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Adaptive consensus filters for collocated infinite dimensional systems

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***Abstract*— This paper considers a class of infinite dimensional systems with structured perturbation. Such a perturbation is assumed to be expressed in terms of the output operator and an unknown matrix. The proposed adaptive observers include a coupling term which penalizes the disagreement of the estimates. The enforcement of consensus is applied to both state and parameter estimates, thereby constituting the main contribution of this work. Due to the specific operator Lyapunov equation that the nominal plant operator satisfies, the convergence of the estimation errors along with the asymptotic convergence of the state and parameter deviations from the mean are established. Extensive simulation studies examine also the case of adapting the consensus gains, which describe the case where the consensus gain is adjusted according to the disagreement of the estimates.**

***Index Terms*— Infinite dimensional systems; adaptive estimation; multi-agent systems; consensus filters.**

I. Introduction

The goal of this work is to introduce adaptive estimation in multi-agent systems in which the underlying process is governed by a class of infinite dimensional systems. The infinite dimensional system under consideration assumes a structured perturbation in which the nominal plant operator generates an exponentially stable C_0 semigroup [1] with a prescribed decay rate. The additive perturbation is assumed to be parameterized by the output operator in a collocated fashion and multiplied by an unknown matrix gain that is desired to be identified on-line.

To aid in the on-line estimation of the process state and the matrix gain, a network of N agents is employed. Using the same output measurements, the N agents generate N interactive adaptive observers. The way that the adaptive observers interact is through a coupling term which penalizes the disagreement of the pairwise difference of all estimates. However, unlike earlier efforts primarily considered by the finite dimensional community [2], the penalization of mismatch is enforced on *both* the state and the parameter estimates. Using a metric for the agreement that is independent of the network topology, as was introduced in the finite dimensional case [3], it is shown that the penalty term in both the state and parameter estimates results in the deviations-from-the-mean-estimate to converge asymptotically to zero. In fact, under certain conditions, one can argue an exponential convergence of the deviations-from-the-mean-estimate to zero.

The rest of the manuscript is as follows: the class of systems under consideration along with a summary of the adaptive observer considered in [4] is presented in Section II. The proposed adaptive consensus observers are presented in Section III along with the relevant convergence results. The special case of the adaptation of the consensus gains, which considers the case wherein the penalty term is weighted by a gain that is adjusted adaptively and dependant on the disagreement of the estimates is considered in Section V. Extensive numerical studies are presented in Section V and conclusions follow in Section VI.

II. Class of Systems and their Adaptive Observers

The class of systems under consideration is described by the following evolution equation in a Hilbert space X

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + Bu(t) + \Delta A x(t), \quad x(0) = x_0 \in X, \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

and which defines systems with structured perturbation. The nominal operator A_0 is assumed to generate an exponentially stable C_0 semigroup $T(t)$, $t \geq 0$ on X [1], which satisfies the following trivial case of operator Lyapunov equation

$$A_0^* + A_0 \leq -\omega I, \quad \omega > 0. \tag{2}$$

Such a condition is typically satisfied by diffusion-advection partial differential equations. The input operator may or may not be collocated to the output operator, but the additive perturbation of the nominal plant operator A_0, given by

$$\Delta A = C^* \Gamma C, \tag{3}$$

assumes a collocated form, and which may represent a passive feedback loop. Of course, when $B = C^*$, we then have the familiar collocated input-output case. The operator $\Gamma \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is unknown and it is desired to be identified on-line using input and output signals. The input operator $B \in \mathcal{L}(\mathbb{R}^n, X)$ and the output operator $C \in \mathcal{L}(X, \mathbb{R}^m)$ may not necessary form a square system, unless the number of inputs n is equal to the number of outputs m. For the purpose of adaptive estimation, the number of inputs is not necessarily required to be the same as m, as long as the perturbation of the operator satisfies the specific structure given in (3)

The above is essentially the same class of systems considered in [4]. Summarizing the result of [4] on the adaptive observer of (1), we arrive at the following adaptive observer

$$\begin{aligned} \dot{\widehat{x}}(t) &= A_0 \widehat{x}(t) + Bu(t) + C^* \widehat{\Gamma}(t) y(t) \\ \widehat{x}(0) &\neq x(0), \end{aligned} \tag{4}$$

where $\widehat{\Gamma}(t)$ is the adaptive estimate of the unknown Γ, and $\widehat{x}(t)$ is the adaptive estimate of the plant state $x(t)$. Central to

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the derivation of adaptive parameter laws, is the appropriate Lyapunov function. Defining the state error $e(t) \triangleq \widehat{x}(t) - x(t)$ we have from (1), (2), (4)

$$\begin{aligned} \dot{e}(t) &= A_0 e(t) + C^* \left(\widehat{\Gamma}(t) - \Gamma\right) y(t) \\ e(0) &\neq 0. \end{aligned} \tag{5}$$

