

Globally Asymptotically Stable Filters for Source Localization and Navigation aided by Direction Measurements

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Abstract—This paper presents a set of filters with globally asymptotically stable error dynamics for source localization and navigation, in 3-D, based on direction measurements from the agent (or vehicle) to the source, in addition to relative velocity readings of the agent. Both the source and the agent are allowed to have constant unknown drift velocities and the relative drift velocity is also explicitly estimated. The observability of the system is studied and realistic simulation results are presented, in the presence of measurement noise, that illustrate the performance of the achieved solutions. Comparison results with the Extended Kalman Filter are also provided and similar performances are achieved.

I. INTRODUCTION

The problem of source localization has been subject of intensive research in recent years [1]. Roughly speaking, an agent has access to a set of measurements and aims to estimate the position of a source. The set of measurements depends on the environment in which the operation occurs and the mission scenario itself. Previous work in the field can be found in [2], where the authors propose a localization algorithm based on the range to the source and the inertial position of the agent. Global exponential stability (GES) is achieved under a persistent excitation condition and the analysis is extended to the case of a non-stationary source, where it is shown that it is possible to achieve tracking up to some bounded error. In [3] the same problem was addressed considering, in addition to range readings to the source, relative velocity readings of the agent. The observability of the system was assessed, including also relative drift velocities, and filtering solutions were proposed with globally asymptotically stable (GAS) error dynamics. More recently, in [4], the same problem was addressed, in 2-D, based on bearing measurements, in addition to the trajectory of the agent. The estimation error dynamics were shown to be GES under an appropriate persistent excitation condition and a circumnavigation control law was also proposed. Earlier work on the observability issues of target motion analysis based on angle readings, in 2-D, can be found in [5], which was later extended to 3-D in [6]. The specific observability criteria thereby derived resort to complicated nonlinear differential equations and some tedious mathematics are needed for the solution, giving conditions that are necessary for system observability. Another related framework in the domain of target motion analysis (TMA) can be found in

[7], where frequency measurements are also included. This topic was further studied in [8], where Cramer Rao analysis revealed the parametric dependencies of TMA with angle-only tracking and angle/frequency tracking, giving also an idea of the increase in estimation accuracy using the later.

Parallel to the topic of source localization based on range or bearing measurements is the topic of navigation aided by these sensors. Previous work by the authors with range measurements can be found in [9], where acceleration readings were also considered. The observability of the system was assessed and conditions were derived that guarantee GAS error dynamics. In [10] a similar design was proposed with two vehicles working in tandem considering relative velocity drifts. Globally asymptotically stable error dynamics were also shown under appropriate observability conditions. In [11] the authors deal with the problem of underwater navigation in the presence of unknown currents based on range measurements to a single beacon. An observability analysis is presented based on the linearization of the nonlinear system which yields local results. Based on the linearized system dynamics, a Luenberger observer is introduced but in practice an Extended Kalman filter (EKF) is implemented, with no warranties of global asymptotic stability. More recently, the same problem has been studied in [12] and [13], where EKFs have been extensively used to solve the navigation problem based on single beacon range measurements. The problem of localization of a mobile robot using bearing measurements was also addressed in [14], where a nonlinear transformation of the measurement equation into a higher dimensional space is performed. This has allowed to obtain tight, possibly complex-shaped, bounding sets for the feasible states in a closed-form representation.

This paper addresses the problem of navigation/source localization based on direction measurements to a single source in the presence of unknown constant drifts. The observability of the system is studied and Kalman filters with GAS error dynamics are proposed, without system linearizations and yielding performances comparable to those of the Extended Kalman Filter but with GAS guarantees. Central to the design is the augmentation of the system state, which allowed to consider linear time-varying (LTV) system dynamics. The observability conditions have clear physical meaning and they are directly related to the motion of the agent/vehicle, hence useful for motion planning and control so that the system is observable.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and \mathbf{I} an identity matrix, both of appro-

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appropriate dimensions. A block diagonal matrix is represented as $\text{diag}(\mathbf{A}_1, \dots, \mathbf{A}_n)$ and the set of unit vectors on \mathbb{R}^3 is denoted by $S(2)$. Finally, $\delta(t)$ corresponds to the Dirac delta function.

II. PROBLEM STATEMENT

A. Source localization

Let $\mathbf{p}(t) \in \mathbb{R}^3$ denote the position of a point-mass agent, in inertial coordinates, moving in a scenario where there is a source whose position, in inertial coordinates, is denoted by $\mathbf{s}(t) \in \mathbb{R}^3$. Suppose that the source is moving with constant unknown velocity $\mathbf{v}_s(t) \in \mathbb{R}^3$ relative to the inertial frame, i.e.,

$$\begin{cases} \dot{\mathbf{s}}(t) = \mathbf{v}_s(t) \\ \dot{\mathbf{v}}_s(t) = \mathbf{0} \end{cases},$$

while the linear motion kinematics of the agent are given by

$$\begin{cases} \dot{\mathbf{p}}(t) = \mathbf{v}_c(t) + \mathbf{v}_r(t) \\ \dot{\mathbf{v}}_c(t) = \mathbf{0} \end{cases},$$

where $\mathbf{v}_c(t) \in \mathbb{R}^3$ is a constant unknown drift velocity of the agent and $\mathbf{v}_r(t) \in \mathbb{R}^3$ is a known input. In the context of the EU project TRIDENT, the source may be an Autonomous Surface Craft (ASC) and the agent an Autonomous Underwater Vehicle (AUV). The ASC is moving with constant unknown velocity $\mathbf{v}_s(t)$ and the AUV is moving with velocity relative to the water $\mathbf{v}_r(t)$, as given by a Doppler Velocity Log (DVL), in the presence of constant unknown ocean currents with velocity $\mathbf{v}_c(t)$. Further consider that the agent measures the direction to the source

