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Connecting Several Stability Criteria for iSS Networks and Their Application to a Network Computing Model

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***Abstract*— Employing the framework of integral input-to-state stability (iISS), this paper studies the problem of verifying stability of dynamical networks. The iISS we impose on the subsystems encompasses a wider variety of nonlinearities than input-to-state stability (ISS) which has been studied extensively in the literature. To go beyond the ISS results, this paper investigates a simultaneous small-gain criterion, a topological separation criterion and a spectral radius formula in view of necessity and suffiency for the stability of the iISS network, and they are related to each other. This work aims at unifying conceivable stability criteria into a single methodology in the iISS formulation which includes ISS as a special case. For illustration, this paper examines the dynamics of network computing for resource utilization. Based on a fluid flow model, the proposed iISS methodology provides us with qualitative and quantitative information on the computing speed, the communication overhead, the number of nodes and the interconnection structure for achieving successful workload distribution across multiple computers.**

I. Introduction

The control-theoretic study of large-scale dynamical systems (dynamical networks) has a long history [22], [20], and the topic has received substantial attention in resent years due to the increased complexity in size and integration of our targets. The concept of input-to-state stability (ISS) has accelerated the utilization of nonlinear gains in analysis and design of interconnected systems [23]. In particular, the tool referred to as the ISS small-gain theorem or the nonlinear small-gain theorem has become popular [16], [26]. Recently, the ISS small-gain theorem has been extended to large-scale systems allowing any number of subsystems and any interconnection structure, and small-gain-type conditions have become available in various forms[5], [6], [17], [18], [19]. The notion of integral input-to-state stability (iISS) accommodates systems which do not have finite ISS gain [24]. Covering iISS systems in the small-gain framework had been considered to be too hard until the recent breakthrough [8], [14]. In fact, an attempt to tackle iISS networks was made in [21], and it has demonstrated that a new technique is required for guaranteeing stability of networks involving non-ISS subsystems. Such a new technique developed in [14] for two subsystems has been extended to iISS networks in the cycle and the cactus graph structures [10]. The structural assumption has been removed in the most recent result [15].

This paper continues this line of developments for iISS systems in two directions. One is to pose several conditions similar to the ISS case and demonstrate their necessity or the sufficiency for the stability of iISS networks. This paper employs the dissipative characterization of subsystems to unify the treatment of iISS and ISS networks. This general formulation, however, has been hampering the extension of the existing techniques focusing on ISS subsystems [4]. This paper makes the full use of [14], [11], [15] as tools to accomplish the goals of this paper. The other direction is to relate these conditions to each other. It is stressed that most of the problems solved in this paper have not been answered even for ISS systems so far. For an illustration by example, this paper analyzes the dynamics of network computing which utilizes resources by workload distribution across multiple computers. In a form of grid computing, the distribution is supposed to be autonomous and require no additional physical infrastructure. Everyone can participate in the grid as long as protocol software is installed. Based on a fluid flow model, it is demonstrated that the presented theory gives qualitative and quantitative information about the adequate balance between individual computing powers, communication overhead and the size and structure of the network to maintain the stability of the workload distribution.

Notation: Let $\mathbb{R}_+$ denote the interval $[0, \infty)$ in the space of real numbers $\mathbb{R}$. A continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be positive definite and denoted by $\omega \in \mathcal{P}$ if it satisfies $\omega(0) = 0$ and $\omega(s) > 0$ holds for all $s > 0$. A function $\omega \in \mathcal{P}$ is said to be of class $\mathcal{K}$ and written as $\omega \in \mathcal{K}$ if it is strictly increasing. A function $\omega \in \mathcal{K}$ is of class $\mathcal{K}_\infty$ if $\lim_{s\to\infty} \omega(s) = \infty$. The symbol $\mathbf{Id}$ denotes the identity map. For a positive definite function $h : \mathbb{R}_+ \to \mathbb{R}_+$, we write $h \in \mathcal{O}(> L)$ with a non-negative real number L if there exists a positive real number $K > L$ such that $\limsup_{s\to 0+} h(s)/s^K < \infty$. We write $h \in \mathcal{O}(L)$ when $K = L$. Note that $\mathcal{O}(L) \subset \mathcal{O}(S)$ holds for $L > S$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. For vectors $a, b \in \mathbb{R}^n$ the relation $a \geq b$ is defined by $a_i \geq b_i$ for all $i = 1, \ldots, n$. The relations $>, \leq, <$ for vectors are defined in the same manner. The negation of $a \geq b$ is denoted by $a \ngeq b$ and this means that there exists an $i \in \{1, \ldots, n\}$ such that $a_i < b_i$. All the detailed proofs are omitted due to the space limitation. A sketch and a key lemma are included in the Appendix.

II. Network of iISS systems

Consider the dynamical network described by

$$\Sigma : \ \dot{x} = f(x, r), \tag{1}$$

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where $f = [f_1^T, \ldots, f_n^T]^T : \mathbb{R}^N \to \mathbb{R}^N$. The state vector of Σ is $x = [x_1^T, \ldots, x_n^T]^T \in \mathbb{R}^N$, where $N := \sum_{i=1}^n N_i$. The vector $r = [r_1^T, \ldots, r_n^T]^T \in \mathbb{R}^K$ represents the disturbance, where $K := \sum_{i=1}^n K_i$. Define the following set:

Definition 1: Given $\alpha_i \in \mathcal{K}$, $\sigma_{ij} \in \mathcal{K} \cup \{0\}$, $\kappa_i \in \mathcal{K}_\infty$, and positive integers n, N_i, K_i for $i, j = 1, 2, ..., n$, $j \neq i$, let $\mathcal{S}(n, N_*, K_*, \alpha_*, \sigma_{*,*}, \kappa_*)$ be the set of networks Σ consisting of subsystems Σ_i, $i = 1, 2, ..., n$, in the form of

$$\dot{x}_i = f_i(x_1, \ldots, x_n, r_i), \quad x_i \in \mathbb{R}^{N_i},\, r_i \in \mathbb{R}^{K_i}, \tag{2}$$

$$f_i(0, ..., 0, 0) = 0, \quad f_i \text{ is locally Lipschitz} \tag{3}$$

for which there exist positive definite and radially unbounded $\mathcal{C}^1$ functions V_i: $\mathbb{R}^{N_i} \to \mathbb{R}_+$ such that

$$\frac{\partial V_i}{\partial x_i} f_i \leq -\alpha_i(V_i(x_i)) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \sigma_{ij}(V_j(x_j)) + \kappa_i(|r_i|) \tag{4}$$

holds for all $x_j \in \mathbb{R}^{N_j}$ and $r_j \in \mathbb{R}^{K_j}$, $j = 1, 2, ..., n$.

The Lipschitzness imposed on f_i is only for guaranteeing the existence of a unique maximal solution of the network Σ. The network Σ is said to be 0-GAS if the equilibrium $x = 0$ is globally asymptotically stable for $r(t) \equiv 0$. The inequality (4) is called a dissipation inequality. For brevity, we write $\mathcal{S}$ instead of $\mathcal{S}(n, N_*, K_*, \alpha_*, \sigma_{*,*}, \kappa_*)$ in the rest of this paper. All the developments in this paper hold true even if N_i and K_i are not given a priori in defining $\mathcal{S}$. Indeed, stability criteria presented in this paper are independent of N_i's and K_i's. The purpose of allowing N_i and K_i to be prescribed in defining $\mathcal{S}$ is to demonstrate that the stability criteria are tight by proving their necessity for such a narrowly specified set $\mathcal{S}$.

The dissipation inequality (4) implies that each subsystem Σ_i with the inputs x_j, $j \neq i$ and r_i is iISS, and that V_i is an iISS Lyapunov function for the disconnected subsystem Σ_i [3]. Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the subsystem Σ_i is guaranteed to be ISS, and the function V_i is an ISS Lyapunov function [25]. By definition [24], an ISS system is iISS. An iISS system is 0-GAS. Note that the function V_i is qualified as an iISS Lyapunov function even when α_i is merely positive definite [3]. Nevertheless, this paper employs $\alpha_i \in \mathcal{K}$ to allow subsystems Σ_i to form loops. It is known that a loop (cycle) of subsystems Σ_i defined with the dissipation inequality (4) can be guaranteed to be 0-GAS only if $\alpha_i \in \mathcal{K}$ holds for all participating subsystems [9]. In this paper, we are interested in investigating criteria for 0-GAS, iISS and ISS of the network $\Sigma \in \mathcal{S}$. In order to demonstrate necessity of the stability criteria within the set $\mathcal{S}$, we assume the following throughout this paper:

Assumption 1: $\alpha_i \in \mathcal{O}(1)$ and $\sigma_{ij} \in \mathcal{O}(>0) \cup \{0\}$ hold for $i, j = 1, 2, ..., n$, $j \neq i$,

We define a directed graph G associated with the network Σ using the vertex set $\mathcal{V}(G)$ and the arc set $\mathcal{A}(G)$ as follows: Elements of $\mathcal{V}(G)$ are subsystems Σ_i, $i = 1, 2, ..., n$. Each element of $\mathcal{A}(G)$ is an ordered pair (i, j) which is directed away from the j-th vertex and directed toward the i-th vertex. The pair (i, j) is an element of $\mathcal{A}(G)$ if and only if $\sigma_{i,j} \neq 0$. Let $\mathcal{C}(G)$ denote the set of all directed cycles contained in the directed graph G. Let $\mathcal{P}(G)$ denote the set of all directed paths contained in the directed graph G. Given a directed path or a directed cycle U of length k, we employ the following notation:

$$|U| = k, \quad U = (u(1), u(2), ..., u(k), u(k+1)),$$

where $u(i)$'s listed in the above are "all" the vertices comprising U and they are listed in the "reversed" order of appearance. If U is a directed cycle, we have $u(1) = u(k+1)$. The starting vertex of the directed path U is $u(k+1)$, and the ending vertex is $u(1)$. Let $\mathcal{I}(G)$ denote the set of all isolated vertices contained in the directed graph G, and we write $|U| = 0$ for $U \in \mathcal{I}(G)$. Define $\mathcal{B}(G) = \mathcal{P}(G) \cup \mathcal{I}(G)$. In the rest of this paper, the term "directed" is omitted in referring to graphs.

III. Topological Separation: Necessity

Define a mapping $M_0 : \mathbb{R}_+^n \to \mathbb{R}^n$ by

$$M_0(\mathbf{s}) := -A(\mathbf{s}) + \Gamma(\mathbf{s}) \tag{5}$$

where A, Γ: $\mathbf{s} \in \mathbb{R}_+^n \mapsto \mathbf{z} \in \mathbb{R}_+^n$ are

$$\mathbf{z} = A(\mathbf{s}) = [\alpha_1(s_1),\, \alpha_2(s_2),\, \ldots, \alpha_n(s_n)]^T,$$

$$\mathbf{z} = \Gamma(\mathbf{s}) = \left[\sum_{j \neq 1} \sigma_{1j}(s_j),\, \sum_{j \neq 2} \sigma_{2j}(s_j), \ldots, \sum_{j \neq n} \sigma_{n,j}(s_j) \right]^T.$$

The following can be proved.