In addition to the state space X, we consider the parameter space Q defined as the space of bounded operators from $\mathbb{R}^m$ to $\mathbb{R}^m$ equipped with the Frobenius norm [5]. Using the following Lyapunov functional for the derivation of the adaptive law of $\widehat{\Gamma}(t)$

$$V(t) = |e(t)|_X^2 + \langle \Phi(t), \Phi(t) \rangle_Q \tag{6}$$

where $\Phi(t) \triangleq \widehat{\Gamma}(t) - \Gamma$ denotes the parameter error, we have

$$\langle \dot{\widehat{\Gamma}}(t), \Psi \rangle_Q = -\langle Ce(t), \Psi y(t) \rangle_{\mathbb{R}^m}, \quad \widehat{\Gamma}(0) \neq \Gamma, \tag{7}$$

where $\Psi \in Q$ is a test function. The above adaptation can be placed in a more familiar form

$$\dot{\widehat{\Gamma}}(t) = -\left(C\widehat{x}(t) - y(t)\right) \cdot y^T(t), \quad \widehat{\Gamma}(0) \neq \Gamma, \tag{8}$$

and which reveals that the proposed adaptation is feasible since it uses the available signals $C\widehat{x}(t)$ and $y(t)$. The time derivative of the Lyapunov functional given by

$$\dot{V}(t) \leq -\omega |e(t)|_X^2 \leq 0, \tag{9}$$

allows one to conclude that $|e(t)|_X$ is uniformly continuous and integrable, and along with the application of Barbălat's lemma [6], ensures that $\lim_{t\to\infty} |e(t)|_X = 0$, [4]. Convergence of $\widehat{\Gamma}(t)$ to Γ can be established with the additional condition of persistence of excitation [7].

III. Adaptive consensus filters

We now assume that there N agents, each of which can implement its own adaptive estimator as summarized in (4), (8) above. Further, it is assumed that each agent can interact and exchange information with the remaining $N-1$ agents, on both its own estimate of the state $x(t)$ and its estimate of the parameter Γ; this means that we have complete graph. The proposed adaptive consensus observers are given by

$$\begin{aligned} \dot{\widehat{x}}_i(t) &= A_0 \widehat{x}_i(t) + Bu(t) + C^* \widehat{\Theta}_i(t) y(t) \\ &\quad - \sum_{j=1}^N \left(\widehat{x}_i(t) - \widehat{x}_j(t)\right), \ \widehat{x}_i(0) \neq x(0), \\ \dot{\widehat{\Gamma}}_i(t) &= -\left(C\widehat{x}_i(t) - y(t)\right) \cdot y^T(t) \\ &\quad - \sum_{j=1}^N \left(\widehat{\Gamma}_i(t) - \widehat{\Gamma}_j(t)\right), \ \widehat{\Gamma}_i(0) \neq \Gamma, i = 1, \ldots, N. \end{aligned} \tag{10}$$

To examine the stability properties of the above adaptive observers, one must consider all of the adaptive estimates collectively. Towards this end, we define the individual state errors $e_i(t) \triangleq \widehat{x}_i(t) - x(t)$ and the individual parameter errors $\Phi_i(t) = \widehat{\Gamma}_i(t) - \Gamma$, $i = 1, \ldots, N$. One immediately has

$$\begin{aligned} \dot{e}_i(t) &= A_0 e_i(t) - \sum_{j=1}^N \left(e_i(t) - e_j(t)\right) + C^* \Phi_i(t) y(t) \\ \dot{\Phi}_i(t) &= -Ce_i(t) \cdot y^T(t) - \sum_{j=1}^N \left(\Phi_i(t) - \Phi_j(t)\right) \\ e_i(0) &\neq 0, \ \ \Phi_i(0) \neq 0, \ \ i = 1 \ldots, N. \end{aligned} \tag{11}$$

Using the above system, we can now state the first result.

Lemma 1: Given the infinite dimensional system (1) where the nominal operator A_0 generates and exponentially stable C_0 semigroup $T(t)$ on X with (2) satisfied, we consider the N interactive consensus filters (10). Then we have:

- If y is bounded, then all signals are bounded and

$$\lim_{t\to\infty} |e_i(t)|_X = 0, \quad i = 1, \ldots, N.$$

Proof: We consider the following Lyapunov functionals

$$V_i(t) = |e_i(t)|_X^2 + \langle \Phi_i(t), \Phi_i(t) \rangle_Q, \quad i = 1, \ldots, N. \tag{12}$$

