Simultaneous Balancing and Model Reduction of Switched Linear Systems

Nima Monshizadeh, Harry L. Trentelman, M. K. Camlibel

Abstract— In this paper, first, balanced truncation of linear systems is revisited. Then, simultaneous balancing of multiple linear systems is investigated. Necessary and sufficient conditions are introduced to identify the case where simultaneous balancing is possible. The validity of these conditions is not limited to a certain type of balancing, and they are applicable for different types of balancing corresponding to different equations, like Lyapunov or Riccati equations. The results obtained are used for model order reduction of switched linear systems (SLS) by simultaneous balanced truncation. Finally, we give conditions under which global uniform exponential stability is preserved after simultaneous balanced truncation of the original switched linear system.

I. Introduction

Seeking for simpler descriptions of highly complex or large scale systems has resulted in the development of many different model reduction techniques. A simpler model provides a simpler description, better understanding, and easier analysis of the system. Two important issues in model reduction are obtaining a small error bound and preserving system properties like stability, passivity, or contractivity. To achieve this, extensive research and many approaches are reported in the literature regarding model reduction of linear time-invariant finite-dimensional systems (see [1]). One of the most well-known techniques is Lyapunov balanced model reduction, first introduced in [11], and later appearing in the control system literature in [10]. In this approach, first the system is transformed into a balanced form, and next a reduced order model is obtained by truncation. There are other types of balancing approaches available in the literature. Instances of those are stochastic and positive real balancing proposed in [2], and bounded real balancing, proposed in [12]. In addition, frequency weighted balancing has been developed to approximate the system over a range of frequencies. [3], [9], [14] and [15] provide different schemes for frequency weighted balancing. Another category of model reduction approaches is Krylov based methods which are based on moment matching. Among the pioneering works in this direction, we refer to [4], [5], and [7].

Despite the considerable research effort on model reduction of ordinary linear systems, developing methods for model reduction of more general classes of linear systems, such as hybrid and switched linear systems (SLS), has only been studied in very few papers up to now (e.g. [13]). A switched linear system, typically, involves switching between a number of linear systems. Hence, to apply balanced truncation techniques to a switched linear system, we need to search for a basis of the (common) state space such that the corresponding linear subsystems are in balanced form. A natural question that arises here is under what conditions such a basis exist. In this paper, necessary and sufficient conditions are derived for the existence of such basis. The results obtained are not limited to a certain type of balancing, and are applicable to different types such as Lyapunov, bounded real, and positive real balancing.

It may happen that some state components are difficult to reach and observe in some modes yet easy to reach and observe in other modes. In that case, deciding how to truncate the state variables and obtain a reduced order model is not trivial. A solution to this problem is proposed in this paper. By averaging the diagonal gramians of the individual modes in the balanced coordinates, a new diagonal matrix is obtained. This average gramian can be used to obtain a reduced order model. In the case of Lyapunov balancing, the average gramian assigns an overall degree of controllability and observability to each state component. In this way, one can decide which state components should be eliminated in order to obtain a reduced order model.

Another interesting issue is that if some information on the stability of the original SLS is available, how to ensure that the reduced order model, which is also in the form of an SLS, retains this stability. It is well-known that the existence of a common quadratic Lyapunov function (CQLF) is a sufficient condition for global uniform exponential stability of the switched linear system, see [8]. In this paper we will establish conditions under which the reduced order switched linear system inherits a CQLF form the original switched linear system, thus preserving global uniform exponential stability.

This paper is organized as follows. In Section 2, some preliminaries and basic materials needed in the rest of the paper are discussed. Balancing transformations for a single system are discussed in Section 3. Simultaneous balancing is the subject of Section 4. In Section 5, model reduction for switched linear systems is discussed. Finally, Section 6 is allocated to conclusions and a summary.

II. Preliminaries

Consider the finite dimensional, linear time-invariant system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \tag{1}$$

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P. O Box 800, 9700 AV Groningen, The Netherlands Phone:+31-50-3633999, Fax:+31-50-3633800 n.monshizadeh@rug.nl, h. l. trentelman@math. rug. nl, and m.k.camlibel@rug.nl

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, and $D \in \mathbb{R}^{l \times m}$. Assume that the system is internally stable, i.e., the matrix A is Hurwitz. We shortly denote this system by $H = (A, B, C, D)$.