Theorem 1: If the network Σ is 0-GAS for all $\Sigma \in \mathcal{S}$, it holds that

$$M_0(\mathbf{s}) \not\geq 0, \quad \forall \mathbf{s} \in \mathbb{R}_+^n \setminus \{0\}. \tag{6}$$

In this paper, the condition (6) is referred to as a topological separation condition in view of its geometric interpretation given in [21], [13]. Theorems 1 and 2 of [11] are straightforward consequences of the above Theorem 1 by taking limiting values toward $s_i \to \infty$, i.e.,

$$\Gamma(\infty) \not> A(\infty), \tag{7}$$

where $A(\infty) := \lim_{\tau \to \infty} A(\mathbf{s})|_{s_1 = \ldots = s_n = \tau}$ and $\Gamma(\infty) := \lim_{\tau \to \infty} \Gamma(\mathbf{s})|_{s_1 = \ldots = s_n = \tau}$. It is not difficult to see that the property (6) is necessary for the comparison system to be 0-GAS without making any connection to the original network defined with the vectors $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2, ..., n$ [21]. This paper not only associates the necessity with networks defined on the original space as in Theorem 1, but also establishes the necessity in the presence of inputs (Theorems 4 and 5).

Define $M_0\langle i_1, i_2, ..., i_m \rangle$ corresponding to the induced subgraph of Σ consisting of vertices $\{i_1, i_2, ..., i_m\}$ by

$$M_0\langle i_1, i_2, ..., i_m \rangle(\hat{\mathbf{s}}) = \begin{bmatrix} -\alpha_{i_1,i_1}(\hat{s}_{i_1}) + \displaystyle\sum_{k \in \{i_1, i_2, ..., i_m\} \setminus \{i_1\}} \sigma_{i_1,k}(\hat{s}_k) \\ \vdots \\ -\alpha_{i_m,i_m}(\hat{s}_{i_m}) + \displaystyle\sum_{k \in \{i_1, i_2, ..., i_m\} \setminus \{i_m\}} \sigma_{i_m,k}(\hat{s}_k) \end{bmatrix},$$

where $1 \leq m \leq n$ and $\hat{\mathbf{s}} = [\hat{s}_{l_1}, \hat{s}_{l_2}, ..., \hat{s}_{l_m}]^T \in \mathbb{R}^m_+$. The following lemma demonstrates that the necessary condition (6) for the 0-GAS holds true even if the network is decomposed into several blocks.

Theorem 2: If (6) is satisfied, then

$$M_0\langle i_1, i_2, ..., i_m\rangle(\hat{\mathbf{s}}) \not\geq 0, \quad \forall \hat{\mathbf{s}} \in \mathbb{R}^m_+ \setminus \{0\}. \tag{8}$$

holds for any integer $m \in \{2, 3, ..., n-1\}$ and any non-repeated sequence $\{i_1, i_2, ..., i_m\}$ in the set $\{1, 2, ..., n\}$.

This theorem answers the question of how many non-ISS subsystems are allowed in a stable network as follows:

Theorem 3: Suppose that the network Σ is 0-GAS for all $\Sigma \in \mathcal{S}$. If there exists $i \in \{1, 2, ..., n\}$ such that

$$\lim_{s\to\infty} \alpha_i(s) < \min_{k\in\{1,2,...,n\}\setminus\{i\}} \lim_{s\to\infty} \sigma_{ik}(s) \tag{9}$$

is satisfied, then either $\lim_{s\to\infty} \alpha_j(s) = \infty$ or

$$\lim_{s\to\infty} \alpha_j(s) \geq \min_{k\in\{1,2,...,n\}\setminus\{j\}} \lim_{s\to\infty} \sigma_{jk}(s) \tag{10}$$

holds for each $j \in \{1, 2, ..., n\} \setminus \{i\}$.

This theorem implies that the number of subsystems which are not ISS *with respect to any single coupling channel* cannot be more than one. Notice that (10) holds if there exists a subsystem Σ_k which does not feed x_k into Σ_j, i.e., $\sigma_{jk} = 0$. Hence, for sets $\mathcal{S}$ which do not form complete graphs, the network Σ can be 0-GAS for all $\Sigma \in \mathcal{S}$ even if we have more than one subsystem which is not ISS respect to each coupling channel. Such an example is a cycle network given in Remark 1 of [11].

We next sharpen the necessary condition given in Theorem 3 by considering stability with respect to external signals. Since we have assumed $\kappa_1, .., \kappa_n \in \mathcal{K}_\infty$ which allows unlimited influence of the external signal r in magnitude on individual subsystems, the following holds true:

Theorem 4: If the network Σ is ISS with respect to input r for all $\Sigma \in \mathcal{S}$, then

$$\lim_{s\to\infty} \alpha_i(s) = \infty, \quad i = 1, 2, ..., n. \tag{11}$$

Using the mappings $M, D : \mathbb{R}^n_+ \to \mathbb{R}^n$ in the form of

$$M(\mathbf{s}) := -D^{-1} \circ A(\mathbf{s}) + \Gamma(\mathbf{s}) \tag{12}$$

$$D(\mathbf{s}) := \begin{bmatrix} s_1 + \beta_1(s_1) \\ s_2 + \beta_2(s_2) \\ \vdots \\ s_n + \beta_n(s_n) \end{bmatrix} \tag{13}$$

Theorem 1 can be generalized to address the stability with respect to the external input.

Theorem 5: If the network Σ is ISS with respect to input r for all $\Sigma \in \mathcal{S}$, there exist $k \in \{1, 2, ..., n\}$ and $\beta_k \in \mathcal{K}_\infty$, $\beta_j \in \mathcal{P}$, $j \in \{1, 2, ..., n\} \setminus \{k\}$ such that

$$M(\mathbf{s}) \not> 0, \quad \forall \mathbf{s} \in \mathbb{R}^n_+ \tag{14}$$

$$\mathbf{Id} + \beta_i \in \mathcal{K}_\infty, \quad i = 1, 2, ..., n. \tag{15}$$

Under the assumption that all subsystems are ISS defined with $\alpha_i \in \mathcal{K}_\infty$ and $\sigma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, and that the graph G is strongly connected, it was proved that the network Σ is 0-GAS if (6) holds [21]. Under the same assumption, the network Σ is guaranteed to be ISS if there exist $\beta_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, such that (14) holds [6]. Except for $n = 2$ in [12], no results allowing for non-ISS subsystems in such a form of sufficient stability criteria have been available. Very recently a sufficient condition [15] has been derived for the stability of networks allowing non-ISS subsystems in a form which differs from the topological separation (6) and (14). This is reviewed in the next section.

