2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Structure Preserving Spatial Discretization of 1D convection-diffusion port-Hamiltonian Systems

T. Voß and S. Weiland

Abstract— **Convection-diffusion is a physical phenomenon that appears in a multitude of dynamical systems, e.g. vibrating string with damping or chemical and thermal systems. This paper focuses on a structure preserving spatial discretization scheme of a general dynamical system with convection and diffusion in the port-Hamiltonian framework. The preservation of the port-Hamiltonian structure ensures that specific properties, such as passivity, of the infinite dimensional system are preserved.**

I. Introduction

Systems with spatial and temporal dynamics occur in many engineering applications where they are typically modeled by partial differential equations or as state space models over infinite dimensional state or phase spaces. The control of infinite dimensional systems is generally troublesome due to the fact that an infinite number of states needs to be controlled through a finite number of control variables. Currently, one can distinguish two common approaches for the control of infinite dimensional systems. The first one, *late lumping* [1], [2], amounts to designing an infinite dimensional control law which renders the infinite dimensional system stable while achieving some desired performance. The second approach, *early lumping* [3], amounts to first spatially discretizing the infinite dimensional system and, subsequently, designing a finite dimensional control law based on the dynamics of the spatially discretized system. All control design approaches based on finite element models belong to this category.

This paper considers the first step in the early lumping control approach, namely the spatial discretization of a 1D convective and diffusive infinite dimensional port-Hamiltonian (pH) system [4]. There are compelling reasons to model convective and diffusive phenomena in infinite dimensional system in the port-Hamiltonian framework. The most important one is that pH systems have structural properties, such as passivity, that make them extremely suitable for control design, see [5], [6]. These properties are ensured by the special mathematical structure of a pH system and are typically lost by arbitrary spatial discretization schemes.

The problem of structure preservation in discretization schemes of pH systems is of paramount importance. Indeed, from the general perspective of modeling, simulation, model approximation, and control system design, there is a need for specialized discretization schemes that preserve the pH structure in spatial discretizations. Classical spatial discretization schemes, e.g. the finite elements method [7], typically destroy the pH structure of the system, and, with it, can not provide guarantees on the preservation of crucial system properties such as passivity, stability, and specific system invariants.

This paper presents a spatial discretization scheme for a 1D convective and diffusive infinite dimensional pH system. The scheme results in a finite dimensional dynamical system in state space form with the key property that the pH structure is preserved in the finite dimensional approximation. For chemical or fluid dynamical systems this scheme implies the preservation of the mass and momentum balance after spatial discretization.

There have been several publications on structure preserving spatial discretizations of lossless pH systems [8], [9], [10], [11]. The spatial discretization of a purely diffusive system has been discussed in [12]. Our contribution differs from the results in [12] in a number of aspects. Firstly, the authors of [12] neglect the kinetic energy of the chemical system. Therefore only the mass balance is ensured. In our approach both energy domains are present, which, for chemical systems, means that the mass *and* momentum balance is preserved. Secondly, we consider general pH systems which can represent a multitude of physical phenomena, e.g. a vibrating string with damping, 1D Navier-Stokes equations or chemical phenomena. Thirdly, in [12] an iterative process is derived which determines the time evolution of the dynamics while our approach results in a finite dimensional dynamical system of ordinary differential equations which can be used with *any* simulation or control method.

II. Short introduction to finite dimensional port-Hamiltonian systems

In this section we briefly introduce the port-Hamiltonian (pH) modeling framework, see [13], [14], [4]. Port-Hamiltonian systems are port-based models. This means that the interconnection of two or more pH systems is defined by connecting ports in a physical way. Hence, the interconnection of pH systems is quite natural and can be exploited for large scale modeling. By this, we mean that a pH system defined on a spatial domain can be viewed as an interconnection of a finite number of finite-dimensional pH systems, each defined on an appropriate sub-domain. This idea was originally developed for the modeling of finite dimensional systems. However, the framework has been extended to the case of infinite dimensional systems, see for example [5], [6].

T. Voß and S. Weiland are with Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, {t.voss,s.weiland}@tue.nl

978-1-61284-799-3/11/$26.00 ©2011 IEEE

A finite dimensional port-Hamiltonian system in local coordinates is a model of the form

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + B(x)u \\ y &= B^\top(x) \frac{\partial H}{\partial x}(x), \end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_n)$ are local state coordinates, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^m$ are the inputs and outputs respectively. $J(x) \in \mathbb{R}^{n\times n}, J(x) = -J(x)^\top$ is the skew symmetric interconnection matrix, $R(x) \in \mathbb{R}^{n\times n}$ is the symmetric positive semi-definite resistance matrix and $B(x) \in \mathbb{R}^{n\times m}$ is the input force matrix. $H(x) : \mathcal{X} \to \mathbb{R}$ with $H(x) > c > -\infty, \forall x \in \mathcal{X}$ is the Hamiltonian which represents the stored energy in the system.

