(x) = x'Px$ and the derivative of $V(x)$

$$\begin{aligned} \dot{V} =& x'(A'P + PA)x + 2x'PBu \\ =& x'(A'P + PA)x + \gamma^2\|u\|^2 + \frac{1}{\gamma^2}x'PBB'Px \\ & - \gamma^2\left\|u - \frac{1}{\gamma^2}B'Px\right\|^2 \end{aligned} \tag{21}$$

Substituting (20) yields

$$\dot{V} \le -\zeta x'Px + \gamma^2\|u\|^2 \tag{22}$$

By *Lemma 3.1*, we have

$$\begin{aligned} V(t) \le& e^{-\zeta t}V(0) + \gamma^2\int_0^t e^{-\zeta\tau}\|u(t-\tau)\|^2\,d\tau \\ \le& e^{-\zeta t}V(0) + \frac{\gamma^2}{\zeta}\sup_{\tau\in[0,t]}\|u(t-\tau)\|^2(1 - e^{-\zeta t}) \end{aligned} \tag{23}$$

Notice that $\lambda_{min}(P)\|x(t)\|^2 \le V(t) \le \lambda_{max}(P)\|x(t)\|^2$, we can obtain that

$$\lambda_{min}(P)\|x(t)\|^2 \le e^{-\zeta t}\lambda_{max}(P)\|x(0)\|^2 + \frac{\gamma^2}{\zeta}\sup_{\tau\in[0,t]}\|u(t-\tau)\|^2(1-e^{-\zeta t}) \quad (24)$$

By considering $\sup_{t\in[0,\infty)}$, the above inequality can be written as

$$\sup_{t\in[0,\infty)}\lambda_{min}(P)\|x(t)\|^2 \le \lambda_{max}(P)\|x(0)\|^2 + \frac{\gamma^2}{\zeta}\sup_{\tau\in[0,\infty)}\|u(t)\|^2 \quad (25)$$

From the definition of L_∞, the above inequality can be rewritten as

$$\lambda_{min}(P)\|x(t)\|_\infty^2 \le \lambda_{max}(P)\|x(0)\|^2 + \frac{\gamma^2}{\zeta}\|u(t)\|_\infty^2 \quad (26)$$

From (26), we have

$$\|x(t)\|_\infty \le \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\|x(0)\| + \sqrt{\frac{\gamma^2}{\zeta\lambda_{min}(P)}}\|u(t)\|_\infty \quad (27)$$

for all $u(t) \in L_\infty$. ■

IV. Main Results

In this section, we develop a CNF control law based on the low-and-high gain control law. Before proceeding with the control design, we describe a polytopic uncertain dynamics. Finally, we derive internal and external stability of the polytopic uncertain dynamics and develop the CNF control law by the solution to an ARI.

A. Dynamics Redesign

Based on *Assumption 3.1*, the relative motion error dynamics (14) can be regarded as a polytopic uncertain dynamics, and it can be written as

$$\dot{\mathrm{e}} = A_1(\xi)\mathrm{e} + A_2(\xi)\mathrm{x}^t + f(\mathrm{x}) + Bsat(\mathrm{u}) + D\mathrm{w} \quad (28)$$

where $(\dot{\nu}, \ddot{\nu}) \in [(\dot{\nu}_1, \ddot{\nu}_1), ..., (\dot{\nu}_N, \ddot{\nu}_N)]$. Then the matrices $A_1(\xi)$ and $A_2(\xi)$ take values in the matrix polytope with N vertices

$$(A_1(\xi)\,,\,A_2(\xi)) \in \left\{ \left(\sum_{k=1}^{N}\xi_k A_{1k}\,,\, \sum_{i=k}^{N}\xi_k A_{2k} \right) : \sum_{k=1}^{N}\xi_k = 1, \xi \ge 0 \right\} \quad (29)$$