Remark 1: In view of the existence of β_i, $i = 1, 2, ..., n$, the condition (14) is equivalent to

$$M(\mathbf{s}) \not\geq 0, \quad \forall \mathbf{s} \in \mathbb{R}^n_+ \setminus \{0\}. \tag{16}$$

Obviously, the fulfillment of (16) implies (14) for the same β_i's. The converse holds if β_i's are divided by two. Notice that, due to $s + 0.5\beta_i(s) = 0.5(s + \beta_i(s)) + 0.5s$, the property $\mathbf{Id} + \beta_i \in \mathcal{K}_\infty$ yields $\mathbf{Id} + 0.5\beta_i \in \mathcal{K}_\infty$.

Remark 2: If the network Σ forms a cycle, it is verified that the statement of Theorem 5 can be strengthen with $\beta_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$. In fact, if (14) and (15) are satisfied by $\beta_k \in \mathcal{K}_\infty$ and $\beta_i \in \mathcal{P}$, $i \neq k$, then we can always pick another set of $\beta_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, achieving (14) and (15).

IV. Simultaneous Small-Gain: Sufficiency

We can always choose $J_U, d_i, d_{i,j} \in \mathbb{R}_+$ satisfying

$$1 = d_i \sum_{U\in\{W\in\mathcal{C}(G)\cup\mathcal{B}(G):\mathcal{V}(W)\ni i\}} J_U, \qquad \forall i \in \mathcal{V}(G) \tag{17}$$

$$1 = d_{i,j} \sum_{U\in\{W\in\mathcal{C}(G)\cup\mathcal{B}(G):\mathcal{A}(W)\ni(i,j)\}} J_U, \quad \forall (i, j) \in \mathcal{A}(G). \tag{18}$$

The set of non-zero J_U's fulfilling (17) and (18) defines a covering of the graph G by cycles, paths and isolated vertices. A subgraph U is adopted to cover a part of G if and only if $J_U \neq 0$. Multiple subgraphs U adopted can overlap each other. Although the set of subgraphs U covering G is not unique, there always exists such a set of subgraphs. For arbitrary real numbers $J_U > 0$ chosen for that set of subgraphs, there always exist $d_i, d_{i,j} > 0$ fulfilling (17) and (18). We define $\hat{\alpha}_i \in \mathcal{K}$ and $\hat{\sigma}_{ij} \in \mathcal{K} \setminus \{0\}$ as

$$\hat{\alpha}_i(s) = d_i\alpha_i(s) \tag{19}$$

$$\hat{\sigma}_{ij}(s) = d_{i,j}\sigma_{ij}(s). \tag{20}$$

using the weights $d_i, d_{i,j} > 0$ of the covering. We employ

$$\hat{\eta}_i(s) = \begin{cases} \hat{\alpha}_i^{-1}(s) & \text{if } \lim_{\tau\to\infty} \hat{\alpha}_i(\tau) > s \\ \infty & \text{otherwise} \end{cases} \tag{21}$$

which is a slightly abused notation of inverse operation on $\hat{\alpha}_i$. Its benefit is discussed in [11] and [15]. The next theorem can be proved as in [15] through the construction of a Lyapunov function for Σ although this paper employs a slightly different formulation of (17), (18). (19) and (20).

Theorem 6: Assume that

$$\left\{ \lim_{s\to\infty} \alpha_j(s) = \infty \vee \lim_{s\to\infty} \sum_{i=1}^{n} \sigma_{i,j}(s) < \infty \right\}, \qquad j = 1, 2, ..., n \tag{22}$$

holds. If there exist $\beta_i \in \mathcal{K}_\infty$ and J_U, d_i, $d_{i,j} \in \mathbb{R}_+$ for $U \in \mathcal{C}(G) \cup \mathcal{B}(G)$ and $i, j = 1, 2, ..., n$ fulfilling (17) and (18) such that

$$\hat{\eta}_{w(1)} \circ (\mathbf{Id} + \beta_{w(1)}) \circ \hat{\sigma}_{w(1),w(2)} \circ \hat{\eta}_{w(2)} \circ (\mathbf{Id} + \beta_{w(2)}) \circ \hat{\sigma}_{w(2),w(3)} \circ \cdots \circ \hat{\eta}_{w(k)} \circ (\mathbf{Id} + \beta_{w(k)}) \circ \hat{\sigma}_{w(k),w(k+1)}(s) \leq s, \ \forall s \in \mathbb{R}_+ \quad (23)$$

holds for all cycles $W \in \mathcal{C}(G)$, where $k = |W|$, then the network Σ is iISS with respect to input r for all $\Sigma \in \mathcal{S}$. Furthermore, it is ISS if $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$.

The inequality (23) is referred to as a small-gain condition. Note that (22) is implied by (11) which is a necessary condition for guaranteeing the network Σ to be ISS for all $\Sigma \in \mathcal{S}$. In the case of $n = 2$, there is only a single element U in $\mathcal{C}(G)$, and the properties (17) and (18) hold if

$$d_i = d_{i,j} = \frac{1}{J_U} > 0, \quad \forall (i, j) \in \mathcal{A}(U).$$

Hence, we can easily verify that (23) is equivalent to (14). In the same way, this remark holds true for general $n \geq 2$ if the network is a cycle. No answers to the equivalence problem for networks of general structure have been known. The next section investigates the problem.