$$\mathbf{d}(t) = \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} \in S(2), \quad (1)$$

with $\mathbf{r}(t) := \mathbf{s}(t) - \mathbf{p}(t) \in \mathbb{R}^3$. The problem of source localization considered here is that of estimating the position of the source relative to the agent, $\mathbf{r}(t)$, and the relative drift velocity $\mathbf{v}_{sa}(t) := \mathbf{v}_s(t) - \mathbf{v}_c(t) \in \mathbb{R}^3$, given direction and relative velocity readings, $\mathbf{d}(t)$ and $\mathbf{v}_r(t)$, respectively. The corresponding system dynamics are given by

$$\begin{cases} \dot{\mathbf{r}}(t) = \mathbf{v}_{sa}(t) - \mathbf{v}_r(t) \\ \dot{\mathbf{v}}_{sa}(t) = \mathbf{0} \\ \mathbf{d}(t) = \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} \end{cases}.$$

The following assumption is required in the sequel.

Assumption 1: The relative velocity is continuous and continuously differentiable. Moreover, both $\mathbf{v}_r(t)$ and $\dot{\mathbf{v}}_r(t)$ are norm-bounded.

This is a mild assumption with clear physical interpretation as the actuation systems of agents or vehicles limit the available force and torque, which implies upper bounds on the velocities and accelerations. In this paper it allows to consider that both $\dot{\mathbf{d}}(t)$ and $\ddot{\mathbf{d}}(t)$ are norm-bounded. The values of the bounds are not required.

B. Navigation

In the context of the EU project TRIDENT, an ASC and an AUV work in close cooperation in order to achieve a certain goal. Assume that the ASC (the source) transmits its inertial position $\mathbf{s}(t)$ and velocity $\mathbf{v}_s(t)$ to the AUV (the agent). In this framework, the goal of the AUV (the agent) is now to determine its own position in inertial coordinates $\mathbf{p}(t)$, as well as its drift velocity $\mathbf{v}_c(t)$, given the information provided

by the ASC (the source), the relative velocity readings $\mathbf{v}_r(t)$, and the direction measurements $\mathbf{d}(t)$. In this framework $\mathbf{v}_s(t)$ is no longer required to be constant and the system dynamics are given by

$$\begin{cases} \dot{\mathbf{p}}(t) = \mathbf{v}_c(t) + \mathbf{v}_r(t) \\ \dot{\mathbf{v}}_c(t) = \mathbf{0} \\ \mathbf{d}(t) = \frac{\mathbf{s}(t) - \mathbf{p}(t)}{\|\mathbf{s}(t) - \mathbf{p}(t)\|} \end{cases}. \quad (2)$$

III. SOURCE LOCALIZATION FILTER DESIGN

This section presents a filter design methodology for the problem of source localization introduced in Section II-A. First, an augmented linear time-varying system is introduced in Section III-A. Afterwards, in Section III-B, the observability of this system is fully characterized and its relation with the original nonlinear system is established. Finally, the filter design is discussed in Section III-C.

A. System dynamics

In order to derive an augmented linear time-varying system for source localization, define the system states as

$$\begin{cases} \mathbf{x}_1(t) = \mathbf{r}(t) \\ \mathbf{x}_2(t) = \mathbf{v}_{sa}(t) \\ x_3(t) = \|\mathbf{r}(t)\| \end{cases}.$$

From (1) it follows that $\mathbf{x}_1(t) - x_3(t)\mathbf{d}(t) = \mathbf{0}$ for all t . Let $\mathbf{x}(t) = [\mathbf{x}_1^T(t) \mathbf{x}_2^T(t) x_3(t)]^T \in \mathbb{R}^7$. Then, the system dynamics are given by the LTV system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}, \quad (3)$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{d}^T(t) & 0 \end{bmatrix} \in \mathbb{R}^{7 \times 7},$$

$$\mathbf{B}(t) = \begin{bmatrix} -\mathbf{I} \\ \mathbf{0} \\ -\mathbf{d}^T(t) \end{bmatrix} \in \mathbb{R}^{7 \times 3},$$

$\mathbf{C}(t) = [\mathbf{I} \ \mathbf{0} \ -\mathbf{d}(t)] \in \mathbb{R}^{3 \times 7}$, and $\mathbf{u}(t) = \mathbf{v}_r(t)$.

B. Observability analysis

The observability of the problem of source localization with relative velocity readings and direction measurements is studied in this section. The following proposition [Proposition 4.2, [15]] is useful in the sequel.

Proposition 1: Let $\mathbf{f}(t) : [t_0, t_f] \subset \mathbb{R} \rightarrow \mathbb{R}^n$ be a continuous and i -times continuously differentiable function on $\mathcal{I} := [t_0, t_f]$, $T := t_f - t_0 > 0$, and such that

$$\mathbf{f}(t_0) = \dot{\mathbf{f}}(t_0) = \dots = \mathbf{f}^{(i-1)}(t_0) = \mathbf{0}.$$

Further assume that there exists a nonnegative constant C such that $\|\mathbf{f}^{(i+1)}(t)\| \leq C$ for all $t \in \mathcal{I}$. If there exists $\alpha > 0$ and $t_1 \in \mathcal{I}$ such that $\|\mathbf{f}^{(i)}(t_1)\| \geq \alpha$, then

$$\exists_{\substack{0 < \delta \leq T \\ \beta > 0}} : \|\mathbf{f}(t_0 + \delta)\| \geq \beta.$$

The following theorem characterizes the observability of the LTV system (3).

