2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Second Order Linear Consensus Protocols with Irregular Topologies and Time Delay

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Abstract— **A methodology for the stability analysis of linear consensus protocols for groups of agents driven by second order dynamics is presented in this paper. It is assumed that the communication topologies are undirected and that the time delays incur between the agents are constant and uniform for all the channels. The proposed technique takes advantage of the general structure of consensus protocols that allows a decomposition of the characteristic equation in a set of factors, facilitating the stability analysis. The factors generated by this procedure are individually studied using the Cluster Treatment of Characteristic Roots paradigm, a recent method which declares the stability features of the system for various compositions of the time delay and other control parameters. Several illustrative examples are provided.**

I. Introduction

Distributed (decentralized) coordination of systems with multiple agents has received a great deal of attention in many recent investigations. This interest is mainly due to the broad spectrum of applications of such systems in many areas, e.g., unmanned search and rescue operations. The problem of consensus generation is one of the most widely studied topics, among many other aspects of the field. The main objective of a consensus protocol is to drive all the agents of the group in a way such that they will reach a common value in some variable of interest, value that usually depends only on the initial conditions of the agents and the communication structure.

The work of Olfati-Saber and Murray [1] is one of the earlier studies published presenting the consensus problem for multi agent coordination. They focus on agents with first order dynamics, considering fixed and switching communications topologies. Under the simplifying features of the first order dynamics, they also studied the behavior of their protocol when communication time delays are present, keeping the communication topology fixed. Several other researchers [2-6, 12-13] have performed further extensions on this earlier work, proposing different protocols for consensus of agents that are driven by second order dynamics.

For the analysis of the stability in the delay space, the previously published works use approximate methods, based on LMIs. Lin [2-4], particularly, studied different protocols and presented stability conditions based on the existence of feasible solutions to LMIs, from which it is very difficult to obtain a clear maximum bound in the time delay for which the system remains stable. None of the previous works offers an exact and determination of the stability boundaries of the system in the time delay space, which is one of the contributions of the CTCR paradigm [7-8].

In this paper, we present a new methodology for the stability analysis of consensus protocols over groups of agents with second order continuous time dynamics, which operate under an undirected and time delayed communication structure. The main contribution is the treatment of the complete stability picture for such systems, taking into account variations of the control parameters as well as the delay. In the rest of the paper, bold face notation is used for vector quantities, bold capital letters for matrices and italic symbols for scalars.

Manuscript received February 28, 2011. This work was supported in part by ARO grant W911NF-07-1-0557.

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II. Graph Theory Review

In order to establish the terminology used for the communication networks within this paper, we present a short review first, following [9].

A graph Γ consists of a set of *vertices* V, a set of *edges* E and an *incidence relation*. The incidence relation is required to be such that an edge is incident with two vertices, and no two edges are incident with the same pair of vertices. Then, the set E can be regarded as a subset of the set of *unordered* pairs of vertices.

If v and w are two vertices of a graph Γ, and $e=\{v,w\}$ is an edge of Γ, then we say that e joins v and w, that v and w are adjacent, and that v and w are the ends of e. The number of edges of which v is an end is called the degree of v. A walk of length ℓ in Γ, from v to w, is defined as a finite sequence of vertices of Γ, $\{v=u_0, u_1, u_2,\ldots, u_\ell=w\}$, such that u_{k-1} and u_k are adjacent for every $1<k<\ell$. A graph is said to be connected if each pair of vertices is joined by a walk.

The adjacency matrix of a graph Γ with vertex set $V=\{v_1, v_2,\ldots,v_n\}$, is the matrix $\mathbf{A}_\Gamma \in \Re^{n\times n}$ whose (i,j) entry is 1 if v_i and v_j are connected and 0 otherwise. From this definition, it follows that $\mathbf{A}_\Gamma$ is a real symmetric matrix. The spectrum of a graph is the set of numbers which are eigenvalues of $\mathbf{A}_\Gamma$ together with their multiplicities. Usually the terminology eigenvalues of the graph Γ is used to refer to the eigenvalues of its adjacency matrix $\mathbf{A}_\Gamma$. A lot of information about a graph can be obtained from its spectrum.

Other important matrices related to a graph are the degree matrix $\mathbf{\Delta}$, which is an n by n diagonal matrix whose (j,j) entry is the degree of the j-th vertex, i.e., $\mathbf{\Delta}$ =diag(Δ_1, Δ_2 ,… Δ_n) and the Laplacian matrix, defined as the difference between the degree matrix and the adjacency matrix of the graph: L= $\mathbf{\Delta}$–$\mathbf{A}_\Gamma$.