Each one has a derivative along (11) given by

$$\begin{aligned} \dot{V}_i(t) \leq & -\omega |e_i(t)|_X^2 - 2\sum_{j=1}^N \langle e_i(t), (e_i(t) - e_j(t)) \rangle_X \\ & -2\sum_{j=1}^N \langle \Phi_i(t), (\Phi_i(t) - \Phi_j(t)) \rangle_Q, \ \ i = 1, \ldots, N. \end{aligned} \tag{13}$$

Since the collective error dynamics must be examined for assessing the stability of the adaptive observers, then we have

$$V(t) = \sum_{i=1}^N V_i(t)$$

which produces

$$\begin{aligned} \dot{V}(t) \leq & -\omega \sum_{i=1}^N |e_i(t)|_X^2 - 2\sum_{i=1}^N \sum_{j=1}^N \langle e_i(t), (e_i(t) - e_j(t)) \rangle_X \\ & -2\sum_{i=1}^N \sum_{j=1}^N \langle \Phi_i(t), (\Phi_i(t) - \Phi_j(t)) \rangle_Q. \end{aligned} \tag{14}$$

Using the following result for the disagreement potential [3],

$$2\sum_{i=1}^N \sum_{j\neq i}^N \langle a_i, a_i - a_j \rangle = \sum_{i=1}^N \sum_{j\neq i}^N |a_i - a_j|^2, \quad i = 1, \ldots, N,$$

we then have that

$$\begin{aligned} \dot{V}(t) \leq & -\omega \sum_{i=1}^N |e_i(t)|_X^2 - \sum_{i=1}^N \sum_{j\neq i}^N |e_i(t) - e_j(t)|_X^2 \\ & -\sum_{i=1}^N \sum_{j\neq i}^N |\Phi_i(t) - \Phi_j(t)|_Q^2. \end{aligned} \tag{15}$$

For shorthand notation, we define the pairwise errors

$$\begin{aligned} e_{ij}(t) &\triangleq e_i(t) - e_j(t), \\ &= \widehat{x}_i(t) - \widehat{x}_j(t) \triangleq \widehat{x}_{ij}(t) \end{aligned}$$

$$\begin{aligned}\Phi_{ij}(t) &\triangleq \Phi_i(t)-\Phi_j(t),\\ &= \widehat{\Gamma}_i(t)-\widehat{\Gamma}_j(t)\triangleq\widehat{\Gamma}_{ij}(t).\end{aligned}$$

This then simplifies the Lyapunov derivative

$$\dot V(t)\le -\omega\sum_{i=1}^N|e_i(t)|_X^2-\sum_{i=1}^N\sum_{j\neq i}^N|e_{ij}(t)|_X^2-\sum_{i=1}^N\sum_{j\neq i}^N|\Phi_{ij}(t)|_Q^2.$$

Similar results to the single adaptive observer are obtained, namely that each $|\Phi_i(t)|_Q$ is uniformly continuous, that each $|e_i(t)|_X^2$ is uniformly continuous and integrable and that

$$\lim_{t\to\infty}|e_i(t)|_X=0,\quad i=1,\ldots,N.$$

Additionally, we have also $|e_{ij}(t)|_X^2$ and $|\Phi_{ij}(t)|_Q^2$ uniformly continuous and integrable. ■

The last result on the pairwise errors provides the necessary conditions for the agreement of the parameter estimates $\widehat{\Gamma}_i(t)$ and the state estimates $\widehat{x}_i(t)$. A way to assess the agreement of the adaptive observers, as was originally proposed for the finite dimensional case [3], is to consider a meaningful metric that is independent of the network topology [3]. We define the *deviation from the mean estimate* for the estimate of the state $x(t)$ and of the unknown parameter Γ as follows

$$\delta_i(t)\triangleq\widehat{x}_i(t)-\frac{1}{N}\sum_{j=1}^N\widehat{x}_j(t),\quad i=1,\ldots,N,\tag{16}$$

and

$$\zeta_i(t)\triangleq\widehat{\Gamma}_i(t)-\frac{1}{N}\sum_{j=1}^N\widehat{\Gamma}_j(t),\quad i=1,\ldots,N.\tag{17}$$

This then leads to the following result on the agreement of the adaptive estimates.