In general, balancing the system H involves finding a state space transformation that simultaneously diagonalizes appropriately chosen positive definite matrices P and Q in a covariant and contravariant manner, respectively. This means that if we denote the state space transformation by T, then P transforms to TPT^T and Q transforms to $T^{-T}QT^{-1}$. Some important concepts of balancing are classical Lyapunov balancing, bounded real (BR) balancing, and positive real (PR) balancing. In Lyapunov balancing the matrices P and Q are the unique solutions of the Lyapunov equations associated with the system H, while in BR and PR balancing P and Q are the minimal real symmetric solutions of a pair of algebraic Riccati equations, see Table 1.

In the first part of the present paper, the exact balancing concept we use is not relevant, and we will just be dealing with finding, for a given pair of real symmetric, positive definite matrices P and Q, a nonsingular matrix T such that TPT^T and $T^{-T}QT^{-1}$ are diagonal, or diagonal and equal.

We will first introduce some basic terminology that will be used in the sequel.

Definition 1 Let $M \in \mathbb{C}^{n \times n}$ have n independent eigenvectors. Then, the nonsingular matrix $V \in \mathbb{C}^{n \times n}$ is called a *diagonalizing transformation* for M if VMV^{-1} is diagonal. In this case, we say V *diagonalizes* M.

Definition 2 Two diagonalizable matrices $X, Y \in \mathbb{C}^{n \times n}$ are said to be *simultaneously diagonalizable* if there exists a nonsingular matrix $V \in \mathbb{C}^{n \times n}$ such that VXV^{-1} and VYV^{-1} are both diagonal.

A necessary and sufficient condition for simultaneous diagonalizability of two given matrices is stated in the following lemma [6]

Lemma 3 *Let $X, Y \in \mathbb{C}^{n \times n}$ be diagonalizable matrices. Then X and Y are simultaneously diagonalizable if and only if they commute, i.e. $XY = YX$.*

Remark 4 The generalization of the above Lemma to the case of three or more matrices is straightforward. In fact, a finite set of matrices is simultaneously diagonalizable if and only if each pair in the set commutes.

Let $P, Q > 0$ be positive definite real symmetric $n \times n$ matrices. Then, the concept of essentially-balancing and balancing transformations are defined as follows.

Definition 5 Let $T \in \mathbb{R}^{n \times n}$ be nonsingular. We call T an *essentially-balancing* transformation for (P, Q) if TPT^T and $T^{-T}QT^{-1}$ are diagonal. In this case, we say T *essentially balances* (P, Q).

Definition 6 Let $T \in \mathbb{R}^{n \times n}$ be nonsingular. We call T a *balancing transformation* for (P, Q) if $TPT^T = T^{-T}QT^{-1} = \Sigma$, where Σ is a diagonal matrix. In this case, we say T *balances* (P, Q).

Remark 7 It is well-known, see for example [16], that for any pair of real symmetric positive definite matrices (P, Q) there exists a balancing transformation. It is clear that the diagonal elements of the matrix Σ in Definition 6 coincide with the square roots of the eigenvalues of PQ. In the case of Lyapunov balancing, where P and Q correspond to the reachability and observability gramians, the diagonal elements of the corresponding Σ are the nonzero Hankel Singular Values (HSV) of the system. Similarly, in the case of bounded real and positive real balancing, the diagonal elements of Σ are the nonzero bounded real and positive real characteristic values, respectively.