The Laplacian matrix of a graph is as important as its adjacency matrix, and its spectrum, known as the Laplacian spectrum of the graph, also provides a good amount of information about the graph. One of the most important properties of the Laplacian matrix is that its rank is always n–r, where r is the number of disconnected subgraphs within the graph, which are connected themselves. These subgraphs are called components. If the graph is connected, i.e., it has only one component, the rank of its Laplacian will be n–1. This implies that zero is always an eigenvalue of the Laplacian of a graph; its corresponding eigenvector is the vector $\mathbf{1}_n$, i.e., an n-D column vector whose elements are all 1. Furthermore, for a connected graph, the zero eigenvalue is unique [9]. The second smallest eigenvalue of the Laplacian matrix is also known as the *algebraic connectivity* of the graph since it is an indication of how connected a graph is. This eigenvalue increases with the number of edges of the graph. [10].

III. Problem Statement

We consider a group of n autonomous agents, which are driven by second order dynamics given by $\ddot{x}_j = u_j$, j=1, 2,…, n, where $x_j \in \Re$ is taken as the scalar position and $u_j \in \Re$ as the control law. Here we treat the motion of the agent as one dimensional, but the entire analysis is still valid for higher dimensions. We declare consensus is achieved when all n agents are at the same position, i.e., $\lim_{t\to\infty}(x_j - x_k) = 0$ for any $j,k \in [1,2,\ldots,n]$. Notice that this consensus definition does not state anything about the value of the final position.

We assume the j-th agent exchanges its position and velocity information with a subset N_j formed by Δ_j agents, $\Delta_j < n$, called informers. Assuming bi-directional channels, the communication network can be described by an undirected graph with n vertices. It is also assumed that all these communication channels have a constant delay of τ seconds, i.e., agent j only knows the τ-seconds-earlier state of its Δ_j informers.

For the consensus creation, several different control laws have been proposed in the literature [1-6, 12-13]. In this paper we consider three of them, presented in the following paragraphs.

A. Protocol A

Introduced by Lin in [2], the control strategy incorporated into the individual dynamics is given as:

$$\ddot{x}(t) = \sum_{k\in N_j}\left[k_1\left(x_k(t-\tau) - x_j(t-\tau)\right) + k_2\left(\dot{x}_k(t-\tau) - \dot{x}_j(t-\tau)\right)\right] \tag{1a}$$

With the state vector $\mathbf{x} = [x_1\ \dot{x}_1\ x_2\ \dot{x}_2\ \ldots\ x_n\ \dot{x}_n]^T \in \Re^{2n}$, the entire group dynamics is given by:

$$\dot{\mathbf{x}}(t) = \left(\mathbf{I}_n \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\mathbf{x}(t) - \left(\mathbf{L} \otimes \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}\right)\mathbf{x}(t-\tau) \tag{1b}$$

where $\mathbf{I}_n$ represents the $n \times n$ identity matrix, $\otimes$ is the Kronecker product [11], $\mathbf{L}$ is the Laplacian of the undirected graph that describes the communication topology and k_1 and k_2 are user-selected positive control gains.

B. Protocol B

The second protocol we consider is a variation of protocol A presented by the same research group in [4]. In order to avoid relative velocity measurements, the state of each agent is augmented with an auxiliary variable, p_j, which works as an observer, increasing the order of the individual dynamics, which results in:

$$\begin{aligned} \dot{p}_j(t) &= -\gamma p_j(t) - \sum_{k\in N_j}\left[x_k(t-\tau) - x_j(t-\tau)\right] \\ \ddot{x}_j(t) &= p_j(t) + k_0\dot{x}_j(t) + k_1\sum_{k\in N_j}\left[x_k(t-\tau) - x_j(t-\tau)\right] \end{aligned} \tag{2a}$$

where the constant parameters γ and k_1 are always positive, and k_0 can be positive, negative or zero. With $\mathbf{x} = [x_1\ \dot{x}_1\ p_1\ x_2\ \dot{x}_2\ p_2 \ldots x_n\ \dot{x}_n\ p_n]^T \in \Re^{3n}$, the state space representation of the complete system is:

$$\dot{\mathbf{x}}(t) = \left(\mathbf{I}_n \otimes \begin{bmatrix} 0 & 1 & 0 \\ 0 & k_0 & 1 \\ 0 & 0 & -\gamma \end{bmatrix}\right)\mathbf{x}(t) - \left(\mathbf{L} \otimes \begin{bmatrix} 0 & 0 & 0 \\ k_1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right)\mathbf{x}(t-\tau) \tag{2b}$$

where $\mathbf{L}$ is again the Laplacian of the communication topology.