Lemma 2: Consider the N adaptive observers given by (10) and assume that the conditions in Lemma 1 are satisfied. Then we have that both pairwise state and parameter errors converge to zero

$$\lim_{t\to\infty}|e_{ij}(t)|_X=0,\quad \lim_{t\to\infty}|\Phi_{ij}(t)|_Q=0,\quad i,j=1\ldots,N$$

A consequence of the above is that the deviations also converge to zero

$$\lim_{t\to\infty}|\delta_i(t)|_X=0,\quad \lim_{t\to\infty}|\zeta_i(t)|_Q=0,\quad i,j=1,\ldots,N.$$

Proof: To show convergence of the deviations, which would demonstrate that all estimates agree with each other, we relate the deviations to the pairwise errors via

$$\begin{aligned}\delta_i(t) &= \widehat{x}_i(t)-\frac{1}{N}\sum_{j=1}^N\widehat{x}_j(t)\\ &= \frac{1}{N}\sum_{j=1}^N(\widehat{x}_i(t)-\widehat{x}_j(t))=\frac{1}{N}\sum_{j=1}^N\widehat{x}_{ij}(t)\\ &= \frac{1}{N}\sum_{j=1}^N e_{ij}(t)\end{aligned}\tag{18}$$

and

$$\begin{aligned}\zeta_i(t) &= \widehat{\Gamma}_i(t)-\frac{1}{N}\sum_{j=1}^N\widehat{\Gamma}_j(t)\\ &= \frac{1}{N}\sum_{j=1}^N(\widehat{\Gamma}_i(t)-\widehat{\Gamma}_j(t))=\frac{1}{N}\sum_{j=1}^N\widehat{\Gamma}_{ij}(t)\\ &= \frac{1}{N}\sum_{j=1}^N\Phi_{ij}(t).\end{aligned}\tag{19}$$

From

$$\dot V(t)\le -\omega\sum_{i=1}^N|e_i(t)|_X^2-\sum_{i=1}^N\sum_{j\neq i}^N|e_{ij}(t)|_X^2-\sum_{i=1}^N\sum_{j\neq i}^N|\Phi_{ij}(t)|_Q^2,$$

we have that each $|\Phi_i(t)|_Q$ is uniformly continuous and that each $|e_i(t)|_X^2$ is uniformly continuous and integrable. Now using the fact that

$$\lim_{t\to\infty}|e_i(t)|_X=0,\quad i=1,\ldots,N,$$

then we also have that

$$\lim_{t\to\infty}|e_{ij}(t)|_X=0,\quad i,j=1,\ldots,N.$$

From (18), the fact that $N<\infty$ and use of triangle inequality, yields

$$\lim_{t\to\infty}|\delta_i(t)|_X=0,\quad i=1,\ldots,N.$$

Now, from (11)

$$\dot\Phi_i(t)=-Ce_i(t)\cdot y^T(t)-\sum_{j=1}^N\Phi_{ij}(t).$$

The fact that $e_i(t)$ is uniformly continuous and that the plant output y is bounded along with the fact that $\Phi_{ij}(t)$ is uniformly continuous allows one to conclude that $\dot\Phi_i(t)$ is uniformly continuous. Similarly

$$\dot\Phi_{ij}(t)=\dot\Phi_i(t)-\dot\Phi_j(t),\ \ i,j=1,\ldots,N,$$

is also uniformly continuous. Application of Barbălat's lemma then yields

$$\lim_{t\to\infty}|\Phi_{ij}(t)|_Q=0,\quad i,j=1,\ldots,N.$$

Equation (19) then provides the requisite convergence

$$\lim_{t\to\infty}|\zeta_i(t)|_Q=0,\quad i=1,\ldots,N.$$

■

Remark 1: The results of Lemma 2 show that all the estimates, both state and parameter, are *cohesive*, i.e. they asymptotically converge

$$\begin{aligned}\widehat{x}_1(t)&=\widehat{x}_2(t)=\ldots=\widehat{x}_N(t),\\ \widehat{\Gamma}_1(t)&=\widehat{\Gamma}_2(t)=\ldots=\widehat{\Gamma}_N(t).\end{aligned}$$

However, only state estimates converge to the true state via

$$\lim_{t\to\infty}|e_i(t)|_X=0,\quad i=1,\ldots,N.$$

Remark 2: The results in Lemma 2 can be strengthened when one considers the pairwise errors in both the state and the parameter errors. From (11) one has

$$\dot{e}_{ij} = A_0 e_{ij}(t) - N e_{ij}(t) + C^* \Phi_{ij}(t) y(t)$$
$$\dot{\Phi}_{ij}(t) = -C e_{ij}(t) \cdot y^T(t) - N \Phi_{ij}(t).$$

From (12), one can define the pairwise Lyapunov function $V_{ij}(t) = |e_{ij}(t)|_X^2 + |\Phi_{ij}(t)|_Q^2$. The derivative of $V_{ij}(t)$ becomes

$$\dot{V}_{ij}(t) \leq -\omega |e_{ij}(t)|_X^2 - N|e_{ij}(t)|_X^2 - N|\Phi_{ij}(t)|_Q^2.$$

Using Bellman-Gronwal Lemma [8], one can argue the exponential convergence of the pairwise errors and via (16), (17), the exponential convergence of the deviations-from-the-mean-estimates (18), (19) in their respective norms.