Theorem 1: The LTV system (3) is observable on $\mathcal{I} := [t_0, t_f]$ if and only if the unit vector $\mathbf{d}(t)$ is not constant on \mathcal{I} or, equivalently,

$$\exists_{t_1 \in \mathcal{I}} \mathbf{d}^T(t_0) \mathbf{d}(t_1) < 1. \quad (4)$$

Proof: The observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C}(t))$ on \mathcal{I} is given by

$$\mathcal{W}(t_0, t_f) = \int_{t_0}^{t_f} \phi^T(\tau, t) \mathbf{C}^T(\tau) \mathbf{C}(\tau) \phi(\tau, t) d\tau,$$

where $\phi(t, t_0)$ denotes the transition matrix associated with $\mathbf{A}(t)$,

$$\phi(t, t_0) = \begin{bmatrix} \mathbf{I} & (t - t_0) \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \int_{t_0}^t \mathbf{d}^T(\tau) d\tau & 1 \end{bmatrix} \in \mathbb{R}^{7 \times 7}.$$

Let $\mathbf{c} = [\mathbf{c}_1^T \quad \mathbf{c}_2^T \quad c_3]^T \in \mathbb{R}^7$, $\mathbf{c}_i \in \mathbb{R}^3$, $i = 1, 2$, $c_3 \in \mathbb{R}$, be a unit vector, i.e., $\|\mathbf{c}\| = 1$. Then,

$$\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} = \int_{t_0}^{t_f} \|\mathbf{f}(\tau)\|^2 d\tau$$

for all $\|\mathbf{c}\| = 1$, where

$$\mathbf{f}(\tau) = \mathbf{c}_1 + \left[(\tau - t_0) \mathbf{I} - \mathbf{d}(\tau) \int_{t_0}^{\tau} \mathbf{d}^T(\sigma) d\sigma \right] \mathbf{c}_2 - c_3 \mathbf{d}(\tau)$$

for all $\tau \in \mathcal{I}$. The first two derivatives of $\mathbf{f}(\tau)$ are given by

$$\frac{d}{d\tau} \mathbf{f}(\tau) = \left[\mathbf{I} - \mathbf{d}(\tau) \mathbf{d}^T(\tau) - \dot{\mathbf{d}}(\tau) \int_{t_0}^{\tau} \mathbf{d}^T(\sigma) d\sigma \right] \mathbf{c}_2 - c_3 \dot{\mathbf{d}}(\tau)$$

and

$$\begin{aligned} \frac{d^2}{d\tau^2} \mathbf{f}(\tau) &= \left[-2\dot{\mathbf{d}}(\tau) \mathbf{d}^T(\tau) - \mathbf{d}(\tau) \dot{\mathbf{d}}^T(\tau) \right. \\ &\quad \left. - \ddot{\mathbf{d}}(\tau) \int_{t_0}^{\tau} \mathbf{d}^T(\sigma) d\sigma \right] \mathbf{c}_2 - c_3 \ddot{\mathbf{d}}(\tau) \end{aligned}$$

for all $\tau \in \mathcal{I}$. Notice that, under Assumption 1, both derivatives are norm-bounded, from above, on \mathcal{I} .

It can be trivially shown that (4) is a necessary condition. Indeed, suppose that (4) is not verified. Then, the unit vector $\mathbf{d}(t)$ is constant on \mathcal{I} , i.e., $\mathbf{d}(t) = \mathbf{d}(t_0)$ for all $t \in \mathcal{I}$. Let $\mathbf{c}_1 = \frac{\sqrt{2}}{2} \mathbf{d}(t_0)$, $\mathbf{c}_2 = \mathbf{0}$, and $c_3 = \frac{\sqrt{2}}{2}$. Then,

$$\mathbf{f}(\tau) = \frac{\sqrt{2}}{2} \mathbf{d}(t_0) - \frac{\sqrt{2}}{2} \mathbf{d}(\tau) = \mathbf{0}$$

for all $\tau \in \mathcal{I}$, which in turn allows to conclude that the observability Gramian $\mathcal{W}(t_0, t_f)$ is not invertible and the LTV system (3) is not observable on \mathcal{I} . Consequently, if the LTV system (3) is observable on \mathcal{I} , it follows that (4) is true.

To show that (4) is also a sufficient condition, suppose first that $c_3 \neq 0$. Then, if $\mathbf{c}_1 \neq c_3 \mathbf{d}(t_0)$, it follows that $\|\mathbf{f}(t_0)\| > 0$ and, from Proposition 1, it must be $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$. Consider now $\mathbf{c}_1 = c_3 \mathbf{d}(t_0)$, with $c_3 \neq 0$. In this case, $\mathbf{f}(t_0) = \mathbf{0}$ and

$$\left. \frac{d}{d\tau} \mathbf{f}(\tau) \right|_{\tau=t_0} = [\mathbf{I} - \mathbf{d}(t_0) \mathbf{d}^T(t_0)] \mathbf{c}_2 - c_3 \dot{\mathbf{d}}(t_0).$$