Throughout, we will use the effort-flow representation of the finite dimensional pH system. That is, the static equations

$$\begin{aligned} f &= (J - R)\, e + Bu \\ y &= B^\top e \end{aligned} \tag{2}$$

with effort variable e and flow variable f represent (1) if and only if the flow variable $f = \dot{x}$ and the effort variable $e = \frac{\partial}{\partial x} H$.

Another concept which we will use is the energy flow, or net power of the system. The energy flow is defined as

$$P_{\text{net}} = e^\top f + y^\top u. \tag{3}$$

III. Short introduction to the differential-geometric setting

In order to provide a coordinate-free treatment of Euclidean differential calculus, we will work with differential forms to represent models. We treat spatial domains $Z \subset \mathbb{R}^n$ mainly of dimension $n = 1$ and distinguish between functions (zero-forms) and distributions (one-forms).

A zero-form is the differential-geometric representation of a smooth function $f : Z \to \mathbb{R}$. Zero-forms can be evaluated points $z \in Z$. A one-form is the differential-geometric representation of a distribution. If $g : Z \to \mathbb{R}$ is a function, then a one-form and expression $g(z)\mathrm{d}z$. One-forms can not be evaluated at points $z \in Z$ but attain their values after integration over a sub-domain of Z as in the line integral $\int_a^b g(z)\mathrm{d}z$.

One can transform a zero-form into a one-form by spatial differentiation. The *exterior derivative* of a zero-form f is denoted by $\mathrm{d}f$ and defined by the one-form $\mathrm{d}f = g(z)\mathrm{d}z$ where $g = \frac{df}{dz}$ is the derivative of f. The *exterior product* (or *wedge product*) of two zero-forms f and g is the zero-form $f \wedge g := h$ where $h(z) = f(z)g(z)$. The exterior product of a zero-from f and a one-form g is the one-form $f \wedge g = f(z)g(z)\mathrm{d}z$. There holds $f \wedge g = -g \wedge f$. Finally, the *Hodge star operator* assigns, in one dimensional domains, zero-forms f to one-forms $*f = f(z)\mathrm{d}z$ and one-forms $g(z)\mathrm{d}z$ to zero-forms $*[g(z)\mathrm{d}z] = g(z)$.

IV. Finite dimensional approximation of a convective-diffusive 1D pH system

In this section we state a finite dimensional approximation of a 1D infinite dimensional pH with convection and diffusion on a given spatial domain $Z_{ac} = [a, c]$. A 1D convective and diffusive pH system is given in the following general form

$$\begin{aligned} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} &= \left(\begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & rd * d \end{bmatrix} \right) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ e_\partial &= e_1|_{\partial Z} \\ f_\partial &= e_2|_{\partial Z} \\ H(x) &= \int_Z \mathcal{H}(x). \end{aligned} \tag{4}$$

Note that we have defined the system in the differential geometric framework. Here $f_i = \dot{x}_i, i \in \{1, 2\}$ are the flows of the system, geometrically they are one-forms. The efforts of the system $e_i = \frac{\delta H}{\delta x_i} = \frac{\partial \mathcal{H}}{\partial x_i}$ are expressed as the variational derivative of the energy function $H(x)$ and are zero-forms. The energy function is defined as the integral of the energy distribution function $\mathcal{H}(x)$ over the spatial domain Z. The boundary energy ports are defined as (e_∂, f_∂). The dissipation parameter $r > 0$ we assume here for simplicity to be a scalar. An example for such a system would be a vibrating string with damping [4] or a 1D representation of Navier-Stokes equations.

By introducing a resistive state and the related equation of motion we can reformulate (4) into the following form

$$\begin{bmatrix} f_1 \\ f_2 \\ f_R \end{bmatrix} = \begin{bmatrix} 0 & d & 0 \\ d & 0 & d \\ 0 & d & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_R \end{bmatrix} \tag{5}$$

where the following relation holds $e_R = *rf_R$. The ports and the energy function are defined in the same way as in (4). It can be shown, see [4], that the systems (4) and (5) are equivalent. Now we are able to state the following theorem.

