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Unconstrained Model Predictive Control and Suboptimality Estimates for Nonlinear Time-Delay Systems

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Abstract— **This work presents an unconstrained model predictive control (MPC) scheme for nonlinear time-delay systems with guaranteed closed-loop stability using neither terminal constraints nor terminal weighting terms. Therefore, we do not require the calculation of control Lyapunov-Krasovskii functionals for the nonlinear time-delay system and obtain a computationally more attractive online optimization problem. Based on similar previous results for discrete-time systems and finite-dimensional continuous-time systems, an extended asymptotic controllability assumption suitable for nonlinear time-delay systems is introduced. Since the stage cost is not positive definite in the full state, but only penalizes the instantaneous state of the system, additional arguments are required in order to guarantee closed-loop stability. It is particularly interesting to note that in contrast to essentially all other MPC schemes with guaranteed stability, the optimal cost is *not* used as Lyapunov function(al) of the closed-loop, and indeed the optimal cost can increase along trajectories of the closed loop due to the influence of the delayed states.**

I. Introduction

Model predictive control (MPC) has been shown to be an effective control method to deal with nonlinear systems, with and without time-delays, subject to input and state constraints. However, it is well known that MPC does not guarantee closed-loop stability in general, which caused significant interest in academic research [1, 2].

There currently exist several MPC schemes for nonlinear time-delay systems which guarantee closed-loop stability. These schemes can be roughly categorized into schemes with terminal constraints [3–7] and unconstrained MPC schemes which use additional terminal weighting functionals [8–11]. All of these schemes require a control Lyapunov-Krasovskii functional, used as a terminal cost functional in the MPC setup, and a positively invariant terminal region. Calculating a control Lyapunov-Krasovskii functional for nonlinear time-delay systems is in general a difficult task. Even if a control Lyapunov-Krasovskii functional is known for the Jacobi linearization of the system about the origin, which by itself is not simple, it is a non-trivial problem to obtain an appropriate terminal cost functional and invariant terminal region for the nonlinear system, see e.g. [4–7]. All existing schemes for calculating these stabilizing design parameters either require restrictive Razumikhin conditions or yield complicated terminal regions and/or terminal cost functionals, which are unattractive for an online implementation.

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In this work, we consider MPC with finite horizon cost functionals containing neither terminal constraints nor terminal penalty terms. The results extend previous results on unconstrained MPC for (finite-dimensional) continuous-time systems [12] and discrete-time systems [13–15]. The results of [14, 15] also hold for infinite-dimensional systems and have been exemplarily applied to certain classes of partial differential equations [16, 17]. However, the results cannot be directly transferred because in our problem setup, as well as in all other MPC schemes for time-delay systems, the stage cost is not positive definite in the full state, but only penalizes the instantaneous state of the system. Hence, additional arguments are required in order to guarantee to closed-loop stability. First, we introduce a modified controllability assumption suitable for time-delay systems. Based on this assumption, we derive conditions on the prediction horizon to guarantee stability of the closed-loop. It is particularly interesting to note that in contrast to essentially all other MPC schemes with guaranteed stability, the optimal cost is *not* used as Lyapunov function(al) of the closed-loop, and indeed the optimal cost can increase along trajectories of the closed loop due to the influence of the delayed states. However, stability is guaranteed because the infinite horizon cost is bounded by a function of the finite horizon optimal cost at initial time and the initial state.

The remainder of this paper is organized as follows. The problem setup considered in this work is described in Section II. Section III introduces a modified controllability assumption appropriate for time-delay systems and provides two intermediate results following from this assumption. The main result, a condition on the prediction horizon for guaranteed nominal stability of the closed-loop, is derived in Section IV and remarks on suboptimality estimates are given. Section V concludes the work with a brief summary and an outlook on future research.

Notation: Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the field of real numbers and the set of non-negative real numbers, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with any norm $|\cdot|$. Given $\tau > 0$, let $\mathcal{C}_\tau = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-\tau, 0] \subset \mathbb{R}$ into $\mathbb{R}^n$. A segment $x_t \in \mathcal{C}_\tau$ is defined by $x_t(s) = x(t + s), s \in [-\tau, 0]$. The norm on $\mathcal{C}_\tau$ is defined as $\|x_t\|_\tau = \sup_{\theta\in[-\tau,0]} |x(t + \theta)|$. $\mathcal{L}^\infty(\mathbb{D}, \mathbb{R}^n)$ is the set of all measurable, essentially bounded functions $\varphi : \mathbb{R} \supset \mathbb{D} \to \mathbb{R}^n$. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to belong to class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $f(0) = 0$ and $f(s) \to \infty$ as $s \to \infty$. ceil(s) denotes the smallest integer larger or equal to s.