If $\left\| \left. \frac{d}{d\tau} \mathbf{f}(\tau) \right|_{\tau=t_0} \right\| > 0$, it follows, using Proposition 1 twice, that $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$. Otherwise, if $\left. \frac{d}{d\tau} \mathbf{f}(\tau) \right|_{\tau=t_0} = \mathbf{0}$, two cases may be considered: i) if $\dot{\mathbf{d}}(t_0) = \mathbf{0}$, it may be $\mathbf{c}_2 = \mathbf{0}$ or $\mathbf{c}_2 = c_2 \mathbf{d}(t_0)$ for some scalar c_2 ; or ii) if $\dot{\mathbf{d}}(t_0) \neq \mathbf{0}$, it must be $\mathbf{c}_2 = c_3 \mathbf{d}(t_0)$, where it is used the fact that

$\mathbf{d}^T(t) \dot{\mathbf{d}}(t) = 0$ for all t . Evaluating $\mathbf{f}(\tau)$ at $\tau = t_1$, when $\mathbf{c}_2 = \mathbf{0}$, yields $\mathbf{f}(t_1) = c_3 \mathbf{d}(t_0) - c_3 \mathbf{d}(t_1)$ which has positive norm if (4) is true. As such, it follows from Proposition 1 that $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$ for $\mathbf{c}_1 = c_3 \mathbf{d}(t_0)$, $\mathbf{c}_2 = \mathbf{0}$, $c_3 \neq 0$. If $\mathbf{c}_2 = c_2 \mathbf{d}(t_0)$, $\mathbf{f}(t_1)$ reads as

$$\begin{aligned} \mathbf{f}(t_1) &= [c_3 + c_2(t_1 - t_0)] \mathbf{d}(t_0) \\ &\quad - \left[c_3 + c_2 \int_{t_0}^{t_1} \mathbf{d}^T(\sigma) \mathbf{d}(t_0) d\sigma \right] \mathbf{d}(t_1). \end{aligned}$$

If (4) is true, and as $\mathbf{d}(t)$ is continuous, it must be $\int_{t_0}^{t_1} \mathbf{d}^T(\sigma) \mathbf{d}(t_0) d\sigma \neq t_1 - t_0$ and therefore it follows that $\|\mathbf{f}(t_1)\| > 0$. As such, from Proposition 1, $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$ for $\mathbf{c}_1 = c_3 \mathbf{d}(t_0)$, $\mathbf{c}_2 = c_2 \mathbf{d}(t_0)$, $c_2 \neq 0$, $c_3 \neq 0$. If $\mathbf{c}_2 = c_3 \mathbf{d}(t_0)$, with $\dot{\mathbf{d}}(t_0) \neq \mathbf{0}$ and $\mathbf{c}_1 = c_3 \mathbf{d}(t_0)$, $c_3 \neq 0$, there exists $\epsilon > 0$ such that

$$\begin{aligned} \mathbf{f}(t_0 + \epsilon) &= c_3 \mathbf{d}(t_0) + c_3 \epsilon \dot{\mathbf{d}}(t_0) \\ &\quad - c_3 \left[1 + \int_{t_0}^{t_0 + \epsilon} \mathbf{d}^T(\sigma) \dot{\mathbf{d}}(t_0) d\sigma \right] \mathbf{d}(t_0 + \epsilon), \end{aligned}$$

where $\mathbf{d}(t_0 + \epsilon)$ cannot be expressed as a linear combination of $\mathbf{d}(t_0)$ and $\dot{\mathbf{d}}(t_0)$. As such, $\|\mathbf{f}(t_0 + \epsilon)\| > 0$ and, using Proposition 1, $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$ for $\mathbf{c}_1 = c_3 \mathbf{d}(t_0)$, $\mathbf{c}_2 = c_3 \dot{\mathbf{d}}(t_0)$, $c_3 \neq 0$. This allows to conclude, so far, that if $c_3 \neq 0$, $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$. It remains to see what happens when $c_3 = 0$. If $\mathbf{c}_1 \neq \mathbf{0}$, it turns out that $\|\mathbf{f}(t_0)\| > 0$ and again, using Proposition 1, it must be $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$ for $\mathbf{c}_1 \neq \mathbf{0}$, $c_3 = 0$. On the other hand, if $\mathbf{c}_1 = \mathbf{0}$, $c_3 = 0$, it follows that $\mathbf{f}(t_0) = \mathbf{0}$ and

$$\left. \frac{d}{d\tau} \mathbf{f}(\tau) \right|_{\tau=t_0} = [\mathbf{I} - \mathbf{d}(t_0) \mathbf{d}^T(t_0)] \mathbf{c}_2.$$

Now, if $\mathbf{c}_2 \neq \pm \mathbf{d}(t_0)$, it follows that

$$\left\| \left. \frac{d}{d\tau} \mathbf{f}(\tau) \right|_{\tau=t_0} \right\| > 0$$

and, using Proposition 1 twice, it must be $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$ for $\mathbf{c}_1 = \mathbf{0}$, $\mathbf{c}_2 \neq \pm \mathbf{d}(t_0)$, $c_3 = 0$. Finally, if $\mathbf{c}_2 = \pm \mathbf{d}(t_0)$, with $\mathbf{c}_1 = \mathbf{0}$ and $c_3 = 0$, it follows that

$$\mathbf{f}(t_1) = \pm (t_1 - t_0) \mathbf{d}(t_0) \mp \int_{t_0}^{t_1} \mathbf{d}^T(\sigma) \mathbf{d}(t_0) d\sigma \mathbf{d}(t_1),$$

which has positive norm. Again, using Proposition 1, it follows that $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$ for $\mathbf{c}_1 = \mathbf{0}$, $\mathbf{c}_2 = \pm \mathbf{d}(t_0)$, $c_3 = 0$. But this concludes the proof, as it is shown that $\mathbf{c}^T \mathcal{W}(t_0, t_f) \mathbf{c} > 0$ for all $\|\mathbf{c}\| = 1$, which means that the observability Gramian is invertible and as such (3) is observable. ■

Before proceeding, it is important to remark that there is nothing in (3) imposing the nonlinear restriction $\|\mathbf{x}_1(t)\| = x_3(t) = \|\mathbf{r}(t)\|$. This is true, by construction, if it is satisfied for $t = t_0$. The following theorem addresses this issue.