IV. Adaptation of consensus weights

In the proposed adaptive observer, the consensus was enforced in both the state and parameter estimation equations. However, both penalty terms, described by the summations in (11), had a uniform-in-time weight. One may elect to adjust the consensus weight adaptively. This would describe the case that as the estimates tend to agree with each other, then the penalty of their disagreement must adjust accordingly. Indeed, one way to ensure that the consensus gain is adjusted according to the distance between any two estimates is to utilize adaptively adjusted consensus gains. We consider the following

$$\begin{aligned}\dot{\widehat{x}}_i(t) &= A_0 \widehat{x}_i(t) + Bu(t) + C^* \widehat{\Theta}_i(t) y(t) \\ &\quad - \eta_i(t) \sum_{j=1}^N \left(\widehat{x}_i(t) - \widehat{x}_j(t)\right), \;\; \widehat{x}_i(0) \neq x(0), \\ \dot{\widehat{\Gamma}}_i(t) &= -\left(C\widehat{x}_i(t) - y(t)\right) \cdot y^T(t) \\ &\quad - \theta_i(t) \sum_{j=1}^N \left(\widehat{\Gamma}_i(t) - \widehat{\Gamma}_j(t)\right), \;\; \widehat{\Gamma}_i(0) = \Gamma.\end{aligned}$$

where $\eta_i(t)$ and $\theta_i(t)$ are the adaptive consensus gains. To derive the adaptation rules of these gains, we follow a procedure similar to the previous case. The associated error equations are given by

$$\dot{e}_i(t) = A_0 e_i(t) - \eta_i(t) \sum_{j=1}^N e_{ij}(t) + C^* \Phi_i(t) y(t), \;\; e_i(0) \neq 0,$$

$$\dot{\Phi}_i(t) = -C e_i(t) \cdot y^T(t) - \theta_i(t) \sum_{j=1}^N \Phi_{ij}(t), \;\; \Phi_i(0) \neq 0.$$

Using the following Lyapunov functionals

$$W_i(t) = |e_i(t)|_X^2 + \langle \Phi_i(t), \Phi_i(t) \rangle_Q + \eta_i^2(t) + \theta_i^2(t),$$

for $i = 1, \ldots N$, we arrive at

$$\begin{aligned}\dot{W}_i(t) &\leq -\omega |e_i(t)|_X^2 - 2\eta_i(t) \sum_{j=1}^N \langle e_i(t), e_{ij}(t) \rangle_X + 2\dot{\eta}_i(t)\eta_i(t) \\ &\quad - 2\theta_i(t) \sum_{j=1}^N \langle \Phi_i(t), \Phi_{ij}(t) \rangle_Q + 2\dot{\theta}_i(t)\theta_i(t)\end{aligned}$$

The choices

$$\left\{\begin{aligned} \dot{\eta}_i(t) &= \sum_{j=1}^N \langle e_i(t), e_{ij}(t) \rangle_X \\ \dot{\theta}_i(t) &= \sum_{j=1}^N \langle \Phi_i(t), \Phi_{ij}(t) \rangle_Q \end{aligned}\right. , \;\; i = 1, \ldots, N$$

produce

$$\dot{W}_i(t) \leq -\omega |e_i(t)|_X^2 \leq 0, \quad i = 1, \ldots, N$$

The above only provides the convergence of the state errors $|e_i(t)|_X$ to zero. The reason is that the adaptation rules eliminate the presence of the pairwise errors in $\dot{W}_i(t)$ thereby removing their integrability property. However, the fact that $|e_i(t)|_X$ goes to zero allows one to conclude that $|e_{ij}(t)|_X$ also goes to zero and therefore one may achieve agreement of the state estimators

$$\lim_{t\to\infty} |\delta_i(t)|_X = 0, \quad i = 1, \ldots, N.$$

To incorporate the benefits of the convergence to the mean estimate and the adaptation of the consensus weights, one may consider adaptation of the consensus gains in the state error equation only. This is summarized in the lemma below.

Lemma 3: Consider the infinite dimensional system (1) with $y \in L_\infty$. If adaptation of the consensus weights is required without any requirements on the convergence of the disagreement of both the state and parameter estimates, then the following adaptive observer

$$\begin{aligned}\dot{\widehat{x}}_i(t) &= A_0 \widehat{x}_i(t) + Bu(t) + C^* \widehat{\Gamma}_i(t) y(t) - \eta_i(t) \sum_{j=1}^N \widehat{x}_{ij}(t), \\ \dot{\widehat{\Gamma}}_i(t) &= -\left(C\widehat{x}_i(t) - y(t)\right) \cdot y^T(t) - \theta_i(t) \sum_{j=1}^N \widehat{\Gamma}_{ij}(t), \\ \dot{\eta}_i(t) &= \sum_{j=1}^N \langle e_i(t), e_{ij}(t) \rangle_X, \;\; \eta_i(0) = \eta_{i0}, \\ \dot{\theta}_i(t) &= \sum_{j=1}^N \langle \Phi_i(t), \Phi_{ij}(t) \rangle_Q, \;\; \theta_i(0) = \theta_{i0}, \\ &\widehat{x}_i(0) \neq x(0), \; \widehat{\Gamma}_i(0) \neq \Gamma, \; i = 1, \ldots, N,\end{aligned}$$