II. PROBLEM SETUP

Consider the nonlinear time-delay system

$$\dot{x}(t) = f(x(t), x(t-\tau), u(t)) \,, \tag{1a}$$

$$x(\theta) = \varphi(\theta) \,, \qquad \forall \theta \in [-\tau, 0] \,, \tag{1b}$$

in which $x(t) \in \mathbb{R}^n$ is the instantaneous state at time t, $u(t) \in \mathbb{R}^m$ is the control input subject to input constraints $u(t) \in \mathcal{U}$ and $\varphi \in \mathcal{C}_\tau$ is the initial function. The time-delay $\tau > 0$ is constant and assumed to be known. The function $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable. The constraint set $\mathcal{U} \subset \mathbb{R}^m$ is compact and contains the origin in its interior. Without loss of generality, $x_t = 0$ is assumed to be an equilibrium of system (1) for $u = 0$, i.e. $f(0,0,0) = 0$. The problem of interest is to stabilize the steady state $x_t = 0$ via model predictive control.

With respect to performance, the goal is to minimize the infinite horizon cost functional

$$J_\infty(\varphi, u) = \int_0^\infty F(x(t), u(t)) \, dt \,. \tag{2}$$

The stage cost $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ is continuous, $F(0,0) = 0$ and there is a class $\mathcal{K}_\infty$ function $\widehat{F} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$F(x,u) \geq \widehat{F}(|x|) \quad \text{for all } x \in \mathbb{R}^n \,,\, u \in \mathbb{R}^m \,. \tag{3}$$

The associated optimal cost of this infinite horizon problem is denoted by $J_\infty^*(\varphi)$.

Since infinite horizon problems are often computationally intractable, finite horizon cost functionals combined with a receding horizon approach are often used instead. The finite horizon cost functional used in our MPC setup at time t_i is given by

$$J_T(x_{t_i}, u) = \int_{t_i}^{t_i+T} F(x(t), u(t)) \, dt \tag{4}$$

in which T is the prediction horizon. The open loop finite horizon optimal control problem is then formulated as

$$\min_{\bar{u} \in \mathcal{L}^\infty([t_i, t_i+T], \mathbb{R}^m)} J_T(x_{t_i}, \bar{u}) = \min_{\bar{u} \in \mathcal{L}^\infty([t_i, t_i+T], \mathbb{R}^m)} \int_{t_i}^{t_i+T} F(\bar{x}(t'; t_i), \bar{u}(t')) \, dt' \tag{5a}$$

subject to

$$\dot{\bar{x}}(t'; t) = f(\bar{x}(t'; t_i), \bar{x}(t' - \tau; t_i), \bar{u}(t')) \,, \tag{5b}$$

$$\bar{u}(t') \in \mathcal{U} \,, \tag{5c}$$

$$\bar{x}(t_i + \theta; t_i) = x(t_i + \theta) \,, \tag{5d}$$

for all $t' \in [0, T]$ and $\theta \in [-\tau, 0]$. Moreover, $\bar{x}(t'; t)$ is the predicted trajectory starting from initial condition x_{t_i} and driven by $\bar{u}(t')$ for $t' \in [t_i, t_i + T]$. We assume that the optimal control which minimizes $J_T(x_{t_i}, \bar{u})$ is given by $u_T^*(t'; x_{t_i}, t_i)$, $t' \in [t_i, t_i + T]$. The associated optimal cost is denoted by $J_T^*(x_{t_i})$ and the associated predicted trajectory is $x_T^*(t'; x_{t_i}, t_i)$, $t' \in [t_i, t_i + T]$. For a given sampling time $\delta > 0$, the control input to the system is defined by the optimal solution of problem (5) at sampling instants $t_i = i\,\delta$, $i \in \mathbb{N}_0$, in the usual receding horizon fashion

$$u_{\mathrm{MPC}}(t) = u_T^*(t; x_{t_i}, t_i) \,, \quad t_i \leq t < t_i + \delta \,. \tag{6}$$

Note that in this work we consider MPC with a finite horizon optimal control problem containing neither terminal constraints nor terminal penalty terms in the cost functional. Thus, we do not require to calculate a local control Lyapunov-Krasovskii functional for the system in a region around the origin. This is in contrast to all previous work on MPC for nonlinear time-delay systems [3–11]. Instead we use a less restrictive controllability assumption along the lines of the work presented in [13–15] for discrete-time systems, which has been recently extended to continuous-time systems in [12].