Theorem 2: Under the terms of Theorem 1, the initial condition of the LTV (3) corresponds to the initial condition of the original nonlinear system, i.e.,

$$\begin{cases} \mathbf{x}_1(t_0) = \mathbf{r}(t_0) \\ \mathbf{x}_2(t_0) = \mathbf{v}_{sa}(t_0) \\ x_3(t_0) = \|\mathbf{r}(t_0)\| \end{cases}. \quad (5)$$

Proof: Under the terms of Theorem 1, the initial condition of the LTV system (3) is uniquely determined by the corresponding system output and input. The proof follows

by showing that (5) explains the system output. As the initial condition is uniquely determined, if (5) explains the output of the system, it must correspond to the initial condition. The output of the LTV system (3) is given by

$$\mathbf{y}(t) = \mathbf{x}_1(t_0) + (t - t_0) \mathbf{x}_2(t_0) - \int_{t_0}^t \mathbf{u}(\tau) d\tau - x_3(t_0) \mathbf{d}(t) - \int_{t_0}^t [\mathbf{x}_2(t_0) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau) d\tau \mathbf{d}(t) = \mathbf{0} \quad (6)$$

for all $t \in \mathcal{I}$, $\mathcal{I} = [t_0, t_f]$. Substituting (5) in (6) gives

$$\mathbf{y}(t) = \mathbf{r}(t_0) - \|\mathbf{r}(t_0)\| \mathbf{d}(t) + \int_{t_0}^t [\mathbf{v}_{sa}(t_0) - \mathbf{u}(\tau)] d\tau - \int_{t_0}^t [\mathbf{v}_{sa}(t_0) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau) d\tau \mathbf{d}(t). \quad (7)$$

It remains only to show that (7) is null for all $t \in \mathcal{I}$. Substituting $t = t_0$ in (7) yields $\mathbf{y}(t_0) = \mathbf{0}$. The time derivative of (7) is given by

$$\dot{\mathbf{y}}(t) = - \left[\|\mathbf{r}(t_0)\| + \int_{t_0}^t [\mathbf{v}_{sa}(t_0) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau) d\tau \right] \dot{\mathbf{d}}(t) + [\mathbf{v}_{sa}(t_0) - \mathbf{u}(t)] - [\mathbf{v}_{sa}(t_0) - \mathbf{u}(t)]^T \mathbf{d}(t) \mathbf{d}(t). \quad (8)$$

As $\mathbf{v}_{sa}(t)$ is constant, it is possible to rewrite (8) as

$$\dot{\mathbf{y}}(t) = - \left[\|\mathbf{r}(t_0)\| + \int_{t_0}^t [\mathbf{v}_{sa}(\tau) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau) d\tau \right] \dot{\mathbf{d}}(t) + [\mathbf{v}_{sa}(t) - \mathbf{u}(t)] - [\mathbf{v}_{sa}(t) - \mathbf{u}(t)]^T \mathbf{d}(t) \mathbf{d}(t). \quad (9)$$

The derivative of $\|\mathbf{r}(t)\|$ given by

$$\frac{d}{dt} \|\mathbf{r}(t)\| = [\mathbf{v}_{sa}(t) - \mathbf{u}(t)]^T \mathbf{d}(t),$$

which allows to write

$$\|\mathbf{r}(t)\| = \|\mathbf{r}(t_0)\| + \int_{t_0}^t [\mathbf{v}_{sa}(\tau) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau) d\tau. \quad (10)$$

On the other hand, the time derivative of (1) is given by

$$\dot{\mathbf{d}}(t) = \frac{[\mathbf{v}_{sa}(t) - \mathbf{u}(t)] - [\mathbf{v}_{sa}(t) - \mathbf{u}(t)]^T \mathbf{d}(t) \mathbf{d}(t)}{\|\mathbf{r}(t)\|}. \quad (11)$$

Substituting (10) and (11) in (9) gives $\dot{\mathbf{y}}(t) = \mathbf{0}$. This concludes the proof, as with $\mathbf{y}(t_0) = \mathbf{0}$ and $\dot{\mathbf{y}}(t) = \mathbf{0}$ it must be $\mathbf{y}(t) = \mathbf{0}$ for all $t \in \mathcal{I}$ and therefore (5) is true. ■

In order to design GAS observers (or filtering) solutions, stronger forms of observability are convenient. The following theorem addresses this issue.