III. BALANCING TRANSFORMATIONS FOR A PAIR OF POSITIVE DEFINITE MATRICES

Let P and Q be two real symmetric positive definite matrices. Throughout this paper, it will be a standing assumption that the eigenvalues of PQ are all distinct. This assumption will simplify the statement and proofs of the results in this paper considerably. The generalization of our results to the case that the eigenvalues of PQ are not necessarily distinct will be treated in a future, full version of this paper. Now, let $\tilde{T}$ be a balancing transformation for the pair (P, Q), and denote the corresponding diagonal matrix Σ by $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_i \neq \sigma_j$ $(i \neq j)$. We note that the diagonal elements σ_i are not necessarily ordered in a decreasing manner. Clearly we have

$$\tilde{T}PQ\tilde{T}^{-1} = \Sigma^2.$$

Hence, the columns of $\tilde{T}^{-1}$ are the eigenvectors of PQ corresponding to the distinct eigenvalues σ_i^2, $i = 1, 2, \ldots, n$. Since the eigenvalues of PQ are real, there exists $V \in \mathbb{R}^{n \times n}$ that diagonalizes PQ. Now, it is easy to observe that $\tilde{T}$ can be written as $\tilde{T} = I_P D^{-1} V$ where $D \in \mathbb{R}^{n \times n}$ is a nonsingular diagonal matrix and I_P is a permutation matrix.

The following lemma indicates that there is a one-to-one correspondence between diagonalizing transformations, and essentially-balancing transformations.

Lemma 8 *Let P and Q be real symmetric positive definite matrices. Assume that the eigenvalues of PQ are all distinct. The matrix $V \in \mathbb{R}^{n \times n}$ is an essentially-balancing transformation for (P, Q) if and only if it is a diagonalizing transformation for PQ.*

Proof: First, assume that V is an essentially-balancing transformation for (P, Q). Then, by definition, VPV^T and $V^{-T}QV^{-1}$ are diagonal. Consequently, the product $VPV^T V^{-T}QV^{-1} = VPQV^{-1}$ is diagonal. Hence, V is a diagonalizing transformation for PQ.

Conversely, suppose V is a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2. Then, there

TABLE I

COMMON TYPES OF BALANCING

Type	Equations
Lyapunov	$AP + PA^\mathsf{T} + BB^\mathsf{T} = 0$ $A^\mathsf{T}Q + QA + C^\mathsf{T}C = 0$
Bounded Real	$AP + PA^\mathsf{T} + BB^\mathsf{T} + (PC^\mathsf{T} + BD^\mathsf{T})(I - DD^\mathsf{T})^{-1}(PC^\mathsf{T} + BD^\mathsf{T})^\mathsf{T} = 0$ $A^\mathsf{T}Q + QA + C^\mathsf{T}C + (QB + C^\mathsf{T}D)(I - D^\mathsf{T}D)^{-1}(QB + C^\mathsf{T}D)^\mathsf{T} = 0$
Positive Real	$AP + PA^\mathsf{T} + (PC^\mathsf{T} - B)(D + D^\mathsf{T})^{-1}(PC^\mathsf{T} - B)^\mathsf{T} = 0$ $A^\mathsf{T}Q + QA + (QB - C^\mathsf{T})(D^\mathsf{T} + D)^{-1}(QB - C^\mathsf{T})^\mathsf{T} = 0$

exists a nonsingular diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that $T = D^{-1}V$ balances (P, Q) with corresponding diagonal matrix Σ. Hence, by Definition 6, we have

$$D^{-1}VPV^\mathsf{T}D^{-1} = \Sigma = DV^{-\mathsf{T}}QV^{-1}D.$$

Consequently, $VPV^\mathsf{T} = D\Sigma D$ and $V^{-\mathsf{T}}QV^{-1} = D^{-1}\Sigma D^{-1}$ are diagonal matrices. This implies that V is an essentially-balancing transformation for (P, Q). $\qquad\square$

Based on the previous discussion and Lemma 8, the following theorem characterizes balancing transformations for a given pair of positive definite matrices (P, Q).

Theorem 9 *Let P and Q be real symmetric positive definite matrices. Assume that the eigenvalues of PQ are all distinct. Let $V \in \mathbb{R}^{n \times n}$ be a diagonalizing transformation for PQ. Then T is a balancing transformation for (P, Q) if $T = D^{-1}V$ where $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix satisfying*

$$D^4 = (VPV^\mathsf{T})(VQ^{-1}V^\mathsf{T}) \tag{2}$$

Moreover, if T is a balancing transformation for (P, Q), then any balancing transformation $\tilde{T}$ can be written as $\tilde{T} = I_P S T$ where S is a sign matrix (i.e. diagonal matrix with $+1$ or -1 on the diagonal), and I_P is a permutation matrix.