Theorem 1: A finite dimension approximation of (5) on Z_{ac} is given as

$$\begin{aligned} \mathbf{f} &= \frac{1}{(b-a)} (J - R)\, \mathbf{e} + B\mathbf{u} \\ \mathbf{y} &= B^\top \mathbf{e} \end{aligned} \tag{6}$$

where

$$\begin{aligned} \mathbf{f} &= \left[f^1_{ab}, f^1_{bc}, f^2_{ac}\right]^\top, \mathbf{e} = \left[e^1_{ab}, e^1_{bc}, e^2_{ac}\right]^\top \\ \mathbf{u} &= [e^2_a, e^2_c]^\top \\ J &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2r \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ r & r \end{bmatrix} \end{aligned}$$

can be achieved by first splitting the discretization interval Z_{ac} into two intervals $Z_{ab} = [a, b]$ and $Z_{bc} = [b, c]$ with $a < b < c$ and then approximating the infinite dimensional efforts and flows on Z_{ab} and Z_{bc} by the following spatial

temporal expansions,

$$\begin{aligned} f_1(z,t) &\approx f^1_{ab}(t)\omega^1_{ab}(z) + f^1_{bc}(t)\omega^1_{bc}(z) && \text{(7a)}\\ f_2(z,t) &\approx f^2_{ac}(t)\omega^2_{ac}(z) && \text{(7b)}\\ f_R(z,t) &\approx f^R_{ab}(t)\omega^R_{ab}(z) + f^R_{bc}(t)\omega^R_{bc}(z) && \text{(7c)}\\ e_1(z,t) &\approx e^1_a(t)\omega^1_a(z) + e^1_b(t)\omega^1_b(z) + e^1_c(t)\omega^1_c(z) && \text{(7d)}\\ e_2(z,t) &\approx e^2_a(t)\omega^2_a(z) + e^2_b(t)\omega^2_b(z) + e^2_c(t)\omega^2_c(z) && \text{(7e)}\\ e_R(z,t) &\approx e^R_a(t)\omega^R_a(z) + e^R_c(t)\omega^R_c(z). && \text{(7f)} \end{aligned}$$

We will state a constructive proof of Theorem 1 in the following section.

V. Proof of Theorem 1

The proof of Theorem 1 is divided into several steps. First we substitute the approximation of efforts and flows (7) into the infinite dimensional equations of motions (5) and integrate over Z_{ac} which will yield a finite dimensional approximation of the dynamics. Then we define a finite dimensional approximation of the efforts on Z_{ac} using the net power of the port-Hamiltonian system. Then we show that the finite dimensional interconnection structure (6) approximates the dynamics of (5) on Z_{ac}. The last step is to calculate a finite dimensional approximation of the energy function by using that $f = \dot{x}$. Recall that $Z_{ab} = [a, b]$ and $Z_{bc} = [b, c]$ with $a < b < c$ partition Z_{ac} according to $Z_{ac} = Z_{bc} \cup Z_{ab}$. Assume that the shape functions satisfy the following conditions

$$\begin{aligned} \omega^1_i(j) &= \omega^R_i(j) = \frac{\omega^2_i(j)}{k_{ij}} = \delta_{ij},\ i,j \in \{a,\, b,\, c\} && \text{(8a)}\\ \int_{Z_l} \omega^1_k &= \int_{Z_l} \omega^R_k = \delta_{kl},\ k,l \in \{ab, bc\} && \text{(8b)}\\ \int_{Z_{ac}} \omega^2_{ac} &\neq 0. && \text{(8c)} \end{aligned}$$

A. Second equation of motion

We first derive a finite dimensional approximation of the second equation of motion

$$f_2 = de_1 + de_R. \tag{9}$$

Substituting the expansion (8) in (9) defines a direct relation between the chosen shape functions . Therefore, the next step is to find a set of conditions to ensure the compatibility of the shape functions. To do this we first assume that $e^1_b = e^1_c = e^R_a = e^R_c = 0$. Then (9) simplifies to

$$f^2_{ac}\omega^2_{ac} = e^1_a d\omega^1_a$$

which, in fact defines a relation between ω^2_{ac} and ω^1_a. Let $k_1 = -f^2_{ac} \cdot \left(e^1_a\right)^{-1}$. Then we obtain

$$-k_1\omega^2_{ac} = d\omega^1_a.$$

Integrating over Z_{ac} and using (8) yields

$$k_1 = \frac{1}{\int_{Z_{ac}} \omega^2_{ac}}.$$

Hence, we have the following relation between ω^2_{ac} and ω^1_a.

$$k_1\omega^2_{ac} = -d\omega^1_a.$$

Similarly we obtain that

$$k_1\omega^2_{ac} = -d\omega^1_a = d\omega^1_c = -d\omega^R_a = d\omega^R_c. \tag{10}$$

Note that without loss of generality we choose here $k_1 = 1$.