III. CONTROLLABILITY ASSUMPTION AND IMPLICATIONS

In the following, we introduce an extended controllability assumption appropriate for the nonlinear time-delay systems considered in this work.

Assumption 1 (Asymptotic Controllability) *For all $\varphi \in \mathcal{C}_\tau$, there exists a input trajectory $\hat{u}(\cdot; \varphi) \in \mathcal{L}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ with $\hat{u}(t; \varphi) \in \mathcal{U}$ for all $t \geq 0$ and with corresponding state trajectory $\bar{x}_{\hat{u}(\cdot;\varphi)}$ such that*

$$F(\bar{x}_{\hat{u}(\cdot;\varphi)}(t; \varphi), \hat{u}(t; \varphi)) \leq \beta(t) \cdot \left(\hat{F}(|\varphi(0)|) + \int_{-\tau}^{0} \hat{F}(|\varphi(t')|) dt' \right) \,, \tag{7}$$

for all $t \in \mathbb{R}^+$ in which $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, positive, and absolutely integrable function with $\lim_{t\to\infty} \beta(t) = 0$.

Note that Assumption 1 is a natural extension of [14, Assumption 3.1] and [12, Assumption 2] for nonlinear time-delay systems and naturally recovers [12, Assumption 2] for $\tau = 0$. The candidate $\hat{u}$ has to be feasible in the sense that it satisfies the input constraints, but it is not required to be optimal.

Remark 1 *Throughout this work, we do not consider state constraints and use a "global" controllability assumption for all $\varphi \in \mathcal{C}_\tau$ for a concise presentation. Modifications and "local" versions using invariant sets containing the initial state φ can be obtained in a straight forward manner.*

A typical example for such a function β is an exponential function $\beta(t) = C\, e^{-\lambda\, t}$ with some overshoot constant $C \geq 1$ and decay rate $\lambda > 0$. This example directly corresponds to the exponential controllability assumption for discrete-time systems in [14, 15] and for continuous-time systems in [12].

Before deriving two intermediate results based on the controllability assumption, we have to introduce some notation necessary for a more concise presentation in the

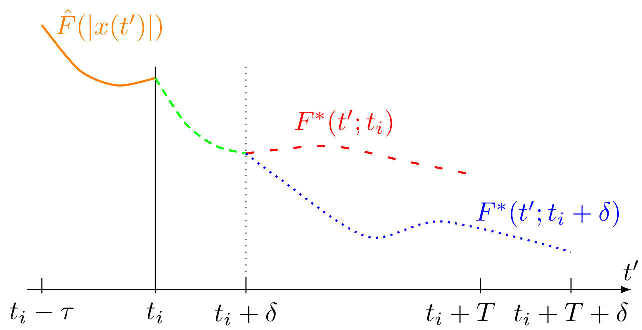

Fig. 1. Sketch for Lemmata 1 and 2. The optimal trajectory calculated at time t_i is depicted by the (green and red) dashed line, the optimal trajectory calculated at $t_i+\delta$ is depicted by the (blue) dotted line.

following. With slight abuse of notation, we use the following abbreviations for the predicted trajectories

$$\begin{aligned}F^*(t;t_i) &= F(x_T^*(t;x_{t_i},t_i),u_T^*(t;x_{t_i},t_i))\,,\\ \hat{F}^*(t;t_i) &= \hat{F}(|x_T^*(t;x_{t_i},t_i)|)\,,\end{aligned}$$

for $t\in[t_i,t_i+T]$ and

$$F^*(t;t_i)=F(x(t),u(t)),\quad \hat{F}^*(t;t_i)=\hat{F}(|x(t)|)$$

for $t<t_i$. Note that $\hat{F}^*(t;t_i)\le F^*(t;t_i)$ trivially due to the definition of $\hat{F}$ in (3). Furthermore, we define

$$B(t)=\int_0^t \beta(t')\,dt'\,,\qquad \mathcal{N}=\text{ceil}\left(\frac{\tau}{\delta}\right)\,,$$

and introduce the following assumption.