B. Finite-gain L_∞ stabilization

To solve *Problem 3.1* based on the polytopic uncertain dynamics (28), we define the CNF control law as

$$\mathrm{u} = \mathrm{u}_1 + \mathrm{u}_2 + \mathrm{u}_3 \quad (30)$$

$$\mathrm{u}_1 = -B'P\mathrm{e} \quad (31)$$

$$\mathrm{u}_2 = -\rho B'P\mathrm{e} \quad (32)$$

$$\mathrm{u}_3 = sign\left(-B'P\mathrm{e}\right)\left|\bar{A}_2\mathrm{x}^t\right|_\infty - \bar{f}(\mathrm{x}) \quad (33)$$

where $P \in \mathbf{R}^{6\times6}$ is a symmetric positive definite matrix, $A_2 = B\bar{A}_2$, $f(\mathrm{x}) = B\bar{f}(\mathrm{x})$ and ρ is any nonnegative function referred to as the high gain parameter and will be discussed later. Next, let us define such a Lyapunov function $V = \mathrm{e}'P\mathrm{e}$ and two sets of states satisfying the following conditions

$$\mathcal{S}_{u_0} = \{\mathrm{e} \in \mathcal{R}^6 : |u_{1(i)} + u_{3(i)}| \le u_{o(i)}, i = 1,2,3\} \quad (34)$$

$$\mathcal{S}_\alpha = \{\mathrm{e} \in \mathcal{R}^6 : \mathrm{e}'P\mathrm{e} \le \alpha\} \quad (35)$$

where α is possibly largest positive constant and will be discussed later, then, under the given CNF control law (30) and two sets (34), (35), the following theorem shows that the polytopic uncertain dynamics (28) is finite-gain L_∞ stable.

Theorem 4.1: If there exist a unique symmetric positive definite matrix $P \in \mathbf{R}^{6\times6}$ that satisfies

$$\mathcal{S}_\alpha \subset \mathcal{S}_{u_0} \quad (36)$$

and the following ARI

$$A_1(\xi)'P + PA_1(\xi) - 2PBB'P + \frac{1}{\gamma^2}PDD'P + \zeta P \le 0 \quad (37)$$

where $\gamma, \zeta > 0$, then, under the given CNF control law (30) and for all $\mathrm{e}_0 \in \mathcal{S}_\alpha$, the polytopic uncertain dynamics (28) guarantees the following stability conditions.

(a) Internal stability : when $\mathrm{w} = 0$, the closed-loop system trajectories converge asymptotically to the origin.

(b) External stability (Finite-gain L_∞ stability) : when $\mathrm{w} \neq 0$, there exist finite scalars $\kappa, \eta > 0$ such that

$$\|\mathrm{e}(t)\|_\infty \le \kappa\|\mathrm{e}(0)\| + \eta\|\mathrm{w}(t)\|_\infty, \;\; \forall t > 0 \quad (38)$$

Proof: Let us consider the polytopic uncertain dynamics (28) and such a Lyapunov function $V = \mathrm{e}'P\mathrm{e}$. The derivative of V can be written as

$$\begin{aligned}\dot{V} &= \mathrm{e}'(A_1(\xi)'P + PA_1(\xi))\mathrm{e} + 2\mathrm{e}'PD\mathrm{w} \\ &\quad + 2\mathrm{e}'PB\left\{sat(\mathrm{u}) + \bar{A}_2(\xi)\mathrm{x}^t + \bar{f}(\mathrm{x})\right\} \\ &= \mathrm{e}'(A_1(\xi)'P + PA_1(\xi))\mathrm{e} \\ &\quad + 2\mathrm{e}'PB\left\{sat(\mathrm{u}) + \bar{A}_2(\xi)\mathrm{x}^t + \bar{f}(\mathrm{x})\right\} \\ &\quad + \gamma^2\|\mathrm{w}\|^2 + \frac{1}{\gamma^2}\mathrm{e}'PDD'P\mathrm{e} - \gamma^2\left\|\mathrm{w} - \frac{1}{\gamma^2}D'P\mathrm{e}\right\|^2 \\ &\le \mathrm{e}'(A_1(\xi)'P + PA_1(\xi) + \frac{1}{\gamma^2}PDD'P)\mathrm{e} \\ &\quad + 2\mathrm{e}'PB\left\{sat(\mathrm{u}) + \bar{A}_2(\xi)\mathrm{x}^t + \bar{f}(\mathrm{x})\right\} + \gamma^2\|\mathrm{w}\|^2 \end{aligned} \quad (39)$$