Assuming that $e^1_a = e^1_c = e^R_a = e^R_c = 0$ and $k = f^2_{ac} \cdot \left(e^1_b\right)^{-1}$ we obtain

$$k\omega^2_{ac} = d\omega^2_b.$$

Integration over Z_{ac} and using (8) yields $k = 0$. Hence, there is no direct relation between ω^2_{ac} and ω^2_b. Substituting (10) into (9) and integrate over Z_{ac} we infer

$$f^2_{ac} = e^1_c - e^1_a + e^R_c - e^R_a.$$

Note that from (8) and (10) one can deduce an expression for all shape function in terms of ω^1_a:

$$\omega^1_a = \omega^R_a = 1 - \omega^1_c = 1 - \omega^R_c \tag{11}$$

B. Third equation of motion

The third equation of motion is given by

$$f_R = de_2. \tag{12}$$

First we make the following observations. It is obvious that the first and third equation of (5) are equivalent. Therefore we chosen the same approximation for f_1 and f_R. Due to the dissipation relation $e_R = *rf_R$ it is clear that the shape functions have to fulfill the following condition

$$\omega^R_{ab} = *\omega^1_a, \quad \omega^R_{bc} = *\omega^1_c.$$

We additionally assume that ω^R_{ab} and ω^R_{bc} fulfill the following conditions

$$\int_{Z_{bc}} \omega^R_{ab} = \int_{Z_{ab}} \omega^R_{bc} = 0. \tag{13}$$

These conditions lead to the following relations

$$\int_{Z_{ab}} \omega^R_{ab} = b - a, \quad \int_{Z_{bc}} \omega^R_{bc} = c - b.$$

Now we can proceed with the spatial discretization of (12). As for the second equation of motion presented in Section V-A, we substitute the approximation of the effort and flow and assume that all the time depending scaling parameters are zero, except two of them — for example we assume that $f^R_{bc} = e^2_b = e^2_c = 0$. Then we obtain

$$f^R_{ab}\omega^R_{ab} = e^2_a d\omega^2_a.$$

If we now assume that $k_2 = -f^R_{ab} \cdot \left(e^2_a\right)^{-1}$ we conclude that

$$-k_2\omega^R_{ab} = d\omega^2_a.$$

The next step would be to integrate over Z_{ac} but since we chose ω^R_{ab} such that (13) is fulfilled we only integrate over Z_{ab}. This integration yields

$$-k_2 = \frac{-k_{aa}}{(b-a)}.$$

Hence, on Z_{ab} we have the following relation between ω^R_{ab} and ω^2_a

$$k_2\omega^R_{ab} = -d\omega^2_a.$$

Similarly, if we assume $k_{aa} = k_{bb}$ we obtain that on Z_{ab}

$$k_2\omega^R_{ab} = d\omega^2_b.$$

Furthermore, since $\int_{Z_{ab}} d\omega^2_c = 0$ we have no relation between ω^2_c and ω^R_{ab}.

Following the exactly same reasoning for $f^R_{bc} \neq 0$ we obtain that on Z_{bc} we have the following relations between the shape functions

$$\begin{aligned} k_3\omega^R_{bc} &= -d\omega^2_b = d\omega^2_c \\ k_3 &= \frac{k_{cc}}{(c-b)} \end{aligned}$$

and that there is no relation between ω^2_a and ω^R_{bc} on Z_{bc}. To simplify calculations we now assume that $k_{aa} = k_{bb} = k_{cc} = (b-a) = (c-b)$ then we have $k_2 = k_3 = 1$. This assumption can be ensured by enforcing that

$$\begin{aligned} \left(\int *\omega^1_a\right)(a) &= -(b-a) \\ \left(\int *\omega^1_a\right)(b) &= 0. \end{aligned} \tag{14}$$

The relations between the shape functions can then be summarized as follows

$$\begin{aligned} \omega^R_{ab} &= -d\omega^1_a = d\omega^1_b \text{ on } Z_{ab} \\ \omega^R_{bc} &= -d\omega^1_b = d\omega^1_c \text{ on } Z_{bc}. \end{aligned} \tag{15}$$

Substituting (15) into (12) yields that on Z_{ab} we have the following equations of motion

$$f^R_{ab}\omega^R_{ab} + f^R_{bc}\omega^R_{bc} = (e^2_b - e^2_a)\omega^R_{ab} + e^2_c d\omega^2_c.$$