Remark 3: The condition (23) cannot be replaced by

$$\hat{\eta}_{w(1)} \circ \hat{\sigma}_{w(1),w(2)} \circ \hat{\eta}_{w(2)} \circ \hat{\sigma}_{w(2),w(3)} \circ \cdots \circ \hat{\eta}_{w(k)} \circ \hat{\sigma}_{w(k),w(k+1)}(s) < s, \ \forall s \in \mathbb{R}_+ \setminus \{0\} \quad (24)$$

even for 0-GAS of Σ in general. In the case where $\hat{\eta}_{w(i)} \circ \hat{\sigma}_{w(i),w(i+1)}$ is linear, i.e., $\hat{\alpha}_{w(i)}^{-1} \circ \hat{\sigma}_{w(i),w(i+1)}$ is linear for all $W \in \mathcal{C}(G)$ and i, there are no differences between (23) and (24) in terms of the existence of β_i's.

Remark 4: Based on ISS gains of subsystems, the cyclic formulation of small-gain conditions which do not involve the decomposition parameters d_i and d_{ij} is developed in [5], [6], [17], [18], [19]. Their criteria are applicable only to ISS subsystems [12]. In contrast, the criteria (23) and (24) starting with (4) and the graph decomposition (19)-(20) allow subsystems to be non-ISS.

V. Spectral Radius: Connection

This section deals with supply rates given in the form of

$$\alpha_i(s) = a_i g_i(s), \qquad i \in \{1, 2, ..., n\} \quad (25)$$

$$\sigma_{i,j}(s) = b_{i,j} g_j(s), \quad i \neq j \in \{1, 2, ..., n\} \quad (26)$$

where $a_i > 0$ and $b_{ij} \geq 0$, $i, j = 1, \ldots, n$ are real numbers and $b_{j,j} = 0$. These functions α_i and $\sigma_{i,j}$ fulfill (22) when $g_i \in \mathcal{K}$. In the case of merely positive definite g_i's, assuming the existence of $\lim_{s\to\infty} g_i(s)$ ensures (22). In contrast to the previous sections, this section does not always assume $g_i \in \mathcal{K}$ corresponding to $\alpha_i \in \mathcal{K}$ and $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$. Such an assumption is stated when needed in this section. Define the following matrices on $\mathbb{R}_+^{n\times n}$:

$$\tilde{A} = \mathrm{diag}[a_1, a_2, ..., a_n], \quad \tilde{\Gamma} = [b_{i,j}]_{i,j=1,2,...,n}. \quad (27)$$

Let $\rho(\cdot)$ denotes the spectral radius of a square matrix. Combining the developments in Section III and a result in [4] via simple computation, we obtain the following:

Theorem 7: Suppose that there exist positive definite functions g_i and constants $a_i > 0$ and $b_{ij} \geq 0$, $i, j = 1, \ldots, n$ such that (25) and (26) hold. Then the following are equivalent:

i) The property (6) holds.
ii) There exist $\beta_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, such that (14) holds.
iii) $\rho(\tilde{A}^{-1}\tilde{\Gamma}) < 1$ holds.
iv) The network Σ is iISS with respect to input r for all $\Sigma \in \mathcal{S}$.

Furthermore, Item **iv)** is replaced by ISS if $g_i \in \mathcal{K}_\infty$ for $i = 1, 2, ..., n$.

The following lemma clarifies that the loop gains for (25) and (26) can be evaluated in the form of linear maps.

Lemma 1: Suppose that there exist $g_i \in \mathcal{K}$ and constants $a_i > 0$ and $b_{ij} \geq 0$, $i, j = 1, \ldots, n$ such that (25) and (26) hold. Let J_U, d_i, $d_{i,j} \in \mathbb{R}_+$ be parameters satisfying (17) and (18) for $U \in \mathcal{C}(G) \cup \mathcal{B}(G)$ and $i, j = 1, 2, ..., n$. Pick a cycle $W \in \mathcal{C}(G)$ arbitrarily. Then there exist $\beta_{w(j)} \in \mathcal{K}_\infty$, $j = 1, 2, ..., |W|$, such that (23) holds if and only if

$$L_W := \prod_{i=1}^{|W|} \frac{d_{w(i),w(i+1)} b_{w(i),w(i+1)}}{d_{w(i)} a_{w(i)}} < 1 \quad (28)$$

Moreover, the above is equivalent to (24).

It is stressed that $d_{w(i)} > 0$ holds in (28) since $w(i) \in \mathcal{V}(G)$. For general networks, the small-gain property for all cycles implies the topological separation due to Theorem 6 and Theorem 1 (or 5). The next lemma establishes its converse for the special supply rates (25)-(26).

Lemma 2: Suppose that there exist $g_i \in \mathcal{K}$ and constants $a_i > 0$ and $b_{ij} \geq 0$, $i, j = 1, \ldots, n$, such that (25) and (26) hold. If (6) is satisfied, then there exist J_U, d_i, $d_{i,j} \in \mathbb{R}_+$ for $U \in \mathcal{C}(G) \cup \mathcal{B}(G)$ and $i, j = 1, 2, ..., n$ fulfilling (17) and (18) such that (28) holds for all cycles $W \in \mathcal{C}(G)$.

Combining Lemma 2 and Theorem 7 yields the main result of this section.

Theorem 8: Suppose that there exist $g_i \in \mathcal{K}$ and constants $a_i > 0$ and $b_{ij} \geq 0$, $i, j = 1, \ldots, n$, such that (25) and (26) hold. Then each of Items **i)**, **ii)**, **iii)** and **iv)** in Theorem 7 and the following are equivalent to each other:

v) There exist J_U, d_i, $d_{i,j} \in \mathbb{R}_+$ for $U \in \mathcal{C}(G) \cup \mathcal{B}(G)$ and $i, j = 1, 2, ..., n$ fulfilling (17) and (18) such that (28) holds for all cycles $W \in \mathcal{C}(G)$.