From the ARI (37), we have

$$\begin{aligned}\dot{V} \le &-\zeta\mathrm{e}'P\mathrm{e} + \gamma^2\|\mathrm{w}\|^2 \\ &+ 2\mathrm{e}'P\left\{sat(\mathrm{u}) + B'P\mathrm{e} + \bar{A}_2(\xi)\mathrm{x}^t + \bar{f}(\mathrm{x})\right\}\end{aligned} \quad (40)$$

Let us define $v = -B'P\mathrm{e}$ and consider the CNF control law (30). Then the above inequality can be rewritten as

$$\begin{aligned}\dot{V} \le &-\zeta\mathrm{e}'P\mathrm{e} + \gamma^2\|\mathrm{w}\|^2 \\ &- 2v'\Big\{sat\left[(1+\rho)v + sign(v)\left|\bar{A}_2\mathrm{x}^t\right|_\infty - \bar{f}(\mathrm{x})\right] \\ &- \left[v - \bar{A}_2(\xi)\mathrm{x}^t - \bar{f}(\mathrm{x})\right]\Big\}\end{aligned} \quad (41)$$

Next, we consider the last term in the right-hand side of (41) defined as

$$\begin{aligned}v'\Big\{&sat\left[(1+\rho)v + sign(v)\left|\bar{A}_2\mathrm{x}^t\right|_\infty - \bar{f}(\mathrm{x})\right] \\ &- \left[v - \bar{A}_2(\xi)\mathrm{x}^t - \bar{f}(\mathrm{x})\right]\Big\}\end{aligned} \quad (42)$$

for three cases of saturation function.

1) $|u_{(i)}| \le u_{o(i)}$.

In this case, (42) can be written as

$$v_{(i)}\left(\rho v_{(i)} + sign(v_{(i)})\left|\bar{A}_{2(i)}\mathrm{x}^t\right|_\infty + \bar{A}_2(\xi)_{(i)}\mathrm{x}^t\right) \quad (43)$$

Since $\left|\bar{A}_{2(i)}\mathrm{x}^t\right|_\infty \geq \left|\bar{A}_2(\xi)_{(i)}\mathrm{x}^t\right|$ for all $\bar{A}_2(\xi)_{(i)}$, we have $sign\left(sign(v_{(i)})\left|\bar{A}_{2(i)}\mathrm{x}^t\right|_\infty + \bar{A}_2(\xi)_{(i)}\mathrm{x}^t\right) = sign(v_{(i)})$. Therefore, (43) ≥ 0.

2) $u_{(i)} > u_{o(i)}$.

In this case, (42) can be written as

$$v_{(i)}\left(u_{o(i)} - v_{(i)} + \bar{A}_{2(i)}(\xi)\mathrm{x}^t + \bar{f}(\mathrm{x})_{(i)}\right) \tag{44}$$

and since $|u_{1(i)} + u_{3(i)}| \leq u_{o(i)}$, $u_{2(i)} = -\rho B'_{(i)}P\mathrm{e} > 0$. Therefore, $v_{(i)} = -B'_{(i)}P\mathrm{e} > 0$, then, we have $u_{o(i)} - v_{(i)} + \bar{A}_2(\xi)_i\mathrm{x}^t + \bar{f}(\mathrm{x})_{(i)} \geq 0$. Thus, (44) ≥ 0.

3) $u_{(i)} < -u_{o(i)}$.

In this case, (42) can be written as

$$v_{(i)}\left(-u_{o(i)} - v_{(i)} + \bar{A}_2(\xi)_{(i)}\mathrm{x}^t + \bar{f}(\mathrm{x})_{(i)}\right) \tag{45}$$

and since $|u_{1(i)} + u_{3(i)}| \leq u_{o(i)}$, $u_{2(i)} = -\rho B'_{(i)}P\mathrm{e} < 0$. Therefore, $v = -B'_{(i)}P\mathrm{e} < 0$, then, we have $-u_{o(i)} - v_{(i)} + \bar{A}_2(\xi)_{(i)}\mathrm{x}^t + \bar{f}(\mathrm{x})_{(i)} \leq 0$. Thus, (45) ≥ 0.