Integration over Z_{ab} results in the following dynamics on Z_{ab}

$$f^R_{ab} = e^2_b - e^2_a.$$

In the same way we obtain that the dynamics on Z_{bc} can be described as

$$f^R_{bc} = e^2_c - e^2_b.$$

Additionally due to the constraint (14) on ω^1_a we are able express all shape functions in terms of ω^1_a

$$\begin{aligned} \omega^2_a &= \begin{cases} -\int *\omega^1_a & \text{on } Z_{ab} \\ 0 & \text{on } Z_{bc} \end{cases} \\ \omega^2_b &= \begin{cases} (b-a) + \int *\omega^1_a & \text{on } Z_{ab} \\ -z + c + \int *\omega^1_a & \text{on } Z_{bc} \end{cases} \\ \omega^2_c &= \begin{cases} 0 & \text{on } Z_{ab} \\ z - b - \int *\omega^1_a & \text{on } Z_{bc} \end{cases}. \end{aligned}$$

Note that the choice $\omega^2_a = 0$ on Z_{bc} is only to simplify the calculations. The same holds for the choice that $\omega^2_c = 0$ on Z_{bc}. The results of this section and Section V-A will then lead to the final approximation of the equations of motion.

C. Spatially discretized equations of motion

We now combine the results of Section V-A and V-B to state a finite dimensional version of (4) on Z_{ac}. In the last two sections we have determined that the finite dimensional approximation of (5) can be stated as

$$\begin{aligned} f^{1/R}_{ab} &= e^2_b - e^2_a \\ f^{1/R}_{bc} &= e^2_c - e^2_b \\ f^2_{ac} &= e^1_c - e^1_a + e^R_c - e^R_a. \end{aligned} \tag{16}$$

From the infinite dimensional system we know that $e_R = *rf_R$. Additionally we know, see Section 12, that e_R and f_R are approximated with the same shape functions. Hence, it must hold that

$$e^R_a = rf^R_{ab}, \quad e^R_c = rf^R_{bc}.$$

If we substitute this relation into (16) we obtain a spatial discretized version of (4) which can be stated as follows

$$\begin{aligned} f^1_{ab} &= e^2_b - e^2_a \\ f^1_{bc} &= e^2_c - e^2_b \\ f^2_{ac} &= e^1_c - e^1_a + r(e^2_a - 2e^2_b + e^2_c). \end{aligned} \tag{17}$$

Next we define a set of efforts such that one can formulate an input-state-output structure of a finite dimensional pH system.

D. Definition of efforts

We define the efforts of the finite dimensional pH system by using the net power of the system. For an infinite dimensional system with two states the net power is defined as

$$P_{net} = \int_Z (e_1 \wedge f_1 + e_2 \wedge f_2) + \int_{\partial Z} e_\partial \wedge f_\partial. \tag{18}$$

Substituting the approximation of the efforts and flows and integrating over Z_{ac} yields an approximation to the net power in the form of (3). For the simplicity of the explanation we will neglect the boundary values during the calculation,

$$\begin{aligned} P^{net}_{ac} &= \int_{Z_{ac}} \left(e^1_a\omega^1_a + e^1_b\omega^1_b + e^1_c\omega^1_c\right) \wedge \left(f^1_{ab}\omega^1_{ab} + f^1_{bc}\omega^1_{bc}\right) \\ &\quad + \left(e^2_a\omega^2_a + e^2_b\omega^2_b + e^2_c\omega^2_c\right) \wedge f^2_{ac}\omega^2_{ac} \\ &= e^1_{ab}f^1_{ab} + e^1_{bc}f^1_{bc} + e^2_{ac}f^2_{ac} \end{aligned}$$

were

$$e^1_{ab} = \alpha^1_{a,ab}e^1_a + \alpha^1_{b,ab}e^1_b + \alpha^1_{c,ab}e^1_c \tag{19a}$$

$$e^1_{bc} = \alpha^1_{a,bc}e^1_a + \alpha^1_{b,bc}e^1_b + \alpha^1_{c,bc}e^1_c \tag{19b}$$

$$e^2_{ac} = \alpha^2_{a,ac}e^2_a + \alpha^2_{b,ac}e^2_b + \alpha^2_{c,ac}e^2_c \tag{19c}$$

$$\alpha^l_{ij} = \int_{Z_{ac}} \omega^l_i \wedge \omega^l_j,\ i \in \{a,b,c\},\ j \in \{ab,bc,ac\} \tag{19d}$$

Using the just defined efforts we can now define the input-state-output structure.