C. Protocol C

The last protocol we consider here was introduced by the authors [12-13]. In this case, the assumption is that the agents compare their own current states with the average of the delayed states of the informers. This protocol is described by:

$$\ddot{x}(t) = P\left(\sum_{k\in N_j}\frac{x_k(t-\tau)}{\Delta_j} - x_j(t)\right) + D\left(\sum_{k\in N_j}\frac{\dot{x}_k(t-\tau)}{\Delta_j} - \dot{x}_j(t)\right) \tag{3a}$$

and expressed in state space as:

$$\dot{\mathbf{x}}(t) = \left(\mathbf{I}_n \otimes \begin{bmatrix} 0 & 1 \\ -P & -D \end{bmatrix}\right)\mathbf{x}(t) + \left(\mathbf{C} \otimes \begin{bmatrix} 0 & 0 \\ P & D \end{bmatrix}\right)\mathbf{x}(t-\tau) \tag{3b}$$

where $\mathbf{x} = [x_1\ \dot{x}_1\ x_2\ \dot{x}_2\ \ldots\ x_n\ \dot{x}_n]^T \in \Re^{2n}$ and $\mathbf{C}=\mathbf{\Delta}^{-1}\mathbf{A}_\Gamma$ is the product of the inverse of the degree matrix, $\mathbf{\Delta}$, and the adjacency matrix, $\mathbf{A}_\Gamma$, of the graph representing the communication network. P and D are user-selected positive control gains.

IV. Stability Analysis of Consensus Protocols with Time Delay

The stability conditions presented in [2] and [4] for protocols A and B, respectively, are based on a Lyapunov-Krasowskii functional and it relies on the solution of a LMI. These stability conditions are cumbersome to apply, since the construction of the LMI is tedious and its solution is non-unique. Furthermore, this path provides only a conservative upper bound for the delay that the system can accommodate before loosing stability. As an alternative, the protocols presented in the previous section, which are in the form $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau)$, can be analyzed for stability using the Cluster Treatment of Characteristic Roots, CTCR, [7-8] paradigm. This procedure provides an exact and exhaustive stability assessment.

However, neither one of the approaches, LMIs or CTCR, requires a significant amount of computational power, especially when the number of agents is large, since the order of the matrices involved in the LMIs and the degree of the characteristic quasi-polynomials of the system are directly related to the size of the group. Just to get an idea of the complexity involved, consider four agents interacting under the fixed communication topology of Fig. 1. The characteristic equations, $\det\left(s\mathbf{I} - \mathbf{A} - \mathbf{B}e^{-\tau s}\right)$, of the resulting system when protocols A, B and C are used, are given, respectively, in (1c), (2c) and (3c) below:

$$\begin{aligned} & s^8 + 6k_2 e^{-\tau s} s^7 + \left(10k_2^2 e^{-2\tau s} + 6k_1 e^{-\tau s}\right)s^6 + \\ & \left(4k_2^3 e^{-3\tau s} + 20k_1 k_2 e^{-2\tau s}\right)s^5 + \left(12k_2^2 k_1 e^{-3\tau s} + \right. \\ & \left. 10k_1^2 e^{-2\tau s}\right)s^4 + 12k_1^2 k_2 e^{-3\tau s} s^3 + 4k_1^3 e^{-3\tau s} s^2 = 0 \end{aligned} \tag{1c}$$

$$\begin{aligned} & s^8 + 4Ds^7 + \left[4P + \left(6 - \frac{5}{4}e^{-\tau s}\right)D^2\right]s^6 + \\ & \left[12DP + 4D^3 - \frac{5}{2}D\left(D^2 - P\right)e^{-2\tau s}\right]s^5 + \\ & \left[D^4 + 6P^2 + 12D^2P - \left(\frac{5}{4}\left(D^4 + P^2\right) + \frac{15}{2}D^2P\right)e^{-2\tau s} + \frac{D^4}{4}e^{-4\tau s}\right]s^4 + \\ & \left[12DP^2 + 4D^3P - \left(\frac{15}{2}DP^2 + 5D^3P\right)e^{-2\tau s} + D^3Pe^{-4\tau s}\right]s^3 + \\ & \left[4P^3 + 6D^2P^2 - \frac{5}{2}\left(3D^2P^2 + P^3\right)e^{-2\tau s} + \frac{3}{2}D^2P^2e^{-4\tau s}\right]s^2 + \\ & \left[4DP^3 - 5DP^3e^{-2\tau s} + DP^3e^{-4\tau s}\right]s + \frac{P^4}{4}\left(1 - 5e^{-2\tau s} + e^{-4\tau s}\right) = 0 \end{aligned} \tag{3c}$$

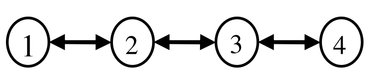

Fig. 1 Communication topology used in the example cases.