Furthermore, Item **iv)** is replaced by ISS if $g_i \in \mathcal{K}_\infty$ for $i = 1, 2, ..., n$.

Without resorting to (25) and (26), the equivalence between Item **i)** and Item **v)** can be verified for cycle networks of $n \geq 2$ by using the results in [14], [12], [11], [15]. For general networks, the equivalence in the form of Theorem 8 has not been established without (25) and (26).

Remark 5: Unless we assume $\alpha_i \in \mathcal{K}$, the inverse $\alpha_i^{-1}(s)$ is not guaranteed to exist even for small value of s. Thus, the small-gain condition (23) cannot be posed. Indeed, we have assumed $g \in \mathcal{K}$ in Lemmas 1, 2 and Theorem 8. The increasing property of α_i is crucial for the sufficiency of the topological separation conditions (6) and (14) as well. It is known that cascades consisting of non-ISS subsystems can fail to be 0-GAS unless $\alpha_i \in \mathcal{K}$ [9]. Note that a cascade (a path graph) always satisfies (6) and (14) without $\alpha_i \in \mathcal{K}$. Thus, the topological separation conditions cannot be sufficient for 0-GAS of the network unless we impose more than the positive definiteness on α_i. The pair (25) and (26) is an example of a constraint ensuring that (6) becomes sufficient for the 0-GAS.

VI. An Illustrative Example: Network Computing

Network computing distributes workload across multiple computers for resource utilization. In the spirit of grid computing, we consider autonomous load distribution which requires no additional physical infrastructure but little overhead incurred by communication interfaces on individual computers. To analyzes it in a macroscopic scale, let $x_i(t) \in \mathbb{R}_+$ denote the queue of tasks to be processed at the i-th node. Consider the following fluid flow model [7], [1], [27]:

$$\dot{x}_i = -a_i(x_i) - \sum_{j\neq i} \underline{h}_{ij}(x_i) + \sum_{j\neq i} \overline{h}_{ji}(x_j) + r_i, \quad i = 1, 2, ..., n, \tag{29}$$

where $a_i \in \mathcal{K} \setminus \mathcal{K}_\infty$ denotes the processing speed of the i-th node and $r_i(t) \in \mathbb{R}_+$ is the queue of tasks requested at the i-th node. A reasonable choice of a_i is a step-like function, where $\alpha_i(\infty) := \lim_{s\to\infty} \alpha_i(s)$ is the computing power. The functions $\underline{h}_{ij} \in \mathcal{K} \cup \{0\} \setminus \mathcal{K}_\infty$, $i \neq j$, decide how much percentage of tasks at the i-th node is split between the other nodes, while the function $\overline{h}_{ji} \in \mathcal{K} \cup \{0\} \setminus \mathcal{K}_\infty$ denotes the amount of workload assigned by the j-th node. Due to the communication overhead for distribution, we have

$$\underline{h}_{ij}(s) < \overline{h}_{ij}(s), \quad \forall s \in (0, \infty) \tag{30}$$

$$\lim_{s\to\infty} \underline{h}_{ij}(s) < \lim_{s\to\infty} \overline{h}_{ij}(s) \tag{31}$$

if $\underline{h}_{ij}(s) \not\equiv 0$. The upper bound of $\overline{h}_{ij}$ and $\underline{h}_{ij}$ corresponds to the link capacity of the communication or the intentional upper bound of the load distribution. The solutions $x(t)$ of (29) for $x(0) \in \mathbb{R}^n_+$ remain in $\mathbb{R}^n_+$. The situation of $x = 0$ indicates that all tasks have been processed. The "little" overhead (30) puts some energies into the network and can lead to an unstable network. No such issue arises if the computers are decoupled, i.e., $\underline{h}_{ij} = \overline{h}_{ij} = 0$ for all i, j. The iISS small-gain methodology developed in this paper provides us with the following observations:

- **i)** If the maximum distribution rates are made large independently of the computing power, the number of nodes and the interconnection structure, the network becomes unstable. $\cdots$ Theorems 1 and 3
- **ii)** Queues grow unboundedly if requested tasks are persistently too large. $\cdots$ Theorem 4
- **iii)** The larger the number of nodes is, the faster the processing speed of individual nodes need to be for reducing the queue lengths. $\cdots$ Theorems 5
- **iv)** Processing speed of one node can be slow if that of other nodes is sufficiently fast. $\cdots$ Theorem 6

The iISS property of (29) with respect to the total load r implies that the queues converge to zero if $r = 0$, and that, in the presence of r, the queue lengths are finite for finite time (forward complete). In addition, the queue lengths are bounded if the total load energy is bounded (Bounded Energy Frequently Bounded State) [2]. The theory developed in this paper applies to

$$\alpha_i(s) = a_i(x_i) + \sum_{j\neq i} \underline{h}_{ij}(x_i), \quad \sigma_{ij}(s) = \overline{h}_{ji}(s).$$

Notice that (29) is a dissipation inequality of the subsystem Σ_i with a storage function $V_i = x_i$ since $x_i(t)$ is guaranteed to be non-negative. It is stressed that the individual subsystems Σ_i are not guaranteed to be ISS although they are iISS.