Proof: Assume V is a diagonalizing transformation for PQ. Let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix satisfying (2). By (2), we have

$$D^{-2}VPV^\mathsf{T} = D^2 V^{-\mathsf{T}}QV^{-1}. \tag{3}$$

According to Lemma 8, V is also an essentially-balancing transformation; hence, VPV^T and $V^{-\mathsf{T}}QV^{-1}$ are diagonal. Therefore, (3) can be written as

$$D^{-1}VPV^\mathsf{T}D^{-1} = DV^{-\mathsf{T}}QV^{-1}D. \tag{4}$$

Since both sides of (4) are diagonal, $T = D^{-1}V$ is a balancing transformation for (P, Q).

Now, assume T and $\tilde{T}$ are two balancing transformations for (P, Q). Then, we have $TPQT^{-1} = \Sigma^2$ and $\tilde{T}PQ\tilde{T}^{-1} = I_P \Sigma^2 I_P^\mathsf{T}$ for some permutation matrix I_P. Clearly, the columns of T^{-1} and $\tilde{T}^{-1}$ are the eigenvectors of PQ. Hence, we have $\tilde{T} = I_P \tilde{D}T$ for some nonsingular diagonal matrix $\tilde{D}$. Since $\tilde{T}$ balances (P, Q), we have $\tilde{T}P\tilde{T}^\mathsf{T} = I_P \Sigma I_P^\mathsf{T}$. Hence,

$$I_P \tilde{D}TPT^\mathsf{T} \tilde{D}I_P^\mathsf{T} = I_P \Sigma I_P^\mathsf{T}. \tag{5}$$

Since T balances (P, Q), we have $TPT^\mathsf{T} = \Sigma$. Therefore, (5) implies that $\tilde{D}^2$ is the identity matrix. Consequently, $\tilde{T}$ can be written as $\tilde{T} = I_P S T$. $\qquad\square$

The above theorem provides a straightforward method for computing balancing transformations. Given positive definite real symmetric matrices P and Q, we compute, firstly, a diagonalizing transformation for PQ. Obviously, this transformation can be obtained directly from the eigenvectors of PQ. Then, as stated in the theorem, a balancing transformation can be obtained by scaling the diagonalizing transformation, and the scaling matrix can be taken as any nonsingular real diagonal matrix satisfying (2). Note that computation of D in (2) is simple and merely requires the multiplication of two diagonal matrices VPV^T and $VQ^{-1}V^\mathsf{T}$.

IV. SIMULTANEOUS BALANCING FOR MULTIPLE PAIRS OF POSITIVE DEFINITE MATRICES

In this section, we will study the question under what conditions multiple pairs of positive definite real symmetric matrices can be simultaneously balanced by one and the same transformation. We will start off by considering the problem for a *pair* of positive definite matrices. Let (P_1, Q_1) and (P_2, Q_2) be two given pairs of positive definite real symmetric matrices. The following theorem gives necessary and sufficient conditions for the existence of a transformation T that simultaneously balances (P_1, Q_1) and (P_2, Q_2). We again assume that the eigenvalues of both P_1Q_1 and P_2Q_2 are distinct.

Theorem 10 *Let (P_1, Q_1) and (P_2, Q_2) be two pairs of real symmetric positive definite matrices. Assume that the eigenvalues of P_1Q_1 and P_2Q_2 are both distinct. Then, there exists a transformation T that balances both (P_1, Q_1) and (P_2, Q_2) if and only if the following two conditions hold:*

1) P_1Q_1 and P_2Q_2 commute,
2) $P_1Q_2 = P_2Q_1$.

Proof: Assume T is a balancing transformation for both (P_1, Q_1) and (P_2, Q_2). Clearly, T is a diagonalizing transformation for both P_1Q_1 and P_2Q_2. Hence, by Lemma 3, P_1Q_1 and P_2Q_2 commute. In addition, by Definition 6, we have

$$\begin{aligned} TP_1T^\mathsf{T} &= T^{-\mathsf{T}}Q_1T^{-1} \\ TP_2T^\mathsf{T} &= T^{-\mathsf{T}}Q_2T^{-1}. \end{aligned} \tag{6}$$

Consequently, $TP_1Q_2T^{-1} = TP_2Q_1T^{-1}$ which yields $P_1Q_2 = P_2Q_1$.