From the above three cases, we can define the following condition

$$\begin{aligned} v'\{sat[(1+\rho)v + sign(v)\left|\bar{A}_2\mathrm{x}^t\right|_\infty - \bar{f}(\mathrm{x})] \\ -[v - \bar{A}_2(\xi)\mathrm{x}^t - \bar{f}(\mathrm{x})]\} \;\; \geq \;\; 0 \end{aligned} \tag{46}$$

Therefore, the equation (41) can be written as

$$\dot{V} \leq -\zeta\mathrm{e}'P\mathrm{e} + \gamma^2\|\mathrm{w}\|^2 \tag{47}$$

Then *Problem 3.1* can be solved such as

(a) $\mathrm{w} = 0$: $\dot{V} \leq -\zeta\mathrm{e}'P\mathrm{e}$. Therefore, the closed-loop system is asymptotically stable.

(b) $\mathrm{w} \neq 0$: By *Lemma 3.2*, we obtain

$$\|\mathrm{e}(t)\|_\infty \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\|\mathrm{e}(0)\| + \sqrt{\frac{\gamma^2}{\zeta\lambda_{min}(P)}}\|\mathrm{w}(t)\|_\infty \tag{48}$$

for all $\mathrm{w}(t) \in L_\infty$. Finally, choosing the L_∞ gains as

$$\kappa = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}, \quad \eta = \sqrt{\frac{\gamma^2}{\zeta\lambda_{min}(P)}} \tag{49}$$

Problem 3.1 is solved. ∎

Remark 4.1: From *Theorem 4.1*, we can see that the high gain controller, u_2, does not play any role in the stability. However, u_2 can be used to achieve a better performance of the system, and design parameter ρ will be presented in *Section IV-C*.

Note that the set $\mathcal{S}_\alpha$ presents a estimated domain of attraction. The parameter α can be chosen such that the condition (36) is satisfied and a positive constant as large as possible to obtain large domain of attraction. For example, the analytic solution to the parameter α can be found in [15].

C. An algorithm for the design of controller

In this section, we will present the CNF controller design algorithm and investigate some properties of the gain parameters. We refer to general low-and-high gain feedback controller and CNF controller design algorithm. The controller design algorithm is carried out in three steps as follows.

Step 1) We design the feedback controller u_1 and u_3 defined as

$$\mathrm{u}_1 + \mathrm{u}_3 = -B'P\mathrm{e} + sign\left(-B'P\mathrm{e}\right)\left|\bar{A}_2\mathrm{x}^t\right|_\infty - \bar{f}(\mathrm{x}) \tag{50}$$

where $P \in \mathbf{R}^{6\times6}$ is the unique symmetric positive definite solution to the ARI (37). The solution can be obtained such that the closed-loop system has certain desired performance, i.e., large domain of attraction and/or small L_∞ gains.

Remark 4.2: [17] Let us consider the ARI (37) and define two design parameters as $\zeta_1 \geq \zeta_2 > 0$. Then, for fixed γ, two solutions, $P_1, P_2 > 0$, of the ARI satisfies $P_1 \geq P_2$.

Remark 4.3: In this paper, we focused on the control problem of SFF in the presence of the input saturation and the disturbance. In this case, as mentioned above, the performance of the system can be considered as a large domain of attraction and small disturbance attenuation level, and they are dependent on the control gain P, i.e., we can estimate the size of the domain of attraction as $tr(P)$. Let us consider the ARI (37). By pre-multiplying P^{-1} and applying the $trace$ operation, provided that

$$tr(2BB'P - \frac{1}{\gamma^2}DD'P) - 2tr(A_1(\xi)) - tr(\zeta I) \geq 0 \tag{51}$$

$$\lambda_{max}(2BB' - \frac{1}{\gamma^2}DD')tr(P) - 2tr(A_1(\xi)) - tr(\zeta I) \geq 0 \tag{52}$$

From the definition of the matrices $A_1(\xi), B, D$, we can see that $tr(A_1(\xi)) = 0, \lambda_{max}(2BB' - \frac{1}{\gamma^2}DD') = 2 - \frac{1}{\gamma^2}$. Then (52) can be written as

$$tr(P) \geq \frac{6\zeta}{2 - \frac{1}{\gamma^2}} \tag{53}$$

(53) presents the lower bounds of $tr(P)$ and γ. Furthermore, from (53), we can see that $tr(P)$ decreases as γ increases and/or ζ decreases. For detailed bound properties of the solution to the ARI (or ARE), see [18].