ensures that $\lim_{t\to\infty} |e_i(t)|_X = 0$, $i = 1, \ldots, N$, and all other signals are bounded. Additionally, all the deviations of the state estimates from the mean estimate converge

$$\lim_{t\to\infty} |\delta_i(t)|_X = 0, \;\; i = 1, \ldots, N.$$

If instead only the adaptation on the state consensus gain is activated, the following adaptive observer

$$\dot{\widehat{x}}_i(t) = A_0\widehat{x}_i(t) + Bu(t) + C^*\widehat{\Gamma}_i(t)y(t) - \eta_i(t)\sum_{j=1}^{N}\widehat{x}_{ij}(t)$$

$$\dot{\widehat{\Gamma}}_i(t) = -\left(C\widehat{x}_i(t) - y(t)\right)\cdot y^T(t) - \sum_{j=1}^{N}\widehat{\Gamma}_{ij}(t)$$

$$\dot{\eta}_i(t) = \sum_{j=1}^{N}\langle e_i(t), e_{ij}(t)\rangle_X, \;\; \eta_i(0) = \eta_{i0},$$

$$\widehat{x}_i(0) \neq x(0),\; \widehat{\Gamma}_i(0) \neq \Gamma,\; i = 1,\ldots,N,$$

ensures that $\lim_{t\to\infty}|e_i(t)|_X = 0$, $i = 1,\ldots,N$, and both deviations of the estimates from the mean estimate converge

$$\lim_{t\to\infty}|\delta_i(t)|_X = 0, \;\; \lim_{t\to\infty}|\zeta_i(t)|_Q = 0, \quad i = 1,\ldots,N.$$

Proof: The proof follows from the arguments made above. However, the proof for the latter part of the lemma is summarized here. Using a Lyapunov functional for the second case

$$Y_i(t) = |e_i(t)|_X^2 + \langle\Phi_i(t),\Phi_i(t)\rangle_Q + \eta_i^2(t), \quad i = 1,\ldots,N$$

we arrive at

$$\dot{Y}_i(t) \leq -\omega|e_i(t)|_X^2 - \sum_{i=1}^{N}\sum_{j\neq i}^{N}|\Phi_{ij}(t)|_Q^2, \quad i = 1,\ldots,N$$

which yields the uniform continuity of $|e_i(t)|_X$, $|\Phi_i(t)|_Q$ and $\eta_i(t)$. Additionally, we have integrability of $|e_i(t)|_X^2$ and $|\Phi_{ij}(t)|_Q^2$ which provide the requisite convergence. ■

V. Numerical Examples

To demonstrate the proposed consensus adaptive filter, we consider the following diffusion equation with collocated input and output

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial \xi^2} + b(\xi)u(t);\;\; x(0,t) = 0 = x(1,t),\;\; x(\xi,0) = x_0(\xi),$$

$$y(t) = \int_0^1 b(\xi)x(\xi,t)\,d\xi,$$

where $b \in L^2(0,1) = X$. The input distribution function is taken as $b(\xi) = 1$, on $[0,\frac{1}{2})$ and $b(\xi) = 0$ elsewhere. We let $\mathcal{D}(A_0) = \{h \in L^2(0,1) : h, \frac{dh}{d\xi}$ are absolutely continuous, $\frac{d^2h}{d\xi^2} \in L^2(0,1)$ and $h(0) = 0 = h(1)\}$ and define $A_0 h = \frac{d^2h}{d\xi^2}$ for $h \in \mathcal{D}(A_0)$. Then A_0 has compact resolvent, eigenvalues $\lambda_n = -n^2\pi^2$, $n \in \mathbb{N}$ and eigenvectors $\phi_n = \sqrt{2}\sin(n\pi\xi)$, $n \in \mathbb{N}$, which form an orthonormal basis for $L^2(0,1)$. A_0 is exponentially stable, self-adjoint and for $x \in \mathcal{D}(A_0)$, $\langle x, A_0 x\rangle \leq -\|x\|^2$, and it generates a contraction semigroup. As a consequence, equation (2) is satisfied with $\omega = 2$. Additionally, $y(t) = \langle b, x(\cdot,t)\rangle = Cx(\cdot,t)$ and $B = C^*$, i.e. we have collocated input and output.