To prove the converse, assume that the conditions *i)* and *ii)* hold. Since P_1Q_1 and P_2Q_2 commute, there exists a nonsingular matrix V that simultaneously diagonalizes P_1Q_1 and P_2Q_2. Clearly,

$$VP_1Q_2V^{-1} = VP_2Q_1V^{-1}$$

which can be rewritten as

$$(VP_1V^{\mathrm{T}})(V^{-\mathrm{T}}Q_2V^{-1}) = (VP_2V^{\mathrm{T}})(V^{-\mathrm{T}}Q_1V^{-1}) \quad (7)$$

By Lemma 8, V is also an essentially-balancing transformation for both (P_1, Q_1) and (P_2, Q_2). Therefore, all four terms in (7) are diagonal, and can be permuted. Consequently, (7) results in

$$(VP_1V^{\mathrm{T}})(VQ_1^{-1}V^{\mathrm{T}}) = (VP_2V^{\mathrm{T}})(VQ_2^{-1}V^{\mathrm{T}}) \quad (8)$$

Now, take D to be a nonsingular real diagonal matrix such that

$$D^4 = (VP_1V^{\mathrm{T}})(VQ_1^{-1}V^{\mathrm{T}}).$$

Then, by Theorem 9, $T = D^{-1}V$ simultaneously balances (P_1, Q_1) and (P_2, Q_2). $\qquad\square$

For the sake of simplicity, the above result and conditions are stated for pairs of matrices (P_1, Q_1) and (P_2, Q_2). The generalization of the results to k pairs of positive definite matrices, with $k \geq 2$ is straightforward, and is stated in the following corollary.

Corollary 11 *Let (P_1, Q_1), (P_2, Q_2), ..., (P_k, Q_k) be k pairs of positive definite real symmetric matrices. Assume for all $i = 1, 2, ..., k$ the eigenvalues of P_iQ_i are distinct. Then, there exists a transformation T that simultaneously balances $(P_1, Q_1), (P_2, Q_2), ..., (P_k, Q_k)$ if and only if the following conditions hold:*
i) P_iQ_i and P_jQ_j commute for all $i, j = 1, 2, ..., k$.
ii) $P_iQ_j = P_jQ_i$ for all $i, j = 1, 2, ..., k$.

Again, we note that this result can be generalized to the case that the product matrices P_iQ_i have repeated eigenvalues. The details will be worked out in a future, full version of this paper.

V. MODEL REDUCTION OF SWITCHED LINEAR SYSTEMS

A. Model reduction by simultaneous balancing

We will now apply our previous results to model reduction by balanced truncation of switched linear systems. In fact, in the context of model reduction, the matrices P and Q can be taken to be the solutions of the Lyapunov equations or the minimal solutions of Riccati equations in Table 1. Consequently, the results stated in the previous sections cover different types of balancing of a single linear system, and, moreover, indicate the possibility of simultaneous balancing for multiple linear systems. Simultaneous balancing, if possible, provides a straightforward approach for model reduction of some hybrid systems. Note that, except from having the

same state space dimension, no assumption is needed regarding the relation of the individual systems. In this section, we will apply our results to classical Lyapunov balanced truncation of switched linear systems. The application to bounded real or positive real balanced truncation will be worked out in a future publication.

A typical SLS is described by (see [8]):

$$\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad (9)$$

where σ is a piecewise constant function of time, t, taking its value from the index set $K = \{1, 2, ..., k\}$, and $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{m \times n}$, $D_i \in \mathbb{R}^{m \times m}$ for all $i \in K$. Let $H_i = (A_i, B_i, C_i, D_i)$ denote the ith mode of the given SLS. Assume that H_i is internally stable, controllable, and observable for every i. Let P_i and Q_i be the reachability and observability gramians of H_i, respectively. Now, if the conditions of Corollary 11 hold with respect to the pairs of gramians (P_1, Q_1), (P_2, Q_2), ..., (P_k, Q_k), then there exists a state space transformation T that simultaneously balances all k modes of the given SLS. Consequently, by applying T to the individual modes of (9) and truncating, reduced order models can be obtained.