Equations (1c), (2c) and (3c) show complex features, even for a small number of agents and a relatively simple communication topology. For higher number of agents, the stability analysis becomes intractable very rapidly. In the following paragraphs, we show that the characteristic equation of this class of systems can be conveniently converted into the product of a set of reduced order factors, simplifying the problem to a level ideally suited for the application of CTCR.

$$\begin{aligned} & s^{12} + 4(\gamma - k_0)s^{11} + \left(6\gamma^2 + 6k_0^2 + 6k_1e^{-\tau s} - 16\gamma k_0\right)s^{10} + \\ & \left[4\gamma^3 + 24\gamma k_0^2 - 24\gamma^2 k_0 - 4k_0^3 + 6(4k_1 - 3k_0k_1 - 1)e^{-\tau s}\right]s^9 + \\ & \left[36\gamma^2k_0^2 - 16\gamma k_0^3 + k_0^4 - 16\gamma k_0^3 - 16\gamma^3k_0 + \gamma^4 + \right. \\ & \left. 18\left(k_0 + k_1k_0^2 - \gamma + 2\gamma^2k_1 - 4\gamma k_1k_0\right)e^{-\tau s} + 10k_1^2e^{-2\tau s}\right]s^8 + \\ & \left[4\left(6\gamma^3k_0^2 + \gamma k_0^4 - \gamma^4k_0 - 6\gamma^2k_0^3\right) + 6\left(4\gamma^3k_1 - k_1k_0^3 + 12\gamma k_1k_0^2 - \right.\right. \\ & \left. 18\gamma^2k_1k_0 + 9\gamma k_0 - 3k_0^2 - 3\gamma^2\right)e^{-\tau s} + 20\left(2\gamma k_1^2 - k_1 - \right. \\ & \left.\left. k_1^2k_0\right)e^{-2\tau s}\right]s^7 + \left[6\gamma^4k_0^2 - 16\gamma^3k_0^3 + 6\gamma^2k_0^4 + 6\left(\gamma^4k_1 - \gamma^3 - \right.\right. \\ & \left. 9\gamma k_0^2 - 12\gamma^3k_1k_0 + 18\gamma^2k_0^2k_1 - 4\gamma k_1k_0^3 + 9\gamma^2k_0 + k_0^3\right)e^{-\tau s} + \\ & 10\left(6\gamma^2k_1^2 + k_1^2k_0^2 - 8\gamma k_1^2k_0 + 4k_1k_0 - 6\gamma k_1 + 1\right)e^{-2\tau s} \\ & \left. + 4k_1^3e^{-3\tau s}\right]s^6 + \left[4\left(\gamma^3k_0^4 - 4\gamma^4k_0^3\right) + 18\left(4\gamma^3k_0^2k_1 + \gamma k_0^3 - \gamma^3k_0 + \right.\right. \\ & \left. 2\gamma^2k_0^3k_1 - 3\gamma^2k_0^2 - \gamma^4k_0k_1\right)e^{-\tau s} + 20\left(2\gamma k_0^2k_1^2 + \gamma - k_0 - k_1k_0^2 - \right. \\ & \left. 6\gamma^2k_1^2k_0 - 3\gamma^2k_1 + 6\gamma k_0k_1 + 2\gamma^3k_1^2\right)e^{-2\tau s} + 4\left(4k_1^3\gamma - k_1^3k_0 - \right. \\ & \left.\left. 3k_1^2\right)e^{-3\tau s}\right]s^5 + \left[k_0^4\gamma^4 + 6\left(3\gamma^2k_0^3 - 3\gamma^3k_0^2 - 4\gamma^3k_1k_0^3 + \right.\right. \\ & \left. 3\gamma^4k_1k_0^2\right)e^{-\tau s} + 10\left(\gamma^4k_1^2 - 2\gamma k_1^3 - 6\gamma k_1k_0^2 - 8\gamma^3k_1^2k_0 - 4\gamma k_0 + \right. \\ & \left. k_0^2 + 12\gamma^2k_1k_0 + 6\gamma^2k_1^2k_0^2 + \gamma^2\right)e^{-2\tau s} + 4\left(3k_1^2k_0 + 3k_1 - 9\gamma k_1^2 + \right. \\ & \left.\left. 6\gamma^2k_1^3 - 4\gamma k_1^3k_0\right)e^{-3\tau s}\right]s^4 + \left[6k_0^3\left(\gamma^3 - \gamma^4k_1\right)e^{-\tau s} + 20\left(\gamma k_0^2 + \right.\right. \\ & \left. 2\gamma^3k_1^2k_0^2 - \gamma^4k_1^2k_0 - \gamma^2k_0 + 2\gamma^3k_1k_0 - 3\gamma^2k_1k_0^2\right)e^{-2\tau s} + 4\left(6\gamma k_1 + \right. \\ & \left.\left. 9\gamma k_1^2k_0 - 3k_1k_0 + 4\gamma^3k_1^3 - 6\gamma^2k_1^3k_0 - 36\gamma^2k_1^2 - 1\right)e^{-3\tau s}\right]s^3 + \\ & \left[10\left(\gamma^2k_0^2 - 2\gamma^3k_1k_0^2 + \gamma^4k_1^2k_0^2\right)e^{-2\tau s} + 4\left(k_0 - 6\gamma k_1k_0 + 3\gamma^2k_1 - \right.\right. \\ & \left.\left. 4\gamma^3k_1^3k_0 + \gamma^4k_1^3 - \gamma + 9\gamma^2k_1^2k_0 - 3\gamma^3k_1^2\right)e^{-3\tau s}\right]s^2 + 4k_0\left(\gamma + \right. \\ & \left. 3\gamma^3k_1^2 - 3\gamma^2k_1 - 3\gamma^4k_1^3\right)e^{-3\tau s}s = 0 \end{aligned} \tag{2c}$$