Step 2) We construct the high gain feedback controller (32)

$$\mathrm{u}_2 = -\rho B'P\mathrm{e} \tag{54}$$

where ρ is a nonnegative function referred to as the high gain parameter and can be chosen such that the function ρ changes from 0 to ρ_0 as e approaches zero. For example, in [12], the scheduled high gain parameter is chosen as

$$\rho = \frac{\rho_0}{1 - \left(\frac{1}{\alpha}\mathrm{e}'P\mathrm{e}\right)^2} \tag{55}$$

where α presents the size of domain of attraction and $\rho_0 > 0$ is a design parameter that can be chosen to yield a desired performance, i.e., fast settling time, small overshoot, small steady-state error. The choice of ρ is nonunique.

Note that from *Remark 4.2* and *Remark 4.3*, we can see that the large domain of attraction lead to the large L_∞ gains. However, in *Section V*, we will see that the L_∞ gains can decreased by the high gain controller. In this paper, we do not provide the analytical relationship between the high gain controller and the L_∞ gains. But we will leave this to future work.

Step 3) We combine the controller u_1, u_2 and u_3 designed in the previous steps as

$$\mathrm{u} = -(1+\rho)B'P\mathrm{e} + sign\left(-B'P\mathrm{e}\right)\left|\bar{A}_2\mathrm{x}^t\right|_\infty - \bar{f}(\mathrm{x}) \tag{56}$$

V. Simulation Results

In this paper, we consider two satellites in a low-earth orbit: semimajor axis of $7178000m$, eccentricity of 0.1. The initial relative positions and velocities are (30, 20, 0) and (1,-2,1) and the desired target positions and velocities are (10,10,10) and (0,0,0) in $\hat{x}$-$\hat{y}$-$\hat{z}$ axes, respectively. The mass of follower satellite is 410 kg, and we are assuming maximum 50 N thrusters are used along all three directions. In this simulation, we consider the disturbance as a J_2 perturbation. The J_2 acceleration can be written in the local

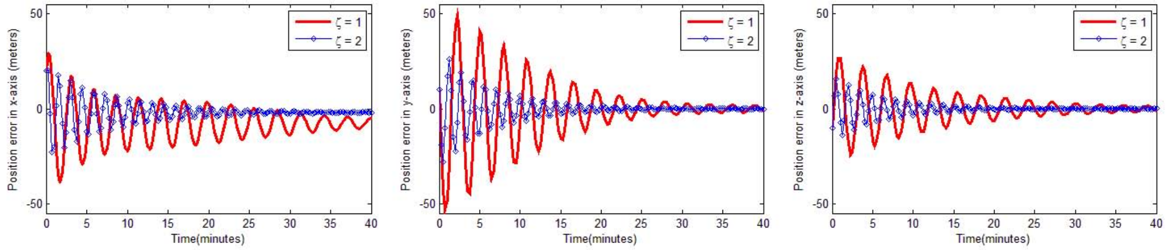

Fig. 2: Relative position errors in x-axis (left), y-axis (center) and z-axis (right)