Theorem 3: The LTV system (3) is uniformly completely observable on $\mathcal{I} = [t_0, t_f]$ if and only if

$$\exists_{\substack{\alpha > 0 \\ \delta > 0}} \forall_{t \geq t_0} \int_t^{t+\delta} \mathbf{d}^T(t) \mathbf{d}(\tau) d\tau \leq \delta(1 - \alpha). \quad (12)$$

Proof: The proof of sufficiency follows similar steps to Theorem 1 considering uniformity bounds that stem from the persistent excitation condition (12). Therefore it is omitted. To show that (12) is also necessary, suppose that (12) does not hold. Then,

$$\forall_{\substack{\alpha > 0 \\ \delta > 0}} \exists_{t^* \geq t_0} \int_{t^*}^{t^*+\delta} \mathbf{d}^T(t^*) \mathbf{d}(t) dt > \delta(1 - \alpha). \quad (13)$$

Let $\mathbf{c} = \left[\frac{\sqrt{2}}{2} \mathbf{d}^T(t^*) \quad \mathbf{0} \quad \sqrt{2}^T/2 \right]^T \in \mathbb{R}^7$. Then,

$$\begin{aligned} \mathbf{c}^T \mathcal{W}(t^*, t^* + \delta) \mathbf{c} &= \frac{1}{2} \int_{t^*}^{t^*+\delta} \|\mathbf{d}(t^*) - \mathbf{d}(\tau)\|^2 d\tau \\ &= \frac{1}{2} \int_{t^*}^{t^*+\delta} [\|\mathbf{d}(t^*)\|^2 + \|\mathbf{d}(\tau)\|^2 - 2\mathbf{d}^T(t^*) \mathbf{d}(\tau)] d\tau. \end{aligned} \quad (14)$$

As $\mathbf{d}(t)$ is a unit vector, it is possible to write (14) as

$$\mathbf{c}^T \mathcal{W}(t^*, t^* + \delta) \mathbf{c} = \delta - \int_{t^*}^{t^*+\delta} \mathbf{d}^T(t^*) \mathbf{d}(\tau) d\tau. \quad (15)$$

Using (13) in (15) allows to conclude that

$$\forall_{\substack{\alpha > 0 \\ \delta > 0}} \exists_{t^* \geq t_0} \mathbf{c}^T \mathcal{W}(t^*, t^* + \delta) \mathbf{c} < \delta\alpha,$$

which means that the LTV system (3) is not uniformly completely observable. Therefore, if the LTV system (3) is uniformly completely observable, (12) is true. ■

C. Kalman filter

Section III-A introduced a LTV system for source localization and its observability was characterized in Section III-B. In particular, it was shown that the LTV system (3) is uniformly completely observable if and only if an appropriate persistent excitation condition, (12), is satisfied. As such, the design of a Kalman filter, with globally asymptotically stable error dynamics, follows naturally. Considering additive system disturbances and sensor noise, the system dynamics are given by

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{n}(t) \end{cases},$$

where $\mathbf{w}(t) \in \mathbb{R}^7$ is zero-mean white Gaussian noise, with $E[\mathbf{w}(t) \mathbf{w}^T(t - \tau)] = \mathbf{\Xi} \delta(\tau)$, $\mathbf{\Xi} \succ \mathbf{0}$, $\mathbf{n}(t) \in \mathbb{R}^3$ is zero-mean white Gaussian noise, with $E[\mathbf{n}(t) \mathbf{n}^T(t - \tau)] = \mathbf{\Theta} \delta(\tau)$, $\mathbf{\Theta} \succ \mathbf{0}$, and $E[\mathbf{w}(t) \mathbf{n}^T(t - \tau)] = \mathbf{0}$. It is important to stress, however, that it is not possible to conclude that this is an optimal solution, as the actual system disturbances and sensor noise may not be additive. Nevertheless, the nominal filter error dynamics are globally asymptotically stable if the LTV system is uniformly completely observable [16]. The design of the Kalman filter is well known and therefore it is omitted.

IV. NAVIGATION FILTER DESIGN

This section presents a solution for navigation based on direction measurements similar to the solution for source localization proposed in Section III. In order to derive an augmented linear time-varying system for navigation based on direction readings, define the system states as

$$\begin{cases} \mathbf{x}_1(t) = \mathbf{p}(t) \\ \mathbf{x}_2(t) = \mathbf{v}_c(t) \\ x_3(t) = \|\mathbf{r}(t)\| \end{cases}.$$

From (1) it follows that $\mathbf{x}_1(t) + x_3(t)\mathbf{d}(t) = \mathbf{s}(t)$ for all t . Let $\mathbf{x}(t) = [\mathbf{x}_1^T(t) \mathbf{x}_2^T(t) x_3(t)]^T \in \mathbb{R}^7$. Then, the system dynamics are given by the LTV system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}, \quad (16)$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{d}^T(t) & 0 \end{bmatrix} \in \mathbb{R}^{7 \times 7},$$

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{d}^T(t) & \mathbf{d}^T(t) \end{bmatrix} \in \mathbb{R}^{7 \times 6},$$

$$\mathbf{C}(t) = [\mathbf{I} \ \mathbf{0} \ \mathbf{d}(t)] \in \mathbb{R}^{3 \times 7}, \text{ and } \mathbf{u}(t) = \begin{bmatrix} \mathbf{v}_r(t) \\ \mathbf{v}_s(t) \end{bmatrix} \in \mathbb{R}^6.$$

In order to characterize the observability of the LTV system (16), consider the Lyapunov state transformation

$$\mathbf{z}(t) = \text{diag}(\mathbf{I}, \mathbf{I}, -1) \mathbf{x}(t).$$

The new system dynamics read as

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}(t)\mathbf{z}(t) + \text{diag}(\mathbf{I}, \mathbf{I}, -1)\mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{z}(t) \end{cases}.$$

Notice that the new system matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are those of the LTV system (3). This immediately allows to characterize the observability of the LTV system (16) with the following two theorems, as both systems are related by a Lyapunov transformation [17].