Assumption 2 *The prediction horizon T is chosen such that*

$$T>\tau+\delta\,.$$

Assumption 2 is essentially needed due to technical reasons in the proof of Lemma 1. In general, this assumption is not restrictive, in particular for small sampling times. In most cases, it is desirable to choose the prediction horizon larger than the time-delay or this might even be required in the case of using terminal constraints.

We can now state two intermediate results in Lemmata 1 and 2 based upon the controllability assumption. Lemma 1 uses the optimality of $J_T^*(x_{t_i+\delta})$ in addition to the controllability assumption in order to derive an upper bound on $J_T^*(x_{t_i+\delta})$ in terms of the endpiece of the predicted trajectory calculated at time t_i. In Figure 1, this can be interpreted as giving an upper bound on the cost of the blue dotted line in terms of the red loosely dashed line. Lemma 2 applies the principle of optimality, i.e. the trajectory $F^*(t;t_i)$ calculated at time t_i is an optimal endpiece on the interval $[t_i+\delta,t_i+T]$. Hence, it is also possible to derive an upper bound based on the controllability assumption. In Figure 1, the result can be interpreted as giving an upper bound on the cost of the red loosely dashed line in terms of the green dashed line and the orange line, which accounts for the influences of the delayed states due to the time-delay τ.

Lemma 1 (Calculation of $\bar{\beta}$) *Consider system* (1) *and let Assumptions 1 and 2 be satisfied. Then,*

$$J_T^*(x_{t_i+\delta})\le\frac{1}{\bar{\beta}}\int_{t_i+\delta}^{t_i+T}F^*(t';t_i)dt' \tag{8}$$

with $\frac{1}{\bar{\beta}}=1+B(T)\,\frac{1+\tau}{T-\tau-\delta}$.

Proof: For any $t\in[\delta,T]$ define the feasible control trajectory $\tilde{u}_t$ as follows

$$\tilde{u}_t(t')=\begin{cases}u_T^*(t';x_{t_i},t_i)\,, & t'\in[t_i+\delta,t_i+t]\\ \hat{u}(t'-t_i-t;x_{t_i+t})\,, & t'\in]t_i+t,t_i+\delta+T]\end{cases}$$

in which $\hat{u}(\cdot;x_{t_i+t})$ is the input trajectory of Assumption 1 starting from initial state $x_{t_i+t}=x_T^*(t_i+t;x_{t_i},t_i)$. Since $\tilde{u}_t$ is a feasible, but not necessarily optimal, solution to the finite horizon optimal control problem (5), we obtain the following

$$\begin{aligned}&J_T^*(x_{t_i+\delta})\le J_T(x_{t_i+\delta},\tilde{u}_t)\\ &\quad\le\int_{t_i+\delta}^{t_i+t}F^*(t';t_i)\,dt'+B(T+\delta-t)\\ &\qquad\times\left(\hat{F}^*(t_i+t;t_i)+\int_{t-\tau}^{t}\hat{F}^*(t_i+t';t_i)\,dt'\right)\,.\end{aligned} \tag{9}$$

The last inequality follows from (7) in Assumption 1 and the definition of B.

Furthermore, for all $t\in[\delta,T]$, we directly obtain the following relations

$$\begin{aligned}\int_{t_i+\delta}^{t_i+t}F^*(t';t_i)\,dt' &\le\int_{t_i+\delta}^{t_i+T}F^*(t';t_i)\,dt'\,,\\ B(T+\delta-t)&\le B(T)\,.\end{aligned}$$

Since (9) holds for all $t\in[\delta,T]$, it results

$$\begin{aligned}J_T^*(x_{t_i+\delta})&\le\int_{t_i+\delta}^{t_i+T}F^*(t';t_i)\,dt'+B(T)\\ &\quad\times\min_{t\in[\delta,T]}\left(\hat{F}^*(t_i+t;t_i)+\int_{t-\tau}^{t}\hat{F}^*(t_i+t';t_i)\,dt'\right)\,.\end{aligned} \tag{10}$$