As mentioned in the introduction, it may occur that some states are relatively difficult to reach and observe in some subsystems yet easy to reach and observe in other subsystems. In order to measure the degree of reachability and observability of each of the state components of the SLS, we propose to take the *average over all subsystems* of the corresponding Hankel singular values. Let Σ_1, Σ_2, ..., Σ_k denote the gramians of the subsystems of (9) in the balanced basis. Then, we define the average gramian, denoted by Σ_{av}, as

$$\Sigma_{av} = \frac{1}{k}(\Sigma_1 + \Sigma_2 + ... + \Sigma_k) \quad (10)$$

Now, the diagonal matrix Σ_{av} indicates the important and negligible states, and can be used to obtain a reduced order model. In fact, the ith diagonal element of Σ_{av} assigns an overall degree of controllability and observability to the ith state component of the balanced representation. Hence, a reduced model of order r can be obtained by maintaining the state components corresponding to the largest r elements of Σ_{av} and eliminating the remaining $n - r$ ones.

The reduced model can also be represented in the form of an SLS as

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}_\sigma x + \bar{B}_\sigma u \\ \bar{y} &= \bar{C}_\sigma x + \bar{D}_\sigma u \end{aligned} \quad (11)$$

Let $\bar{H}_i = (\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i)$ denote the ith mode of the reduced order model. Clearly, the reachability and observability gramians are diagonal in the balanced basis, and internal stability of H_i is preserved in $\bar{H}_i$ for all $i = 1, 2, ..., k$.

An infinity norm error bound for Lyapunov balanced truncation of any of the subsystems is known to be twice the sum of the neglected Hankel Singular values (i.e. those corresponding to the eliminated state components)([1], p. 212). Hence, the definition of Σ_{av} in (10) is well justifiable based on the model reduction errors of the different

subsystems. In fact, eliminating the states based on Σ_{av} corresponds to minimizing the sum of the error bounds involved in approximating the individual subsystems. Of course, it remains an open problem how to measure the error between the original overall SLS and its reduced r^{th} order SLS model.

Remark 12 If the subsystems of the given SLS are not of equal importance or some information regarding the switching signal is available, the overall gramian can be defined as a *weighted* average of the gramians of individual subsystems in balanced coordinates. Then, one can take into account the importance of the different subsystems by adjusting the weighting coefficients.

Remark 13 Although the above results are stated in terms of classical Lyapunov balancing, they can be adopted for different types of balancing mentioned in Table 1 as well. In particular, in positive real and bounded real balancing, (P, Q) corresponds to the minimal solutions of the Riccati equations in Table 1 (see [1]). Clearly, depending on the type of balancing, H_i and $\bar{H}_i$ will then share the property of contractivity or passivity.

B. Stability of the reduced order SLS model under arbitrary switching

As already observed in the previous subsection, simultaneous balanced truncation of the individual modes of a switched linear system yields a reduced order switched linear system whose individual modes are internally stable. Of course, this does not mean that the SLS itself is stable. In the present subsection we will find conditions under which simultaneous balanced truncation preserves the stability of the SLS. The concept of stability that we will use here is that of *global uniform exponential stability*. We call the SLS given by (9) globally uniformly exponentially stable if there exist positive constants K and α such that the solution $x(t)$ of $\dot{x} = A_\sigma x$ for any initial state $x(0)$ and any switching signal σ satisfies $\|x(t)\| \leq Ke^{-\alpha t}\|x(0)\|$ for all $t \geq 0$ (see [8]). A sufficient condition for global uniform exponential stability of an SLS is that the state matrices of the individual modes share a common quadratic Lyapunov function (CQLF)[8]. Assuming that the state matrices of the modes of the given SLS enjoy this property, we seek for conditions under which this property is preserved in the reduced order SLS. This leads us to the following theorem.