Theorem 4: The LTV system (16) is observable on $\mathcal{I} := [t_0, t_f]$ if and only if the unit vector $\mathbf{d}(t)$ is not constant on \mathcal{I} or, equivalently, (4) is true.

Theorem 5: The LTV system (16) is uniformly completely observable on $\mathcal{I} = [t_0, t_f]$ if and only if (12) holds.

It remains to see that, as in the solution for source localization, the initial condition of the LTV system, uniquely determined under the observability condition expressed in the previous theorems, matches the initial condition of the original system. This is shown in the following theorem.

Theorem 6: Under the conditions of Theorem 4, the initial condition of the LTV (16) corresponds to the initial condition of the original nonlinear system, i.e.,

$$\begin{cases} \mathbf{x}_1(t_0) = \mathbf{p}(t_0) \\ \mathbf{x}_2(t_0) = \mathbf{v}_c(t_0) \\ x_3(t_0) = \|\mathbf{r}(t_0)\| \end{cases}. \quad (17)$$

Proof: Under the terms of Theorem 4, the initial condition of the LTV system (16) is uniquely determined by the corresponding system output and input. The proof follows by showing that (17) explains the system output. The output of the LTV system (16) is given by

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{x}_1(t_0) + (t - t_0)\mathbf{x}_2(t_0) + \int_{t_0}^t \mathbf{v}_r(\tau) d\tau + x_3(t_0) \mathbf{d}(t) \\ &\quad + \int_{t_0}^t [\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(t_0)]^T \mathbf{d}(\tau) d\tau \mathbf{d}(t) \\ &= \mathbf{s}(t) \end{aligned} \quad (18)$$

for all $t \in \mathcal{I}$, $\mathcal{I} = [t_0, t_f]$. Substituting (17) in (18) gives

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{p}(t_0) + (t - t_0)\mathbf{v}_c(t_0) + \int_{t_0}^t \mathbf{v}_r(\tau) d\tau + \|\mathbf{r}(t_0)\| \mathbf{d}(t) \\ &\quad + \int_{t_0}^t [\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(t_0)]^T \mathbf{d}(\tau) d\tau \mathbf{d}(t) \end{aligned} \quad (19)$$

It remains only to show that (19) is equal to $\mathbf{s}(t)$ for all $t \in \mathcal{I}$. Substituting $t = t_0$ in (19) yields

$$\mathbf{y}(t_0) = \mathbf{p}(t_0) + \|\mathbf{r}(t_0)\| \mathbf{d}(t_0) = \mathbf{p}(t_0) + \mathbf{r}(t_0) = \mathbf{s}(t_0).$$

The time derivative of (19) is given by

$$\begin{aligned} \dot{\mathbf{y}}(t) &= \|\mathbf{r}(t_0)\| \dot{\mathbf{d}}(t) \\ &\quad + \int_{t_0}^t [\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(t_0)]^T \mathbf{d}(\tau) d\tau \dot{\mathbf{d}}(t) \\ &\quad + \mathbf{v}_r(t) + \mathbf{v}_c(t) \\ &\quad + [\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t_0)]^T \mathbf{d}(t) \mathbf{d}(t). \end{aligned} \quad (20)$$

As $\mathbf{v}_c(t)$ is constant, it is possible to rewrite (20) as

$$\begin{aligned} \dot{\mathbf{y}}(t) &= \|\mathbf{r}(t_0)\| \dot{\mathbf{d}}(t) \\ &\quad + \int_{t_0}^t [\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(\tau)]^T \mathbf{d}(\tau) d\tau \dot{\mathbf{d}}(t) \\ &\quad + \mathbf{v}_r(t) + \mathbf{v}_c(t) \\ &\quad + [\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)]^T \mathbf{d}(t) \mathbf{d}(t). \end{aligned} \quad (21)$$

The derivative of $\|\mathbf{r}(t)\|$ given by

$$\frac{d}{dt} \|\mathbf{r}(t)\| = [\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)]^T \mathbf{d}(t),$$

which allows to write

$$\|\mathbf{r}(t)\| = \|\mathbf{r}(t_0)\| + \int_{t_0}^t [\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(\tau)]^T \mathbf{d}(\tau) d\tau. \quad (22)$$

On the other hand, the time derivative of (1) is given by

$$\begin{aligned} \dot{\mathbf{d}}(t) &= \frac{\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)}{\|\mathbf{r}(t)\|} \\ &\quad - \frac{[\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)]^T \mathbf{d}(t)}{\|\mathbf{r}(t)\|} \mathbf{d}(t). \end{aligned} \quad (23)$$

Substituting (22) and (23) in (21) gives $\dot{\mathbf{y}}(t) = \mathbf{v}_s(t)$. This concludes the proof, as with $\mathbf{y}(t_0) = \mathbf{s}(t_0)$ and $\dot{\mathbf{y}}(t) = \mathbf{v}_s(t) = \dot{\mathbf{s}}(t)$ it must be $\mathbf{y}(t) = \mathbf{s}(t)$ for all $t \in \mathcal{I}$ and therefore (17) is true. ■

The design of a Kalman filter with globally asymptotically stable error dynamics for navigation based on direction measurements follows naturally as in Section III-C.

V. SIMULATION RESULTS

This sections presents realistic simulation results in order to evaluate the performance achieved with the proposed solutions. Due to the lack of space, results are only shown for the source localization problem. However, tests revealed that similar performances are achieved for the navigation problem based on direction measurements.