A. Main Result: factorization of the Characteristic Equation

Protocols A, B, and C share a common structure in their state space representations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau)$. It is straightforward to demonstrate that the corresponding **B** matrices, from (1b), (2b) and (3b), are always block diagonalizable, since the Laplacian matrix used in (1b) and (2b) is always a real symmetric matrix [9] and the matrix $\mathbf{C}=\mathbf{\Delta}^{-1}\mathbf{A}_{\Gamma}$, used in (3b) is a symmetrizable matrix, and thus always diagonalizable [14]. Also, the self evident **A** matrices of those equations are block diagonal, and the size of the blocks is always equal to the order of the individual dynamics, i.e., 2 in (1) and (3), and 3 in (2). These properties allow the introduction of the following lemma:

Main Lemma: The characteristic equation of linear consensus systems with time delay $\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}e^{-\tau s}\right)=0$, operating under protocols *A*, *B*, *C* or similar, can be expressed as a product of n factors whose degrees are all equal to the order of the dynamics of the individual agents.

Proof: Consider the following state-space representation of the three protocols *A*, *B* and *C*:

$$\dot{\mathbf{x}}(t)=\left(\mathbf{I}_n\otimes\mathbf{F}_1\right)\mathbf{x}(t)+\left(\mathbf{M}\otimes\mathbf{F}_2\right)\mathbf{x}(t-\tau) \quad (4)$$

for a set of *n* agents. The matrices $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{M}$ are self-evident from (1b), (2b) and (3b). If *r* is the order of the dynamics of each individual agent (3 for protocol *B* and 2 for protocols *A* and *C*), then $\mathbf{F}_1,\mathbf{F}_2\in\Re^{r\times r}$, and $\mathbf{M}\in\Re^{n\times n}$. Since $\mathbf{M}$ is diagonalizable, there is a nonsingular matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{M}\mathbf{T}=\mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is a diagonal matrix whose non-zero entries, λ_1, λ_2,…, λ_n, are the eigenvalues of $\mathbf{M}$. Introducing a state transformation $\mathbf{x}=\left(\mathbf{T}\otimes\mathbf{I}_n\right)\boldsymbol{\xi}\in\Re^n$ into (4), we obtain:

$$\dot{\boldsymbol{\xi}}=\left(\mathbf{T}^{-1}\otimes\mathbf{I}_\gamma\right)\left(\mathbf{I}_n\otimes\mathbf{F}_1\right)\left(\mathbf{T}\otimes\mathbf{I}_\gamma\right)\boldsymbol{\xi}(t)+\left(\mathbf{T}^{-1}\otimes\mathbf{I}_\gamma\right)\left(\mathbf{M}\otimes\mathbf{F}_2\right)\left(\mathbf{T}\otimes\mathbf{I}_\gamma\right)\boldsymbol{\xi}(t-\tau) \quad (5)$$

Using the following property of the kronecker multiplication [11]:

$$\left(\mathbf{U}\otimes\mathbf{V}\right)\left(\mathbf{W}\otimes\mathbf{Z}\right)=\mathbf{UW}\otimes\mathbf{VZ} \quad (6)$$