Theorem 14 *Consider the switched linear system* (9) *with subsystems* $H_i = (A_i, B_i, C_i, D_i)$, $i = 1, 2, \ldots, k$. *Assume that there exists* $X > 0$ *such that* $A_i^T X + X A_i < 0$ *for all* $i = 1, 2, ..., k$. *Let* P_i *and* Q_i *be the reachability and observability Gramians, respectively, of the* i^{th} *subsystem* H_i. *Assume that for all* $i = 1, 2, ..., k$, *the eigenvalues of* $P_i Q_i$ *are distinct. Then there exists a state space transformation that simultaneously balances all subsystems* H_i *for* $i = 1, 2, ..., k$, *and, moreover, for each positive integer* $r \leq n$ *the* r^{th} *order truncated SLS given by* (11) *is globally*

uniformly exponentially stable, if the following conditions hold:

i) $P_i Q_i$ and $P_j Q_j$ commute for all $i, j = 1, 2, ..., k$.
ii) $P_i Q_j = P_j Q_i$ for all $i, j = 1, 2, ..., k$.
iii) $X P_i Q_i = Q_i P_i X$ for all $i = 1, 2, ..., k$.

Proof: Based on Corollary 11, simultaneous balancing is possible upon satisfaction of the first two conditions. By the third condition we have

$$X^{\frac{1}{2}} P_i Q_i X^{\frac{-1}{2}} = X^{\frac{-1}{2}} Q_i P_i X^{\frac{1}{2}}.$$

Hence, $X^{\frac{1}{2}} P_i Q_i X^{\frac{-1}{2}}$ is a symmetric matrix. In addition, the first condition implies that $X^{\frac{1}{2}} P_i Q_i X^{\frac{-1}{2}}$ and $X^{\frac{1}{2}} P_j Q_j X^{\frac{-1}{2}}$ commute for all $i, j = 1, 2, ..., k$. Therefore, there exists an orthogonal matrix U which diagonalizes $X^{\frac{1}{2}} P_i Q_i X^{\frac{-1}{2}}$, for all $i = 1, 2, ..., k$ (see [6], p. 103). Hence, $U X^{\frac{1}{2}}$ is a diagonalizing transformation for $P_i Q_i$, $i = 1, 2, ..., k$. Consequently, a simultaneous balancing transformation T can be obtained as $T = D^{-1} U X^{\frac{1}{2}}$ for some nonsingular real diagonal matrix D. Applying T to the individual subsystems of the given SLS, the state matrices in the new coordinates are given by

$$\tilde{A}_i = D^{-1} U X^{\frac{1}{2}} A_i X^{\frac{-1}{2}} U^{\mathsf{T}} D \qquad (12)$$

By our assumption regarding the CQLF, we have $A_i^T X + X A_i < 0$ for all $i = 1, 2, ..., k$. Hence, for all $i = 1, 2, ..., k$, we have

$$DU X^{\frac{-1}{2}} (A_i^T X + X A_i) X^{\frac{-1}{2}} U^{\mathsf{T}} D < 0$$

which yields

$$DU X^{\frac{-1}{2}} A_i^T X^{\frac{1}{2}} U^{\mathsf{T}} D + DU X^{\frac{1}{2}} A_i X^{\frac{-1}{2}} U^{\mathsf{T}} D < 0.$$

This can be rewritten as

$$DU X^{\frac{-1}{2}} A_i^T X^{\frac{1}{2}} U^{\mathsf{T}} D^{-1} D^2 + D^2 D^{-1} U X^{\frac{1}{2}} A_i X^{\frac{-1}{2}} U^{\mathsf{T}} D < 0.$$

which, based on (12), is simplified to

$$\tilde{A}_i^T D^2 + D^2 \tilde{A}_i < 0. \qquad (13)$$

Since D^2 is a positive definite diagonal matrix, for any $r \leq n$ the state matrices of the reduced model obtained by balanced truncation of the individual subsystems, share a CQLF. Consequently, the SLS r^{th} order reduced model is globally uniformly exponentially stable. $\square$

Note that the first two conditions in Theorem 14 implies the possibility of simultaneous balancing whereas the third condition guarantees the existence of a CQLF for the reduced order model.