Consider the homogenous distribution defined as

$$\underline{h}_{ji} = h, \quad \overline{h}_{ji} = kh, \quad \forall i \neq j \tag{32}$$

for a constant $k > 1$ and a function $h \in \mathcal{K} \setminus \mathcal{K}_\infty$. Then the conditions (14) and (23) are satisfied if

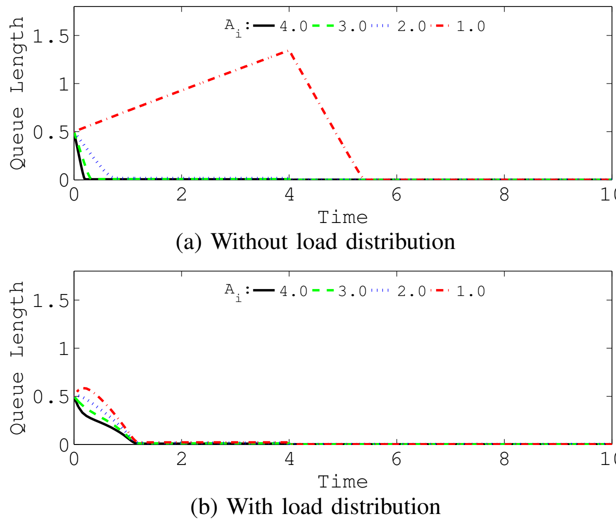
$$\sum_{i=1}^{n} a_i(s_i) > (k-1) \sum_{i=1}^{n} \sum_{j\neq i} h(s_j), \quad \forall \mathbf{s} \in (0, \infty]^n \tag{33}$$

is met. Note that the fulfillment of $a_i(\infty) > (k-1)(n-1)h(\infty)$, $i = 1, 2, ..., n$, is neither necessary nor sufficient either for 0-GAS or for iISS of the network. The inequality (33) shows that the stability is achievable by decreasing load sharing and increasing processing speed, or by reducing the overhead and the number of nodes. The iISS small-gain approach not only justifies the intuitive constraint (33), but also suggests that achieving (33) is not necessary. The condition (14) (or (23)) allows a small a_i to be compensated by larger a_j's. This philosophy also applies to the general case of heterogeneous distribution and overhead which do not meet the simplifying assumption (32).

Figures 1 and 2 show numerical simulations for

$$a_i(s) = \frac{A_i s}{0.01 + s}, \quad \underline{h}_{ij} = \frac{4.5s}{0.2 + s}, \quad \overline{h}_{ij}(s) = 1.1\underline{h}_{ij}(s)$$
$$x_i(0) = 0.5, \quad i = 1, 2, ..., n$$
$$r_i(t) = \begin{cases} 1.2, & i = 1, 2, ..., n, \quad 0 \le t < 4 \\ 0, & i = 1, 2, ..., n, \quad 4 \le t \end{cases}.$$

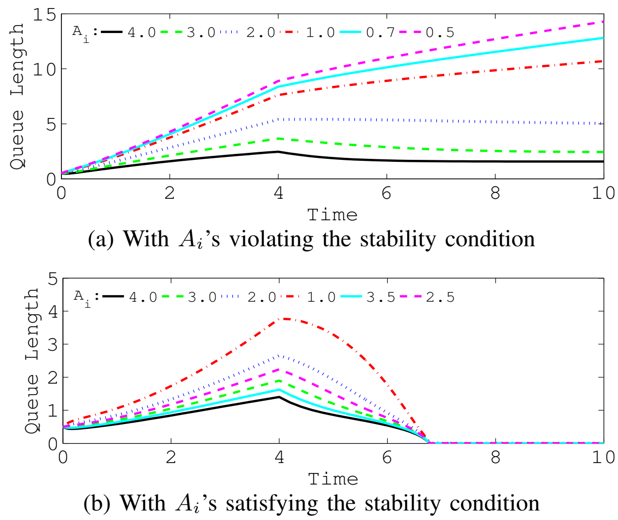
The set of A_i's used in Fig.2(a) yields $-A(s) + \Gamma(s) = [0.258, 0.248, 0.504, 0.056, 0.185, 0.096]^T$ for $\mathbf{s} = [1.1, 1.5, 2, 5, 6, 9]^T$, so that (6) and (14) are violated. Indeed, the queue lengths plotted in Fig.2(a) grow unboundedly. In contrast, the parameters yielding the bounded convergent response in Fig.2(b) fulfill (14) although the simplest sufficient condition (33) is not satisfied. Due to $s/(0.01 + s) \ge s/(0.2 + s)$, Theorem 8 establishes that $\rho(\tilde{A}^{-1}\tilde{\Gamma}) < 1$ guarantees the iISS of the network, where $\tilde{A} = \mathrm{diag}[\alpha_i(\infty)] \in \mathbb{R}^{n\times n}_+$ and $\tilde{\Gamma} = [\sigma_{ij}(\infty)] \in \mathbb{R}^{n\times n}_+$. The spectral radius computed for Fig.2(b) is 0.985. Figure 3 depicts the response



(a) Without load distribution

(b) With load distribution

Fig. 1. Queue length at $n = 4$ nodes.



(a) With A_i's violating the stability condition

(b) With A_i's satisfying the stability condition

Fig. 2. Queue length at $n = 6$ nodes.

for connection and disconnection of nodes with

$$r_i(t) = \begin{cases} 1.70, & i = 1, 5, \quad 0 \le t < 5 \\ 1.23, & i = 2, 3, 4, 6, \quad 0 \le t < 5 \\ 0, & i = 1, 2, ..., 6, \quad 5 \le t \end{cases} .$$

The response exhibits a typical iISS property guaranteed.

Remark 6: If the participation in the computing power sharing is completely free, the iISS of the network can be ensured by applying the small-gain condition to the complete graph. The network remains iISS even if some nodes are disconnected. Note that disconnection always implies decrease (to zero) of loop gains of cycles.

Remark 7: The model (29) is defined with a scalar non-negative $x_i(t)$. The necessity of the small-gain criterion for such a network can be proved without invoking technique of constructing artificial destabilizers via Lemma 1 in [11] .

VII. Conclusions

In this paper, stability and robustness of nonlinear dynamical networks in arbitrary interconnection graph structure have been investigated. This paper allows subsystems to be iISS which is much more general than ISS which has been studied extensively in the literature. Although the dissipative characterization of subsystems can unify the treatment of iISS and ISS properties, the general formulation has been hampering the extension of the existing ISS results [4], [21]. This paper has been focused on the validity of the topological separation, the simultaneous small-gain criterion and the spectral radius condition in the iISS setup. Some relationships between them have been established. The equivalence between those conditions has been demonstrated for matched supply rates. The author is currently investigating the possibility of extending the equivalence to the case of general supply rates.