E. Input-state-output structure

We choose the following input vector field $\tilde{\mathbf{u}} = [e^2_a, e^2_c, e^1_b]^\top$, note that this is different to the definition in Theorem 1 because we will show that the input $\tilde{\mathbf{u}}_3$ is obsolete. With this definition of the input vector field and the definition of the efforts (19a) we can express the first two equations of motions of (17) as follows

$$\begin{aligned} f^1_{ab} &= \frac{1}{\alpha^2_{b,ac}} e^2_{ac} - \left(1 + \frac{\alpha^2_{a,ac}}{\alpha^2_{b,ac}}\right) e^2_a - \frac{\alpha^2_{c,ac}}{\alpha^2_{b,ac}} e^2_c \\ f^1_{bc} &= -\frac{1}{\alpha^2_{b,ac}} e^2_{ac} + \frac{\alpha^2_{a,ac}}{\alpha^2_{b,ac}} e^2_a + \left(1 + \frac{\alpha^2_{c,ac}}{\alpha^2_{b,ac}}\right) e^2_c. \end{aligned}$$

To have a skew symmetric interconnection matrix J we need that we can find a parameter k such that

$$e^1_c - e^1_a = -\frac{1}{\alpha^2_{b,ac}} e^1_{ab} + \frac{1}{\alpha^2_{b,ac}} e^1_{bc} + k e^1_b$$

Hence, we need that

$$\frac{1}{\alpha^2_{b,ac}} \left(-\alpha^1_{a,ab} + \alpha^1_{a,bc}\right) = -1 \tag{20a}$$

$$\frac{1}{\alpha^2_{b,ac}} \left(-\alpha^1_{c,ab} + \alpha^1_{c,bc}\right) = 1 \tag{20b}$$

$$-\alpha^1_{b,ab} + \alpha^1_{b,bc} = -k. \tag{20c}$$

Because we are able to express all shape function in terms of ω^1_a it is clear we are able to express all α's in terms of $\alpha^1_{a,ab}$,

$$\begin{aligned} \alpha^1_{a,bc} = \alpha^1_{c,ab} &= -\alpha^1_{a,ba} + (b-a) \\ \alpha^1_{c,bc} = \alpha^2_{b,ac} &= \alpha^1_{a,ba} \\ \alpha^2_{a,ac} &= -\int_{Z_{ab}} \omega^1_a \wedge \omega^1_a + (b-a) \\ \alpha^2_{c,ac} &= -\int_{Z_{bc}} \omega^1_a \wedge \omega^1_a. \end{aligned}$$

Hence, (20a) and (20b) are equivalent. This means that we only have to ensure that

$$\frac{1}{\alpha^2_{b,ac}} \left(-\alpha^1_{c,ab} + \alpha^1_{c,bc}\right) = 1.$$

After simple calculations we obtain an additional condition for ω^1_a which ensures that (20) is fulfilled, namely

$$\alpha^1_{a,ab} = b - a.$$

Enforcing this extra condition results into $\alpha^1_{a,bc}$ and $\alpha^1_{c,ab}$ being zero. If we additionally assume that

$$\begin{aligned} \int_{Z_{ab}} \omega^1_a \wedge \omega^1_a &= (b-a) \\ \int_{Z_{bc}} \omega^1_a \wedge \omega^1_b &= \alpha^1_{b,ab} = \alpha^1_{b,cb} = 0 \end{aligned}$$

then the definition of our efforts (19) simplifies to

$$e^1_{ab} = (b-a)e^1_a, \quad e^1_{bc} = (b-a)e^1_c, \quad e^2_{ac} = (b-a)e^2_b. \tag{21}$$

Note that the following two shape functions fulfill all constraints we have enforced until now on ω^1_a and ω^1_b

$$\omega^1_a = \begin{cases} 1 & z \in Z_{ab} \\ 0 & z \in Z_{bc} \end{cases}, \quad \omega^1_b = 0.$$

With the new definition of the efforts (21) we can express (17) as

$$\begin{aligned} f^1_{ab} &= \frac{1}{(b-a)} e^2_{ac} - \tilde{\mathbf{u}}_1 \\ f^1_{bc} &= -\frac{1}{(b-a)} e^2_{ac} + \tilde{\mathbf{u}}_2 \\ f^2_{ac} &= -\frac{1}{(b-a)} e^1_{ab} + \frac{1}{(b-a)} e^1_{bc} + \frac{2r}{(b-a)} e^2_{ac} - r\tilde{\mathbf{u}}_1 - r\tilde{\mathbf{u}}_2 \end{aligned}$$

which is equivalent to (6). Next we have to show how to derive a finite dimensional expression of the energy function.