Equation (5) becomes:

$$\dot{\boldsymbol{\xi}}=\left(\mathbf{I}_n\otimes\mathbf{F}_1\right)\boldsymbol{\xi}(t)+\left(\mathbf{\Lambda}\otimes\mathbf{F}_2\right)\boldsymbol{\xi}(t-\tau) \quad (7)$$

which is a block-diagonal expression, since $\mathbf{I}_n$ and $\mathbf{\Lambda}$ are diagonal matrices. Then, (7) can be expressed as a set of *n* decoupled systems of order *r*, each one with dynamics:

$$\dot{\boldsymbol{\xi}}=\mathbf{F}_1\boldsymbol{\xi}_j(t)+\lambda_j\mathbf{F}_2\boldsymbol{\xi}_j(t-\tau)\in\Re^r \quad j=1,2,\ldots,n \quad (8)$$

The characteristic equation of each subsystem is given as $\det(s\mathbf{I}_r-\mathbf{F}_1-\lambda_j\mathbf{F}_2e^{-\tau s})=0$, so the characteristic equation of the complete system is the product of the n individual factors:

$$\det\left(s\mathbf{I}_{nr}-\mathbf{A}-\mathbf{B}e^{-\tau s}\right)=\prod_{j=1}^{n}\det\left(s\mathbf{I}_r-\mathbf{F}_1-\lambda_j\mathbf{F}e^{-\tau s}\right)=0 \quad (9)$$

QED

Remark: In all of these protocols, one of the factors in (9) is always representative of the dynamics of the group decision value, which would be the common value of the state of the agents if consensus is reached. This factor appears because the vector $\mathbf{1}_n$, an *n*-dimensional column vector with all elements equal to 1, is always an eigenvector of **M.** In protocols *A* and *B* the corresponding eigenvalue is 0 [2-4], whereas in protocol *C* it is 1 [12]. The remaining *n*–1 factors are related to the disagreement dynamics, and they dictate whether the consensus is reached or not. This fact has already been proven in [2-4] for protocols *A* and *B*, and in [12] for protocol *C*. The contribution of this paper is in the stability analysis of these three dynamics in the domain of the delay and control parameters.

In order to show the simplification created by Main Lemma, we present the outlook of the factorized characteristic equations of protocols *A* (10), *B* (11) and *C* (12):

$$\begin{aligned}&s^2\left(s^2+0.59(k_2s+k_1)e^{-\tau s}\right)\times\\&\left(s^2+2(k_2s+k_1)e^{-\tau s}\right)\left(s^2+3.41(k_2s+k_1)e^{-\tau s}\right)=0\end{aligned} \quad (10)$$

$$\begin{aligned}&\left(s^3+(\gamma-k_0)s^2-\gamma k_0s\right)\times\\&\left(s^3+(\gamma-k_0)s^2-\gamma k_0s+0.59(k_1s+k_1\gamma-1)e^{-\tau s}\right)\times\\&\left(s^3+(\gamma-k_0)s^2-\gamma k_0s+2(k_1s+k_1\gamma-1)e^{-\tau s}\right)\times\\&\left(s^3+(\gamma-k_0)s^2-\gamma k_0s+3.41(k_1s+k_1\gamma-1)e^{-\tau s}\right)=0\end{aligned} \quad (11)$$

$$\begin{aligned}&\left(s^2+(Ds+P)\left(1-e^{-\tau s}\right)\right)\left(s^2+(Ds+P)\left(1-0.5e^{-\tau s}\right)\right)\times\\&\left(s^2+(Ds+P)\left(1+e^{-\tau s}\right)\right)\left(s^2+(Ds+P)\left(1+0.5e^{-\tau s}\right)\right)=0\end{aligned} \quad (12)$$

As a clarification to the reader, the eigenvalues of the Laplacian of the topology used are 0, 0.59, 2 and 3.41, and the eigenvalues of the corresponding $\mathbf{C}$ matrix are ±1 and ±0.5. These eigenvalues are already incorporated in equations (10), (11) and (12).

The comparison between (1c) and (10), (2c) and (11), (3c) and (12); clearly shows the considerable reduction in the complexity of the problem created by the application of the Lemma 1. Instead of dealing with the cumbersome stability analysis of (1c), (2c) and (3c), we solve the problem for each one of the factors in (10), (11) and (12) separately and superpose them. This point constitutes the main contribution of the present study.

B. CTCR Deployment

After the simplification, the CTCR paradigm is now used for the stability analysis of each individual factor in (9). Since the $\mathbf{F}_2$ matrices, for all three cases, are of rank 1, none of the factors exhibits commensuracy (integer multiplicity) in the delay terms. Then, the first step of the CTCR methodology, the exhaustive determination of the finite number of crossing frequencies (Proposition I in [7]), is straightforward for any of the protocols. It is performed using a procedure similar to the one presented in the analysis of the Delayed Resonator active vibration suppression system [15]. The second step uses the invariance property of the root crossing tendency at these imaginary roots as the delay increases (Proposition II in [7]).