This paper has illustrated the usefulness of the proposed methodology for iISS networks through the analysis of a network computing dynamics. Due to the limitation of computing power, subsystems can never be ISS with respect to arbitrarily large task request. Based on a fluid flow model, the stability conditions give qualitative and quantitative information on the computing speed, the communication overhead, the number of nodes and the interconnection structure for maintaining the stability of the load distribution.

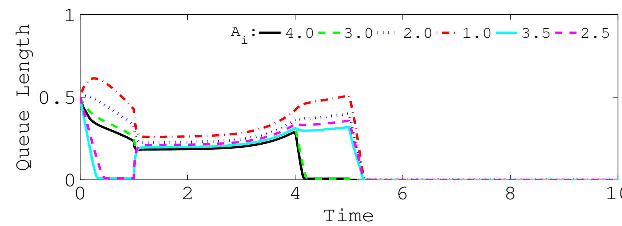


Fig. 3. Queue length at $n = 6$ nodes:
Connecting $A_5 = 3.5$ and $A_6 = 2.5$ at $t = 1$;
Disconnecting $A_1 = 4.0$ and $A_2 = 3.0$ at $t = 4$.

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Appendix

A. Sketch of the proof of Theorem 5

Theorem 4 yields (11). In order to prove the claim by contradiction, assume that (14) is violated whatever $\beta_k \in \mathcal{K}_\infty$ we select for any choice of $k \in \{1, 2, ..., n\}$. Since ISS implies 0-GAS, Theorem 1 implies $M_0(\ell) \not\geq 0$ for all $\ell \in \mathbb{R}^n_+ \setminus \{0\}$, i.e, (6). This leads to

$$\lim_{s\to\infty} \alpha_i(s) = \lim_{s\to\infty} \sum_{j\neq i} \sigma_{ij}(s) = \infty, \;\; i = 1, 2, ..., n. \quad (34)$$

Pick $\underline{l} \in (0, \infty)$. there exist constants $R_i \in [0, \infty)$, $i = 1, 2, ..., n$, which is independent of l_1, such that

$$\exists l_2, l_3, ..., l_n \in (0, \infty) \text{ s.t.}$$
$$\left\{ \alpha_i(l_i) \le \sum_{j\neq i} \sigma_{ij}(l_j) + R_i, , \;\; i = 1, 2, ..., n \right\}$$

holds for each $l_1 \in [\underline{l}, \infty)$. Thus, for each $l_1 \in [\underline{l}, \infty)$ we have

$$\alpha_i(l_i) \leq \sum_{j \neq i} \sigma_{ij}(l_j) + \kappa_i(|r_i|), \ i = 1, 2, ..., n \qquad (35)$$

for the constant signal $|r_i(t)| = \kappa_i^{-1}(R_i)$ which is independent of l_1. Lemma 1 in [11] guarantees the existence of a time-invariant system $\Sigma \in \mathcal{S}$ and positive definite radially unbounded functions V_i, $i = 1, 2, ..., n$, such that

$$\alpha_i(V_i(x_i)) \leq \sum_{j \neq i} \sigma_{ij}(V_j(x_j)) + \kappa_i(|r_i|) \ \Rightarrow \ \frac{\partial V_i}{\partial x_i} f_i \geq 0$$

is achieved as long as $V_i(x_i) \geq \underline{l}$, $i = 1, 2, ..., n$. By virtue of (35), the set

$$\mathbf{U}(\ell) = \{x \in \mathbb{R}^N : V_i(x_i) \geq l_i, \ i = 1, 2, ..., n\} \qquad (36)$$

defined with $\ell = [l_1, l_2, ..., l_n]^T$ is forward invariant for the selected $\Sigma \in \mathcal{S}$ with the fixed $|r_i(t)| = \kappa_i^{-1}(R_i)$, $i = 1, 2, ..., n$, no matter how large l_k is. Therefore, the network is not ISS with respect to input r.

B. Sketch of the proof of Lemma 2

Lemma 3: Assume that there exist $g_i \in \mathcal{K}$ and constants $a_i > 0$ and $b_{ij} \geq 0$, $i, j = 1, \dots, n$, such that (25) and (26) are satisfied, and that the graph G is strongly connected. If, for each choice of J_U, d_i, $d_{i,j} \in \mathbb{R}_+$ for $U \in \mathcal{C}(G) \cup \mathcal{B}(G)$ and $i, j = 1, 2, ..., n$ fulfilling (17) and (18), there exists a cycle $W \in \mathcal{C}(G)$ such that

$$\prod_{i=1}^{|W|} \frac{d_{w(i),w(i+1)} b_{w(i),w(i+1)}}{d_{w(i)} a_{w(i)}} \geq 1 \qquad (37)$$

holds, then there exists $\ell \in \mathbb{R}_+^n \setminus \{0\}$ such that $M_0(\ell) \geq 0$ holds.

The above proves Lemma 2 by contradiction when G is strongly connected. If the graph G is not strongly connected, then the graph can be decomposed into strongly connected subgraphs which are connected so that no cycles of the subgraphs are formed. In other words, the adjacency matrix of G can be brought in upper block triangular form via a permutation of the vertices. Decreasing J_U of the arcs connecting the strongly connected subgraphs to which the argument of Lemma 3 is applicable, we can prove Lemma 2 by contradiction without assuming the strong connectivity. As regards Lemma 3, altering the weights J_U in the covering of the graph G needs to investigated. Addressing the capability and limitation of such a modification of the weights appropriately, we can arrive at Lemma 3.

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