A finite element Galerkin approximation scheme based on spline elements [9] with 50 elements was used for the spatial discretization of the PDE. The resulting finite dimensional ODE systems were integrated in time using a Fehlberg fourth-fifth Runge-Kutta method in the Fortran® code `rkf45.f`.

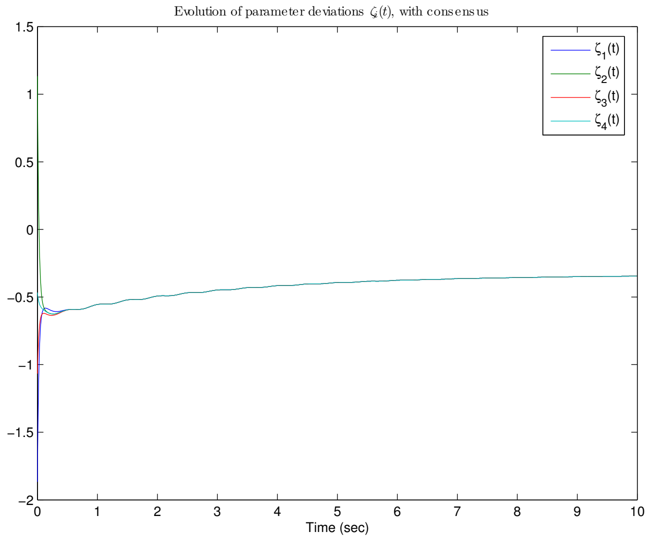

Fig. 1. Case 1: Evolution of parameter deviations with consensus.

A. Case 1: consensus filters with 4 agents

The number of agents was taken to be $N = 4$. The initial condition for the state was taken as $x(\xi,0) = \sin(\pi\xi)$, whereas the initial guesses for the four state estimates were $\widehat{x}_1(\xi,0) = \cos(2\pi\xi) - 1$, $\widehat{x}_2(\xi,0) = \sin(2\pi\xi)$, $\widehat{x}_3(\xi,0) = 0.5(1-\cos(2\pi\xi))$, $\widehat{x}_4(\xi,0) = 0$, $0 \leq \xi \leq 1$. The unknown gain was chosen as $\Gamma = 1$ and the initial guesses for the four adaptive estimates were $\widehat{\Gamma}_1(0) = 0.4$, $\widehat{\Gamma}_2(0) = 3.4$, $\widehat{\Gamma}_3(0) = 1.2$, $\widehat{\Gamma}_4(0) = 1.8$, with an adaptive gain of 10 used in (10).

The measure of disagreement $\zeta_i(t)$ is presented in Figures 1 and 2 with and without consensus, where one may observe the fast convergence of the parameter deviations when consensus is enforced..

Figures 3 and 4 depict the evolution of the output estimation error $y(t) - \widehat{y}_i(t)$ and the output state deviations given by $C\delta_i(t)$. In both cases, it can be observed that when consensus is enforced, then the estimation errors converge to zero faster.

B. Case 2: consensus filters with 3 agents and adaptation of consensus gains

Following Lemma 3, we consider the adaptation of the consensus gains $\eta_i(t)$. Using the same infinite dimensional system with only three agents–where for initial conditions of $\widehat{x}_i(t)$ and $\widehat{\Gamma}_i(t)$, we consider the first three from the previous case–we simulated it with $\eta_i(0) = 1$, $i = 1,2,3$. Figure 5 depicts the evolution of the three adaptive estimates of the consensus gains $\eta_i(t)$. It is observed that the gains settle to a value less than one when the disagreement between the estimates gets smaller.

VI. Conclusions

We have considered a special case of structurally perturbed infinite dimensional systems and proposed adaptive consensus observers to estimate both the unknown state and the unknown parameters in the structured perturbation. The use of consensus enforcement which penalized the disagreement

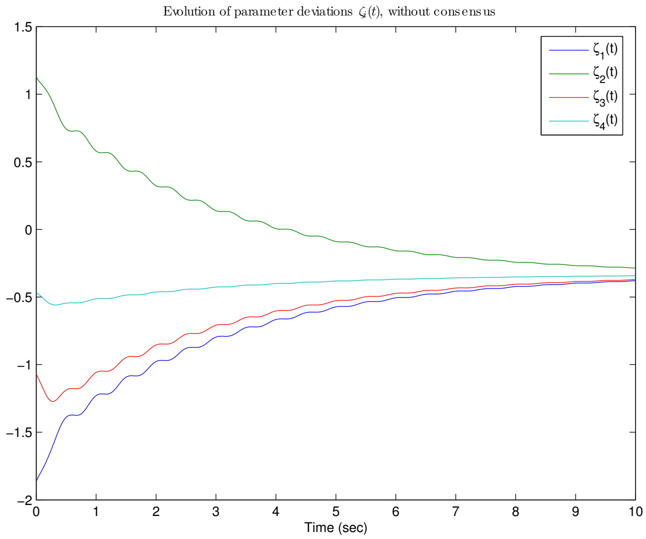

Fig. 2. Case 1: Evolution of parameter deviations without consensus.