In the simulations, the source and the agent trajectories are those depicted in Fig. 1. Clearly, the persistent excitation condition (12) is satisfied, which allows to apply the solutions proposed in the paper. The drift velocity of the source was set to $\mathbf{v}_s(t) = [1 \ 0 \ 0]^T$ (m/s), while the drift velocity of the agent was set to $\mathbf{v}_c(t) = [-0.5 \ 0 \ 0]^T$ (m/s), which gives $\mathbf{v}_{sa}(t) [1.5 \ 0 \ 0]^T$ (m/s) for the relative drift velocity.

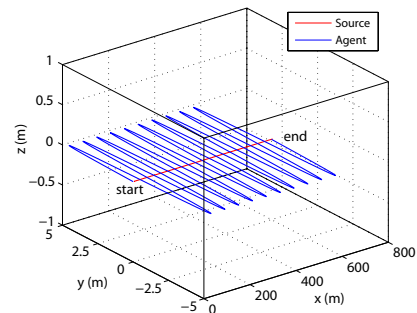


Fig. 1. Trajectory described by the agent and the source

Noise was considered for both the directions measurements and the relative velocity of the agent $\mathbf{v}_r(t)$. In particular, additive zero mean white Gaussian noise was considered

TABLE I

STANDARD DEVIATION OF THE STEADY-STATE ESTIMATION ERROR,
AVERAGED OVER 1000 RUNS OF THE SIMULATION

	Proposed solution	EKF
$\sigma_{\hat{x}_{11}}$ (m)	8.5×10^{-3}	19.6×10^{-3}
$\sigma_{\hat{x}_{12}}$ (m)	3.3×10^{-3}	11.7×10^{-3}
$\sigma_{\hat{x}_{13}}$ (m)	1.2×10^{-3}	9.2×10^{-3}
$\sigma_{\hat{x}_{21}}$ (m/s)	4.8×10^{-4}	5.6×10^{-4}
$\sigma_{\hat{x}_{22}}$ (m/s)	4.8×10^{-4}	5.8×10^{-4}
$\sigma_{\hat{x}_{23}}$ (m/s)	4.6×10^{-4}	5.8×10^{-4}
$\sigma_{\hat{x}_3}$ (m)	1.1×10^{-2}	not explicitly estimated

for $\mathbf{v}_r(t)$, with standard deviation of 0.01 m/s, while the direction readings were assumed perturbed by rotations about random vectors of an angle modeled by zero-mean white Gaussian noise, with standard deviation of 1° . The Kalman filter parameters were set to $\Xi = \text{diag}(10^{-2}\mathbf{I}, 10^{-5}\mathbf{I}, 10^{-2})$ and $\Theta = \mathbf{I}$. The initial estimates were all set to zero.

The evolution of the state estimates, along with the actual values, are depicted in Fig. 2. As it is possible to see, the initial transients due to the mismatch of the initial conditions quickly fade out, resulting in state estimates very close to the true values.

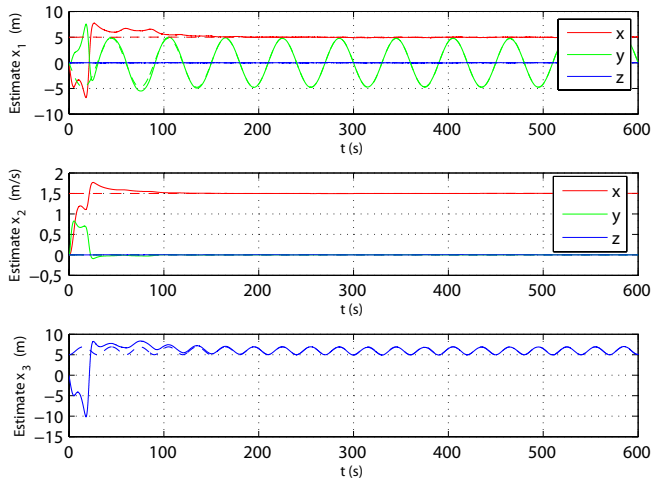


Fig. 2. Evolution of the state estimates (in solid lines) and the actual variables (in dashed lines)

In order to better evaluate the performance of the proposed solution, the Monte Carlo method was applied. The simulation was carried out 1000 times with different, randomly generated noise signals. The mean and standard deviation was computed for each simulation and averaged over the 1000 simulations. The results are depicted in Table I, where the results obtained with an Extended Kalman Filter with similar parameters are also presented. As the initial estimate for the source location cannot be set to zero with the EKF (in the linearization there appear terms divided by the norm of this estimate), the initial source position estimate was set to $[1 \ 0 \ 0]$ m. The convergence speed results slightly smaller. As it is possible to observe, the proposed solutions achieve similar performance to the EKF, while providing, at the same time, global asymptotic stability guarantees.

VI. CONCLUSIONS

This paper presented a set of globally asymptotically stable Kalman filters for the problems of source localization and navigation based on direction measurements to a single source. The observability of the systems was fully characterized, which allowed to conclude about the asymptotic stability of the Kalman filters. The observability conditions that were derived are directly related to the motion of the agent/vehicle and as such they are useful for motion planning and control. Simulations results were presented that illustrate the good performance achieved by the proposed solutions, which were also compared with the EKF, achieving similar performance but with global asymptotic stability guarantees. Future work includes the extension of the present work to the case where directions to multiple sources are available for navigation purposes.

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