Using Lemma 4 in the Appendix, it holds that

$$\begin{aligned}&\min_{t\in[\delta,T]}\left(\hat{F}^*(t_i+t;t_i)+\int_{t-\tau}^{t}\hat{F}^*(t_i+t';t_i)\,dt'\right)\\ &\le\min_{t\in[\delta+\tau,T]}\left(\hat{F}^*(t_i+t;t_i)+\int_{t-\tau}^{t}\hat{F}^*(t_i+t';t_i)\,dt'\right)\\ &\le\frac{1+\tau}{T-\tau-\delta}\int_{t_i+\delta}^{t_i+T}F^*(t';t_i)\,dt'\,,\end{aligned}$$

and we finally arrive at

$$J_T^*(x_{t_i+\delta}) \leq \underbrace{\left(1 + B(T)\,\frac{1+\tau}{T-\tau-\delta}\right)}_{=\frac{1}{\bar\beta}} \int\limits_{t_i+\delta}^{t_i+T} F^*(t';t_i)\,dt' \,.$$

This completes the proof. ■

Note that $\bar\beta \to 1$ for $T \to \infty$. This property will later be useful to show that given Assumption 1, there always exists a finite prediction horizon T large enough such that the closed-loop using the MPC controller is asymptotically stable.

Lemma 2 (Calculation of γ) *Consider system* (1) *and let Assumption 1 be satisfied. Then,*

$$\int_{t_i+\delta}^{t_i+T} F^*(t';t_i)dt' \leq \gamma \left(\int_{t_i}^{t_i+\delta} F^*(t';t_i)\,dt' + \int_{t_i-\tau}^{t_i} \hat F(|x(t')|)\,dt'\right) \quad (11)$$

with $\gamma = B(T)\,\frac{1+\tau}{\delta}$.

Proof: Let Assumption 1 be satisfied. Then the inequality

$$\int_{t_i+t}^{t_i+T} F^*(t';t_i)dt' \leq B(T-t) \times \left(\hat F^*(t_i+t;t_i) + \int\limits_{t-\tau}^{t} \hat F^*(t_i+t';t_i)\,dt'\right) \quad (12)$$

holds for all $t \in [0,T]$. This result is similar to [12, Lemma 3] and to the first part in the proof of Lemma 1. It is a direct consequence of the priniciple of optimality (endpieces of optimal trajectories are optimal), the Controllability Assumption 1 and the definition of B. Furthermore,

$$\int_{t_i+\delta}^{t_i+T} F^*(t';t_i)dt' \leq \int_{t_i+t}^{t_i+T} F^*(t';t_i)dt' \quad (13)$$

holds for all $t \in [0,\delta]$, Equation (12) and $B(T-t) \leq B(T)$ hold for all $t \in [0,T]$. Putting all of these intermediate results together, we obtain

$$\int_{t_i+\delta}^{t_i+T} F^*(t';t_i)dt' \leq B(T) \times \min_{t\in[0,\delta]} \left(\hat F^*(t_i+t;t_i) + \int\limits_{t-\tau}^{t} \hat F^*(t_i+t';t_i)\,dt'\right) \,.$$

Using Lemma 4 in the Appendix, it follows that

$$\min_{t\in[0,\delta]} \left(\hat F^*(t_i+t;t_i) + \int\limits_{t-\tau}^{t} \hat F^*(t_i+t';t_i)\,dt'\right) \leq \frac{1+\tau}{\delta} \int\limits_{t_i-\tau}^{t_i+\delta} \hat F^*(t';t_i)\,dt' \,,$$

and we finally arrive at

$$\int_{t_i+\delta}^{t_i+T} F^*(t';t_i)dt' \leq \underbrace{B(T)\,\frac{1+\tau}{\delta}}_{=\gamma} \times \left(\int_{t_i}^{t_i+\delta} F^*(t';t_i)\,dt' + \int_{t_i-\tau}^{t_i} \hat F(|x(t')|)\,dt'\right) \,.$$

This completes the proof. ■

IV. Asymptotic Stability and Suboptimality Estimate

Based on the results of the previous section, we can state our main result regarding asymptotic stability of the closed-loop using the unconstrained MPC as follows.