As discussed in the previous subsection, the actual algebraic structure of the factors depends on the protocol being used. Here we study the factors generated by protocol *A* just to demonstrate the steps of the suggested method. A similar procedure can be used in all other cases. Its application to protocols *B*, and *C,* for example, can be found in [13] and [12] respectively.

The factors generated by protocol *A* display the form:

$$s^2 + \lambda_j\left(k_2 s + k_1\right)e^{-\tau s} = 0 \tag{13}$$

where λ_j are the eigenvalues of the Laplacian matrix corresponding to the underlying communication topology. Following the graph theory conventions [9], we name these eigenvalues as $0=\lambda_1<\lambda_2<\ldots<\lambda_n$. As it was mentioned before, it can be proven that the factor corresponding to the single zero eigenvalue, $\lambda_1=0$, describes the dynamics of the group decision value. For this $\lambda_1=0$, equation (13) becomes $s^2=0$. It indicates that consensus, if reached, would be at a constant velocity and linearly increasing position.

For the nonzero eigenvalues of the Laplacian, $\lambda_j\neq 0$, (13) can be written as:

$$s^2 = -\lambda_j\left(k_2 s + k_1\right)e^{-\tau s} \tag{14}$$

For equation (14) to have an imaginary root at $s=\omega i$, the magnitudes and phases of both of its members must be equal when the suggested root is substituted in place. The magnitude equation yields:

$$\omega^4 - k_2^2\lambda_j^2\omega^2 - \lambda^2 k_1^2 = 0 \tag{15}$$

The solutions of (15) are given by:

$$\omega^2 = \frac{k_2^2\lambda_j^2 \pm \sqrt{k_2^4\lambda_j^4 + 4\lambda^2 k_1^2}}{2} \tag{16}$$

Since ω^2 should be a positive quantity, the only feasible solution for the imaginary solutions of (14) is:

$$\omega = \left(\frac{k_2^2\lambda_j^2 + \sqrt{k_2^4\lambda_j^4 + 4\lambda^2 k_1^2}}{2}\right)^{1/2} \tag{17}$$

The phase equality condition of (14) produces:

$$\pi = \pi + \arctan\left(\frac{k_2\omega}{k_1}\right) - \tau\omega \tag{18}$$

which defines infinitely many equidistant delays that generate the only imaginary crossing as the delay reaches the values:

$$\tau_{c,k} = \frac{1}{\omega}\arctan\left(\frac{k_2\omega}{k_1}\right) + k\frac{2\pi}{\omega} \tag{19}$$

At these imaginary crossings, the root tendency, i.e., the direction of crossing for increasing τ values is defined as [7]:

$$RT = \operatorname{sgn}\left[\operatorname{Re}\left(\frac{ds}{d\tau}\right)\Bigg|_{s=\omega_c i\ \tau=\tau_{c,k}}\right] \tag{20}$$

and this property is invariant vis-à-vis the counter k for a given imaginary root ωi, (Proposition II in [7]). For this particular case, using the characteristic equation (14), we obtain the roots sensitivity as:

$$\frac{ds}{d\tau} = \frac{-s\left(k_2 s + k_1\right)\lambda e^{-\tau s}}{2s + \left[k_2 - \tau\left(k_2 s + k_1\right)\right]\lambda e^{-\tau s}} \tag{21}$$

At the only imaginary root of the generic factor (14), defined by (17), and the corresponding time delays of (19), the root tendency is always destabilizing ($RT=+1$), which means that the roots move to the right half of the complex plane. This statement can be verified by the following logical sequence. First, consider the stability of (14) when $\tau = 0$. If the control gains k_1 and k_2 are positive, the factor is stable and all the characteristic roots would be on the left half of the complex plane. As the delay τ increases these roots could only move to the right at the first (and the only) imaginary crossing, making only $RT = +1$ possible.

The stability posture in the parametric space of the individual factors (14) can be determined with the use of (17), (19), and (21). These stability tables are exact and exhaustive. The individual results can then be superposed to obtain a combined stability picture for the complete multi-agent system. Those combinations of parameters and time delays that bring stability to all the factors of the form (14) for a certain communication structure generate consensus among the agents when protocol *A* is in use.

An important observation in this procedure is that the factors of the characteristic equation, such as (14), require only the eigenvalues of a known matrix (Laplacian for protocols *A* and *B*, **C** for protocol *C*). Since these matrices are easily obtained from the definitions of the protocols and the respective communication topologies, the stability problem reduces to repeated applications of the above described stability study. Superposition of the stable regions will reveal the system stability. Some example case studies are presented in the next section.