VI. Conclusion

In this paper, a generalization of the balanced truncation scheme is investigated for model reduction of subclasses of hybrid systems, and in particular switched linear systems. A straightforward formula to compute a balancing transformation for a single linear system is suggested. Clearly, making multiple linear systems balanced in general is not possible

with a single state space transformation. Hence, necessary and sufficient conditions for simultaneous balancing of multiple linear systems are derived. These conditions do not depend on the particular type of balancing, and are in terms of commutativity of product of positive definite matrices. The obtained results are applied to balanced truncation of switched linear systems. Obviously, after truncation, each individual subsystem of the reduced SLS model inherits the property of internal stability, contractivity or passivity depending on the type of balancing. the property of internal stability. a desired property such as stability, contractivity, or passivity. Furthermore, we have considered the case where the stability of a given SLS is verified by the existence of a CQLF. Consequently, the possibility for preserving this property in the reduced order model is investigated. As we observed, this is possible by imposing an additional constraint on those already obtained for simultaneous balancing. It is, however, clear that stability of the reduced order SLS is a more delicate issue. Starting from the assumption that the original SLS has a common quadratic Lyapunov function, we establish a condition under which global uniform exponential stability of the SLS is preserved after balanced truncation. Recall that the results reported in this paper are based on the standing assumption that the eigenvalues of the product of the gramians are distinct. However, it can be shown that similar results can be obtained for the case where the eigenvalues are not distinct. These results will be reported in a future publication.

REFERENCES

[1] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM, Philadelphia,Pa,USA, 2005.

[2] U. B. Desai and D. Pal. A transformation approach to stochastic model reduction. *IEEE Transactions on Automatic Control*, AC-29, 1984.

[3] D. Enns. Model reduction with balanced realizations: An error bound and a frequency weighted generalization. In *Proceedings of the 23rd IEEE conference on Decision and Control*, Las Vegas, Nevada, USA, 1984.

[4] P. Feldman and R. W. Freund. Efficient linear circuit analysis by Pade approximation via a Lanczos method. *IEEE Transactions on Computer-Aided design*, 14:639–649, 1995.

[5] E. J. Grimme. *Krylov Projection Methods for Model Reduction*. PhD thesis, ECE Dept., U. of Illinois, Urbana-Champaign, 1997.

[6] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, NY, USA, 1988.

[7] I. M. Jaimoukha and E. M. Kasenally. Implicitly restarted Krylov subspace methods for stable partial realizations. *SIAM Journal on Matrix Analysis and Applications*, 18:633–652, 1997.

[8] D. Liberzon. *Switching in Systems and Control*. Birkhauser, Boston, 2003.

[9] C.-A. Lin and T.-Y. Chiu. Model reduction via frequency weighted balanced realization. *Control Theory and Advanced Technology*, 8:341–351, 1992.

[10] B. C. Moore. Principle component analysis in linear system: controllability, observability and model reduction. *IEEE Transactions on Automatic Control*, AC-26, 1981.

[11] C. T. Mullis and R. A. Roberts. Synthesis of minimum roundoff noise fixed point digital filters. *IEEE Transactions on Circuits and Systems*, CAS-23:551–562, 1976.

[12] P. C. Opdenacker and E. A. Jonckheere. A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds. *IEEE Transactions on Circuits and Systems*, CAS-35:184–189, 1988.

[13] H. R. Shaker and R. Wisniewski. Switched systems reduction framework based on convex combination of generalized gramians. *Journal of Control Science and Engineering*, 2009(Article ID 710478):11 pages, 2009.

[14] G. Wang, V. Sreeram, and W. Q. Liu. A new frequency weighted balanced truncation method and an error bound. *IEEE Transactions on Automatic Control*, 44:1734–1737, 1999.

[15] K. Zhou. Frequency-Weighted L_∞ norm and optimal Hankel norm model reduction. *IEEE Transactions on Automatic Control*, 40:1687–1699, 1995.

[16] K. Zhou and J. C. Doyle. *Essentials of robust control*. Prentice Hall, New Jersey, 1998.