Theorem 3 (Asymptotic Stability) *Consider the system* (1) *and let Assumptions 1 and 2 be satisfied. Define*

$$\alpha := 1 - (\mathcal{N}+1)\left(\frac{1}{\bar\beta} - 1\right)\gamma \quad (14)$$

with $\bar\beta$ and γ defined in Lemmata 1 and 2, respectively, i.e.

$$\alpha = 1 - (\mathcal{N}+1)\,\frac{B(T)^2\,(1+\tau)^2}{(T-\tau-\delta)\,\delta} \,. \quad (15)$$

If $\alpha > 0$, then the closed-loop using the MPC controller (6) *is asymptotically stable.*

Proof: Consider the optimal cost J_T^* at two arbitrary sampling instants t_i and t_0, for which $t_i > t_0$. Adding zero and reordering terms yields directly

$$J_T^*(x_{t_i}) - J_T^*(x_{t_0}) = \sum_{j=0}^{i-1} J_T^*(x_{t_{j+1}}) - J_T^*(x_{t_j}) \,. \quad (16)$$

Using Lemma 1, we obtain

$$J_T^*(x_{t_{j+1}}) - J_T^*(x_{t_j}) = J_T^*(x_{t_j+\delta}) - J_T^*(x_{t_j}) \leq \left(\frac{1}{\bar\beta} - 1\right) \int_{t_j+\delta}^{t_j+T} F^*(t';t_j)dt' - \int_{t_j}^{t_j+\delta} F^*(t';t_j)dt' \,.$$

Moreover, it follows from Lemma 2 that

$$J_T^*(x_{t_{j+1}}) - J_T^*(x_{t_j}) \leq \left(\left(\frac{1}{\bar\beta} - 1\right)\gamma - 1\right) \int\limits_{t_j}^{t_j+\delta} F^*(t';t_j)\,dt' + \left(\frac{1}{\bar\beta} - 1\right)\gamma \int\limits_{t_j-\tau}^{t_j} \hat F(|x(t')|)\,dt' \quad (17)$$

Because we consider no model plant mismatch, the predicted trajectories and the actual trajectories of the closed-loop system coincide until the next sampling instant. Thus, $F^*(t';t_j) = F(x(t'), u_{\mathrm{MPC}}(t'))$ for $t' \in [t_j, t_j+\delta]$. Furthermore, $\hat F(|x(t')|) \leq F(x(t'), u_{\mathrm{MPC}}(t'))$ by definition of of $\hat F$

in (3). Careful inspection of the sum in (16) in combination with (17) then yields

$$
\begin{aligned}
&J_T^*(x_{t_i}) - J_T^*(x_{t_0}) \\
&\leq \left(\left(\frac{1}{\bar{\beta}} - 1 \right) \gamma - 1 \right) \int_{t_0}^{t_i} F(x(t'), u_{\mathrm{MPC}}(t'))\, dt' \\
&\qquad + \mathcal{N} \left(\frac{1}{\bar{\beta}} - 1 \right) \gamma \int_{t_0}^{t_i} F(x(t'), u_{\mathrm{MPC}}(t'))\, dt' \\
&\qquad + \mathcal{N} \left(\frac{1}{\bar{\beta}} - 1 \right) \gamma \int_{t_0-\tau}^{t_0} \hat{F}(|x(t')|)\, dt' \\
&= \underbrace{\left((\mathcal{N}+1) \left(\frac{1}{\bar{\beta}} - 1 \right) \gamma - 1 \right)}_{=-\alpha} \int_{t_0}^{t_i} F(x(t'), u_{\mathrm{MPC}}(t'))\, dt' \\
&\qquad + \mathcal{N} \left(\frac{1}{\bar{\beta}} - 1 \right) \gamma \int_{t_0-\tau}^{t_0} \hat{F}(|x(t')|)\, dt' \,.
\end{aligned}
$$

Since $J_T^*(x_{t_i}) > 0$ and $J_T^*(x_{t_0})$ is finite, it follows for any arbitrary $t_i > t_0$

$$
\begin{aligned}
&\int_{t_0}^{t_i} F(x(t'), u_{\mathrm{MPC}}(t'))\, dt' \\
&\leq \frac{1}{\alpha} J_T^*(x_{t_0}) + \mathcal{N} \left(\frac{1}{\bar{\beta}} - 1 \right) \frac{\gamma}{\alpha} \int_{t_0-\tau}^{t_0} \hat{F}(|x(t')|)\, dt' \qquad (18) \\
&< \infty \,.
\end{aligned}
$$