V. Example Cases

After performing the analysis of the previous section, the stability posture of the system with respect to (k_1, k_2, τ) is obtained. Figure 2 shows the composition of the stability boundaries for each one of the 3 factors generated. The plot was obtained keeping constant k_1=5 and changing k_2. The red thick lines indicate points at which each factor has its first stability change, and the shaded region represents stable operation zone: parametric selections inside this region render the system stable, bringing agents to a consensus.

In order to verify the results, Fig. 3 and 4 present the time history of the individual positions and velocities of the agents for different parametric settings, showing consensus for points inside the region and divergence for points outside.

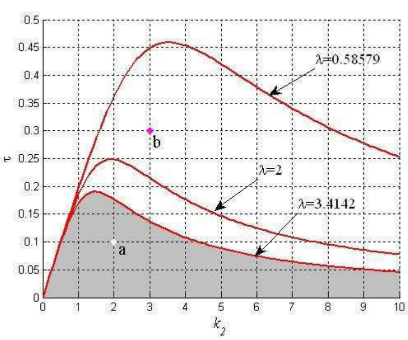

Fig. 2: Stability boundaries generated by protocol A with k_1=5. Shaded zone depict stability region for the complete system.

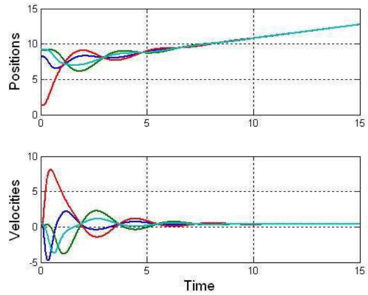

Fig. 3: Example stable behavior of 4 agents under protocol *A*. Point **a** in Fig. 2.

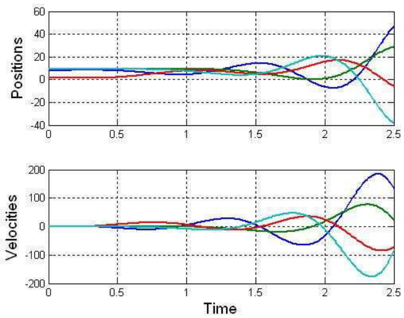

Fig 4: Example unstable behavior of 4 agents under protocol *A*. Point **b** in Fig. 2.

The parametric selection used to generate Fig. 3 is k_1=5, k_2=2 and τ=0.1, corresponding to point **a** in Fig. 2; for Fig. 4 k_1=5, k_2=3 and τ=0.3 where used, corresponding to point **b** in Fig. 2. To finish this section, we wish to point out, as a very interesting feature of the protocol studied, that there is always a factor of the characteristic equation introducing the most restrictive stability region. The eigenvalue corresponding to that factor turns out to be very easy to identify, and it is the largest eigenvalue of **L**, for the protocol *A* we have studied here. The proof of this property is non-trivial and is suppressed here for space considerations, but we refer the reader to [12], where a similar property is shown for protocol *C*. This feature introduces a wonderful simplification to the stability problem, in that, it is necessary to assess the stability outlook of only one of the factors of the characteristic equation. Furthermore this factor is *a priori* determined from the distribution of the eigenvalues of a given matrix. Consequently, the stability variations of the complete swarm dynamics for different parameters are obtained expeditiously.

VI. Conclusions

This paper presents a general procedure for the stability analysis of linear consensus protocols of multi-agent swarms with fixed communication topology and uniform and fixed time-delayed information exchange. The methodology developed here, which can be applied to different control laws, allows a significant reduction in the complexity of the problem by means of a crucial factorization of the characteristic equation of the system. The stability of the resulting factors can be exactly and exhaustively analyzed in the space of the time delay by using the Cluster Treatment of Characteristic Roots paradigm.

The strategy presented has several advantages over other contemporary techniques, which are mainly based on the solution of LMIs. For example, once the control gains are set, the bounds obtained for the maximum time delay tolerable by the system are exact not approximate. The effect of variations in different parameters on the stability is easier to study. These are the results of the use of the CTCR paradigm and the reduction in complexity introduced by the factorization is the key advantage of the new analysis procedure, especially when the number of agents is large. In such cases the size of the LMIs used by other methods can increase to prohibitively large dimensions, whereas the new method only requires the knowledge of the eigenvalues of a matrix related to the communication topology.

Another interesting observation from this methodology is that there is only one eigenvalue creating the most restrictive stability boundary. This idea introduces an even more dramatic simplification of the problem, transforming it into the study of only one factor. The identification of this "most exigent eigenvalue" is the main objective of our on-going work.

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