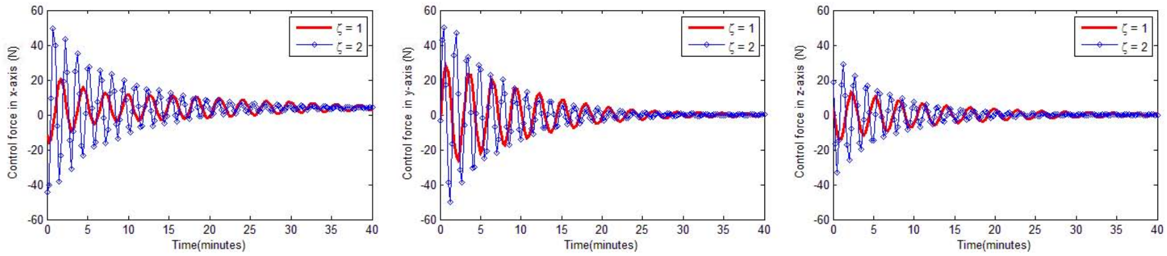

Fig. 3: Control forces in x-axis (left), y-axis (center) and z-axis (right)

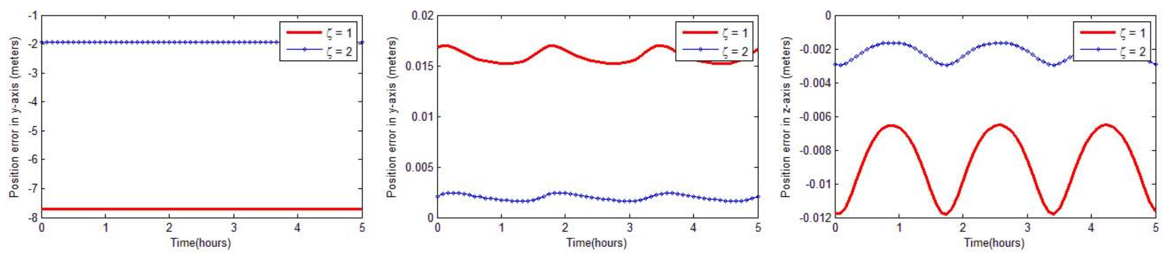

Fig. 4: Steady-state relative position errors in x-axis (left), y-axis (center) and z-axis (right)

reference frame $(\hat{x}, \hat{y}, \hat{z})$ as [7]

$$\mathrm{w}_{J_2} = -\frac{3}{2}\frac{\mu \mathrm{R}_e^2}{\mathrm{r}^4} J_2 [\,(1 - 3\sin^2 i \sin^2\theta)\,\hat{x} + (2\sin^2 i \sin\theta \cos\theta)\,\hat{y} + (2\sin i \cos i \sin\theta)\,\hat{z}\,] \tag{57}$$

Where R_e is the radius of the Earth, $J_2 = 1082.63 \times 10^{-6}$, i is orbital inclination, and $\theta = \nu + w$ (true anomaly and argument of perigee, respectively).

To compare the size of the domain of attraction,$tr(P)$, and the L_∞ gains, we consider the solution to ARI (37) as change the value of ζ for fixed value of γ^2. Table 1 shows that the size of the domain of attraction and the L_∞ gains obtained by the solution to ARI. Fig.2, Fig.3 and Fig.4 show the simulation results without high gain controller. In the simulation, we use $\gamma^2 = 10$, $\zeta = 1, 2$. The simulation results also show that as ζ decreases, we can achieve the large domain of attraction, but the poor transient response and the poor disturbance attenuation level.

Next, we consider the proposed CNF controller and the high gain controller as (55). We choose the parameters as $\gamma^2 = 10$, $\zeta = 1$, $\rho_0 = 10, 1000$, and $\alpha = 10^4$. The simulation results are shown in Fig.5, Fig.6, and Fig.7. From the simulation results, we can see that as the high gain increases, the system has not only the fast transient response, but also the small steady-state error.

VI. Conclusion

In this paper, we proposed a formation control law which guarantees finite-gain L_∞ stability. The proposed feedback controller, CNF controller, was developed using algebraic Riccati inequality (ARI). Using the CNF controller, we derived internal and external stability (finite-gain L_∞ stability). Furthermore, we investigated the relationship between the solution to the ARI, the design parameters and the performance of the system. From the numerical example, it was derived, and the efficacy of the CNF feedback controller was demonstrated.