Asymptotic stability follows directly from standard arguments in optimal control and Barbalat's Lemma [18]. This completes the proof. ■

Remark 2 *Equation* (18) *and* $J_T^*(\varphi) \leq J_\infty^*(\varphi)$ *yield for initial condition* φ *in* (1b)

$$
\begin{aligned}
J_\infty^*(\varphi) \leq J_\infty^{\mathrm{MPC}}(\varphi) &= \int_0^\infty F(x(t'), u_{\mathrm{MPC}}(t'))\, dt' \\
&\leq \frac{1}{\alpha} J_T^*(\varphi) + \mathcal{N} \left(\frac{1}{\bar{\beta}} - 1 \right) \frac{\gamma}{\alpha} \int_{-\tau}^{0} \hat{F}(|\varphi(t')|)\, dt' \qquad \text{(19a)} \\
&\leq \frac{1}{\alpha} J_\infty^*(\varphi) + \mathcal{N} \left(\frac{1}{\bar{\beta}} - 1 \right) \frac{\gamma}{\alpha} \int_{-\tau}^{0} \hat{F}(|\varphi(t')|)\, dt' \,. \qquad \text{(19b)}
\end{aligned}
$$

Thus, an upper bound on the infinite horizon performance of the MPC controller can be given in terms of the finite horizon optimal cost at initial time $J_T^*(\varphi)$ *and on an additional term depending on the initial state* φ*. This second term is not necessary in the case of delay-free systems as can be directly seen for* $\tau = 0$*.*

Remark 3 *Note that* $\bar{\beta} \to 1$ *for* $T \to \infty$*, which directly implies* $\alpha \to 1$ *for* $T \to \infty$*. Hence, there always exists a finite prediction horizon* T *chosen suitably large such that the closed-loop using the MPC controller is asymptotically stable. Furthermore, Equation* (19) *shows that* $J_\infty^{\mathrm{MPC}}(\varphi) \to J_\infty^*(\varphi)$ *for* $T \to \infty$*, i.e. infinite horizon optimal performance is recovered for large enough enough prediction horizon and the influence of the second term in* (19b)*, which depends on the initial condition* φ*, vanishes.*

Remark 4 *Note that* $\alpha \to -\infty$ *for* $\delta \to 0$*, which means that asymptotic stability of the closed-loop cannot be guaranteed for arbitrarily small sampling times. This result is somewhat counterintuitive and it was shown in [19, 20] that with an additional condition, the so-called* growth condition*, this effect can be avoided. The estimate obtained in [12] satisfies* $\bar{\beta} \to 1$ *for* $\delta \to 0$*, which allows to cancel the effect of* $\gamma \to \infty$ *for* $\delta \to 0$ *when using the growth condition. Unfortunately, this is not possible for the results presented in this work. The estimate for* $\bar{\beta}$ *in Lemma 1 is more conservative and it is not based on an optimization problem in contrast to [12, 19, 20], but only uses the estimate based on Lemma 4. It is simple to see in Lemma 1, that* $\bar{\beta} \nrightarrow 1$ *for* $\delta \to 0$ *and independent of* $B(t)$*. Hence, the growth condition is not applicable to avoid the poor estimates for small sampling times.*

Remark 5 *Stability is not proven by a decrease of the optimal cost function from one sampling instant to the next one, but only by a decrease in the long run. The optimal cost can indeed increase along trajectories of the closed-loop due to the effect of the delayed states.*

Remark 6 *Note that for* $\tau = 0$*, we obtain* $\mathcal{N} = 0$ *and* $\alpha = 1 - \left(\frac{1}{\bar{\beta}} - 1 \right) \gamma$*. This directly recovers the stability condition of [12, Theorem 6] for delay-free continuous-time systems. Furthermore, Equation* (18) *becomes*

$$
\int_{t_0}^{\infty} F(x(t'), u_{\mathrm{MPC}}(t'))\, dt' \leq \frac{1}{\alpha} J_T^*(x_{t_0}) \leq \frac{1}{\alpha} J_\infty^*(x_{t_0}) \,,
$$

which recovers the suboptimality estimate of the infinite horizon performance of the MPC controller.