F. Discretization of the energy function

The discretization of the energy function is compared to the discretization of the input-state-output structure rather straightforward. As was discussed in Section (II) is the flow f of a pH system related to the state x of the pH system via the time derivative ($f = \dot{x}$). And because our shape functions are time independent it is clear that the approximation of the state has to be done in the same way as for the flows. Hence, we have the following approximation of the states,

$$\begin{aligned} x_1 &\approx x^1_{ab}\omega^1_{ab} + x^1_{bc}\omega^1_{bc} \\ x_2 &\approx x^2_{ac}\omega^2_{ac}. \end{aligned}$$

If we now substitute this approximation in the definition of our energy function $H(x)$ and then integrate over Z_{ac} we obtain the finite dimensional approximation of the energy function below

$$H_{ac}(x^1_{ab}, x^1_{bc}, x^2_{ac}) = \int_{Z_{ac}} \mathcal{H}(x^1_{ab}\omega^1_{ab} + x^1_{bc}\omega^1_{bc}, x^2_{ac}\omega^2_{ac}).$$

Moreover, since the efforts of the finite dimensional are defined as the gradient of the energy function with respect to the state it follows immediately that

$$\begin{aligned} e^1_{ab} &= \frac{\partial}{\partial x^1_{ab}} H_{ac}(x^1_{ab}, x^1_{bc}, x^2_{ac}) \\ e^1_{bc} &= \frac{\partial}{\partial x^1_{bc}} H_{ac}(x^1_{ab}, x^1_{bc}, x^2_{ac}) \\ e^2_{ac} &= \frac{\partial}{\partial x^2_{ac}} H_{ac}(x^1_{ab}, x^1_{bc}, x^2_{ac}). \end{aligned}$$

This result combined with (6) is the a finite dimensional approximation of the dynamics of (4) on Z_{ac} in pH form.

VI. Simulation results

In this section we present results from a numerical simulation of the dynamics of a vibrating string including structural damping [4]. We assume that all physical parameters are 1 and that the energy function is given as $H(p, \varepsilon) = \int_Z T * \varepsilon \wedge \varepsilon + \rho^{-1} * p \wedge p$, where ε is the strain, p is the momentum, T is the elastic modulus and ρ is the density of the string. A finite dimensional model of one finite element is given by (6). We choose a mesh of 21 finite elements and compare the results of the pH discretization scheme stated in Theorem 1 with a classical finite difference (FD) approach. We assume that the initial value for the strains is zero in all elements and that the initial value for all momenta is zero except for the 11th element where we choose an initial value of $500[Ns]$. To be able to see the diffusive effect we have set all inputs to zero, which means that there is no energy transfer over the boundaries and only dissipation in the spatial domain through structural damping. As

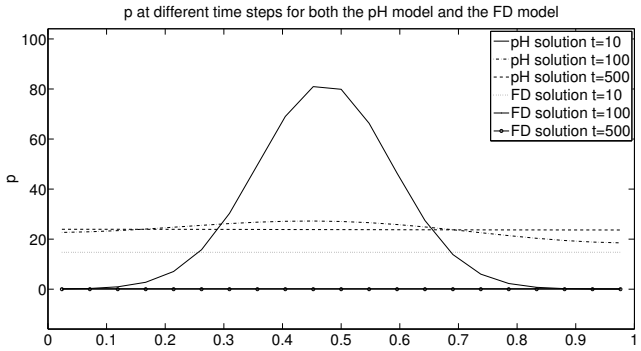

Fig. 1. Momenta for both methods at specific times

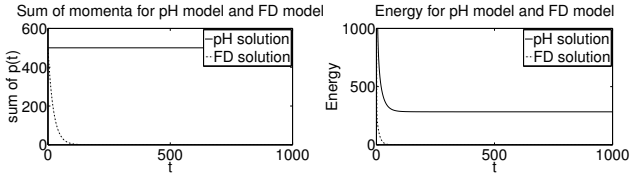

Fig. 2. a) Sum of Momenta for both methods b) Energy plot for the both methods

numerical solver we used the `ode23t` method in Matlab to ensure that no numerical damping is introduced.