TABLE I: Comparison of size of the domain of attraction and L_∞ gains

γ^2	ζ	$tr(P)$	κ	η
1	0.1	0.6081	20.1762	141.5985
	1	9.0066	2.6196	1.6180
	5	405.1214	5.2092	0.2041
10	0.1	0.3200	20.1763	617.2134
	1	4.7403	2.6196	7.0528
	2	18.9549	2.6189	2.4936
	5	213.2218	5.2092	0.8897
50	0.1	0.3071	20.1762	1.4089×10^3
	1	4.5488	2.6196	16.0992
	5	204.6066	5.2092	2.0309

VII. Acknowledgements

This research was supported by the MKE (The Ministry of Knowledge Economy), Korea, under the ITRC (Information Technology Research Center) support program supervised by the NIPA (National IT Industry Promotion Agency) (NIPA-2011-C1090 - 1131- 0006).

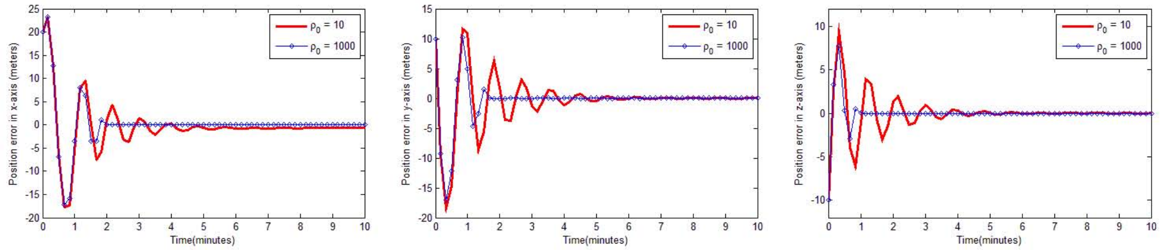

Fig. 5: Relative position errors in x-axis (left), y-axis (center) and z-axis (right)

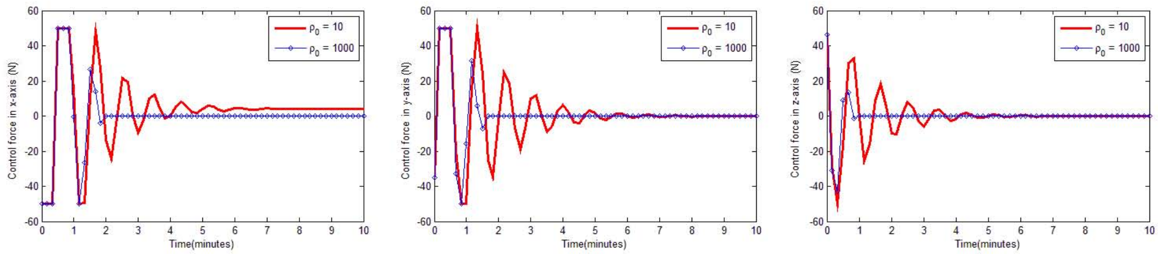

Fig. 6: Control forces in x-axis (left), y-axis (center) and z-axis (right)

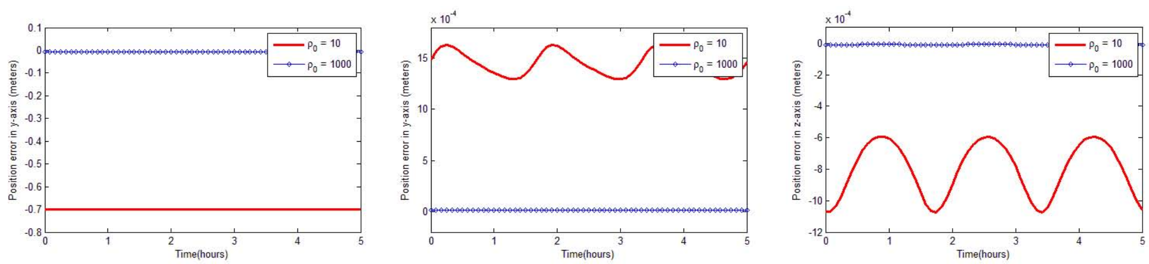

Fig. 7: Steady-state relative position errors in x-axis (left), y-axis (center) and z-axis (right)

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