V. Conclusions

In this work we considered model predictive control for nonlinear time-delay systems using neither terminal constraints nor control Lyapunov-Krasovskii functionals as terminal weighting terms. First, we proposed an extended asymptotic controllability assumption, which was necessary because the stage cost only penalizes the instantaneous state instead of the full delayed state. Based on this assumption, we provided conditions on the length of the prediction horizon to guarantee nominal asymptotic stability of the closed-loop. In contrast to most other results on stability of MPC, the optimal cost is not used as Lyapunov function or Lyapunov Krasovskii functional. The optimal cost can indeed increase along trajectories of the closed-loop due to the effect of the delayed states.

Future research will be dedicated to improving the estimates given in this work. In particular, the estimates for $\bar{\beta}$ have to be improved in order to possibly use a certain growth condition similar to previous results for discrete-time systems and finite-dimensional continuous-time systems. This might enable us to avoid the poor estimates for small sampling times and remove the counterintuitive lack of guaranteed stability for arbitrary small sampling times.

Appendix

Lemma 4 *For any $t_1, t_2, \tau \in \mathbb{R}$ with $t_1 < t_2$ and $\tau > 0$, and for any positive integrable function $F : [t_1 - \tau, t_2] \to \mathbb{R}^+$, the following holds*

$$\min_{t\in[t_1,t_2]} \left(F(t) + \int_{t-\tau}^{t} F(t')\,dt' \right) \leq \frac{1+\tau}{t_2 - t_1} \int_{t_1-\tau}^{t_2} F(t')\,dt' \,.$$

Proof: Due to fundamental properties of integrals and positivity of F, it follows that

$$\begin{aligned}
&\min_{t\in[t_1,t_2]} \left(F(t) + \int_{t-\tau}^{t} F(t')\,dt' \right) \\
&\leq \frac{1}{t_2-t_1} \int_{t_1}^{t_2} \left(F(t) + \int_{t-\tau}^{t} F(t')\,dt' \right) dt \\
&\leq \frac{1}{t_2-t_1} \int_{t_1-\tau}^{t_2} F(t')\,dt' + \frac{1}{t_2-t_1} \int_{t_1}^{t_2} \int_{t-\tau}^{t} F(t')\,dt'dt \\
&\overset{(*)}{\leq} \frac{1}{t_2-t_1} \int_{t_1-\tau}^{t_2} F(t')dt' + \frac{1}{t_2-t_1} \int_{t_1-\tau}^{t_2} \int_{t'}^{t'+\tau} F(t')dtdt' \\
&= \frac{1}{t_2-t_1} \int_{t_1-\tau}^{t_2} F(t')\,dt' + \frac{\tau}{t_2-t_1} \int_{t_1-\tau}^{t_2} F(t')dt' \,.
\end{aligned}$$

For the interchange of the order of integration and the enlarged domain of integration in inequality $(*)$, see Figure 2. The term $\int_{t_1}^{t_2} \int_{t-\tau}^{t} F(t')\,dt'dt$ results from integration over the domain within the black solid line, whereas $\int_{t_1-\tau}^{t_2} \int_{t'}^{t'+\tau} F(t')dtdt'$ corresponds to the larger domain additionally including the areas given by the orange dashed lines. ■

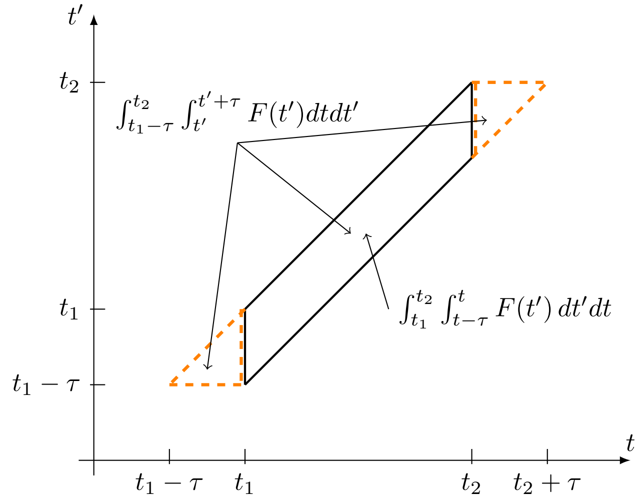

Fig. 2. Sketch for interchange of the order of integration in the proof of Lemma 4.

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