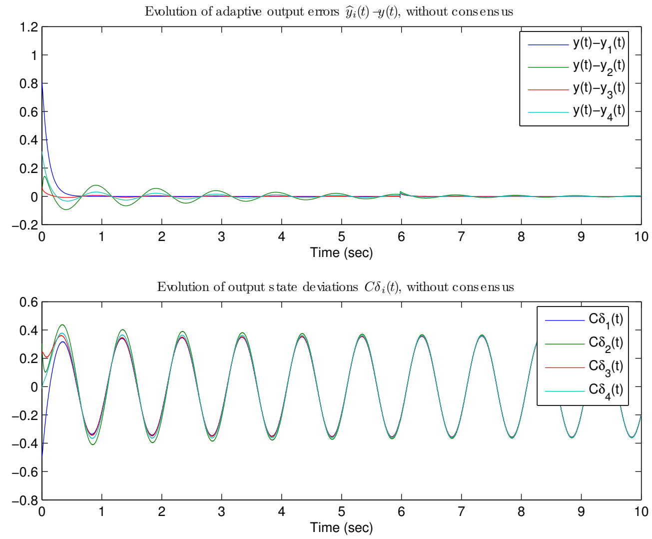

Fig. 4. Case 1: Evolution of state deviations without consensus.

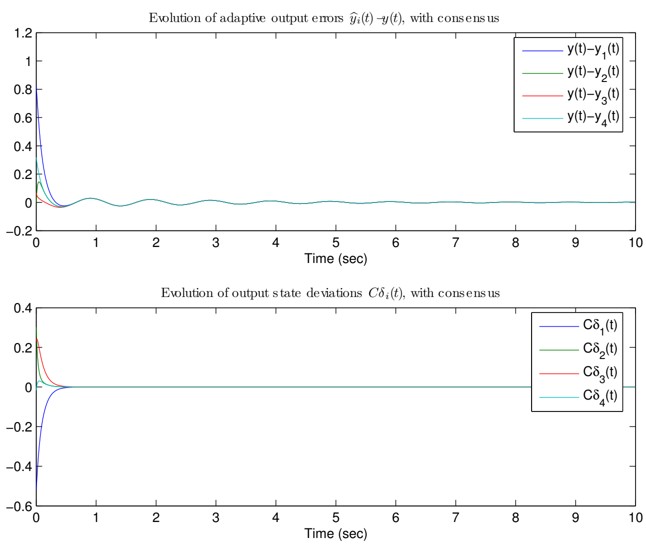

Fig. 3. Case 1: Evolution of state deviations with consensus.

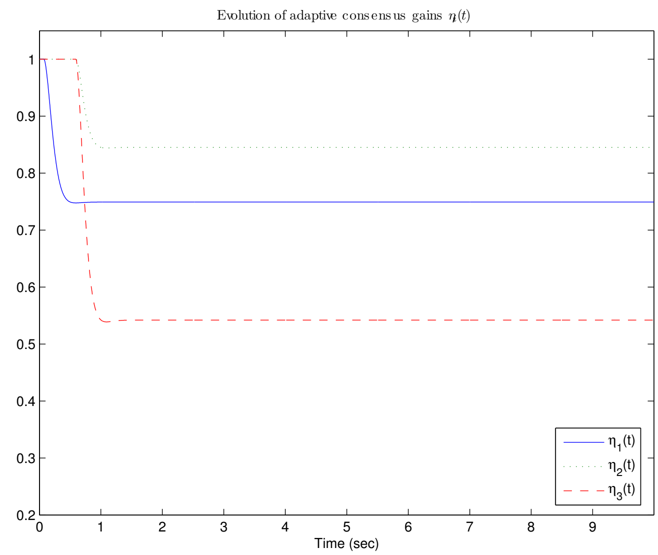

Fig. 5. Case 2: Evolution of consensus adaptive gains.

between the estimates, ensured that the deviation-from-the-mean-estimate converges to zero. Simulation studies showed that such convergence is exponential, as predicted from Remark 2.

A class of systems where the nominal plant operator may satisfy a more general version of the operator Lyapunov equation is warranted, as it would allow non-collocated systems to be considered. However this may come at a price, as the solutions to Lyapunov equations may not be coercive and therefore may weakened the stability arguments for both the state errors and the deviations-from-the-mean-estimate. These are currently being pursued by the author.

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