In Figure 1 one can clearly see that the FD solution is not capturing the diffusive effect because it completely damps out any momenta for $t \to \infty$, while the pH system clearly captures the diffusive behavior and achieves an equal momentum for $t \to \infty$ as one would expect. The difference between the methods becomes even clearer in Figure 2a where we plot the sum of the momenta in all elements. For an autonomous system with diffusive dynamics one expects that the sum of all momenta will be equal for all t, because the diffusion results in an equalization of all momenta in all elements without dissipating any momenta. This is clearly the case for the pH model while the FD model totaly fails in preserving this equality because the sum of the momenta will be zero for $t \to \infty$. This can also be seen in the energy plot. Of course the diffusive effect will dissipate energy while achieving the equality of the momenta in all elements but only while equality is not achieved. After the equality has been achieved the system will be in rest and will not dissipate any energy. Again the pH model is able to capture this effect and the FD model fails again. So one can see that the derived finite dimensional pH model is far superior to the FD model when it comes to capturing the dynamics more accurately.

VII. Conclusions

We have shown how one can approximate the dynamics of a 1D port-Hamiltonian system with convective and diffusive effect on a given interval with a finite dimensional port-Hamiltonian system. The resulting finite dimensional approximation can then be used for various tasks. For example based on the finite dimensional model one could design a stabilizing controller by applying an energy based control scheme, e.g. [15], or one could use the so derived model for simulation purposes.

In future work we will expand the discretization to 2D port-Hamiltonian systems with diffusion.

References

[1] A. Macchelli, C. Melchiorri, and L. Bassi, "Port-based modelling and control of the Mindlin plate," in *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05*, 2005, pp. 5989–5994.

[2] J. Schröck, T. Meurer, and A. Kugi, "Motion planing for a Flexible Beam structure with Macro-Fiber Composite Acutators," in *Proceedings of the 10th European Control Conference*, 2009, pp. 2193–2198.

[3] T. Voß and J. Scherpen, "Stabilization and shape control of a 1-D piezoelectric Timoshenko beam," *accepted for publication in Automatica*.

[4] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx, Eds., *Modeling and Control of Complex Physical Systems*. Springer-Verlag Berlin Heidelberg, 2009.

[5] B. Maschke and A. van der Schaft, "Hamiltonian formulation of distributed parameter systems with boundary energy flow," *Journal of Geometry and Physics*, vol. 42, pp. 166 – 194, 2002.

[6] A. Macchelli, "Port Hamiltonian systems. A unified approach for modeling and control finite and infinite dimensional physical systems," Ph.D. dissertation, University of Bologna–DEIS, Bologna, Italy, 2003.

[7] O. Zienkiewicz and R. Taylor, *The Finite Element Method*. A Butterworth-Heinemann Title, 2005, vol. 6.

[8] R. Moulla, L. Lefére, and B. Maschke, "Geometric pseudo-spectral method for spatial integration of dynamical systems," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 17, no. 1, pp. 85–104, 2011.

[9] G. Golo, V. Talasila, A. van der Schaft, and B. Maschke, "Hamiltonian discretization of boundary control systems," *Automatica*, vol. 40, no. 5, pp. 757–771, 2004.

[10] R. Pasumarthy and A. van der Schaft, "A finite dimensional approximation of the shallow water equations: The port-Hamiltonian approach," in *Proceedings of the 45th IEEE Conference on Decision and Control*, 2006.

[11] T. Voß and J. Scherpen, "Structure preserving spatial discretization of a 1D piezoelectric Timoshenko beam," *SIAM Journal on Multiscale Modeling and Simulation*, vol. 9, pp. 129–154, 2011.

[12] A. Baaiu, F. Couenne, L. Lefevre, Y. L. Gorrec, and M. Tayakout, "Structure-preserving infinite dimensional model reduction: Application to adsorption processes," *Journal of Process Control*, vol. 19, no. 3, pp. 394 – 404, 2009. [Online]. Available: http://www.sciencedirect.com/science/article/B6V4N-4TKBP4K-1/2/9bf2bca8884dad55388a1623fc35c113

[13] A. Van der Schaft and B. Maschke, "The Hamiltonian formulation of energy conserving physical systems with external ports," *AEU. Archiv für Elektronik und Übertragungstechnik*, vol. 49, no. 5-6, pp. 362–371, 1995.

[14] M. Dalsmo and A. van der Schaft, "On the representation and integrability of mathematical structures in energy-conserving physical systems," *SIAM Journal On Control and Optimization*, vol. 37, pp. 54–91, 1998.

[15] R. Ortega, A. Van Der Schaft, B. Maschke, and G. Escobar, "Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems* 1," *Automatica*, vol. 38, no. 4, pp. 585–596, 2002.