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# Stochastic Approximation Algorithm with Application to Event-triggered Filtering

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***Abstract*— This paper is focused on the problem of event-triggered filtering which has various applications such as sensor networks and data sampling/acquisition. An estimator may get a "sparse" sequence of observations: the observations may arrive only when some events trigger the sensor. In this paper, a series of stopping times is used to model the times when the sensors are triggered. Based on this model, a filtering problem is formulated as to estimate the true state of the dynamic system using the information from both the new observations and their corresponding stopping times. This filtering problem is numerically solved by a stochastic approximation algorithm which uses a Markov chain to approximate the evolution of the system.**

## I. INTRODUCTION

The theory of stochastic filtering has various applications in target tracking [1], financial engineering [2], fault detection and isolation [3], etc. In the "conventional" filtering theory, the observations are often assumed to be available at each sampling time. However, this is not the case in many applications. For example, in many space projects, e.g., the NASA Phoenix mission [4], the data link resources between a ground control station and a space probe are very limited. To transmit observations from a probe back to the Earth continuously in time may exhaust the bandwidth of a data link. Thus, to tradeoff the limited communication resources and estimation accuracy, a better choice is to send out observations at some "critical" time when some events happen (see also Figure 1). Other event-triggered observation/filtering models can be found in [5], [6], [7], [8].

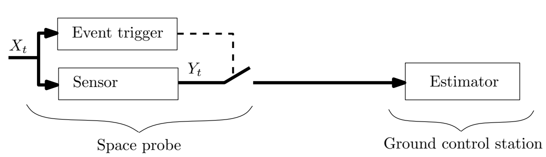


Fig. 1. Illustration of triggered observations of a space probe.

This paper presents our initial results in the event-triggered filtering problem. A mathematical model is proposed in which new observations are available when a set of sensors are triggered by some events. The times when the sensors are triggered are modeled by *stopping times*. The filtering problem is to use the information from both the stopping times and the observations at the stopping times to estimate the true state of the system. To solve this filtering problem, the probability distribution function (pdf) of the system's state conditioned on both the stopping times and corresponding observations needs to be computed. This is done via two steps:

- *Propagation:* Between two consecutive stopping times, the pdf of the system's state can be propagated given that the next stopping time has not arrived.
- *Correction:* When the next stopping time arrives, the pdf of the system's state is corrected according to the stopping time and new observation.

In this paper, the event-triggered filtering problem is numerically solved by a stochastic approximation algorithm. The proposed algorithm uses a Markov chain defined on discretized time and state space to approximates the evolution of the pdf of the system's state conditioned on the stopping times and corresponding observations. Unlike the approach presented in [9] which approximates the pdf of the system's state using a series of eigenfunctions, the stochastic approximation approach approximates the whole stochastic process weakly, i.e., the distribution of the Markov chain converges weakly to the distribution of the system's state when its grid size and time step approach zero. Thus, it can be guaranteed that the pdf approximation error is bounded at each time and the error is not propagated unboundedly over time. The proposed algorithm is validated through simulations.

This paper is organized as follows: Section II presents the mathematical formulation and theoretical solution to the event-triggered filtering problem; a stochastic approximation algorithm is presented in Section III to solve the event-triggered filtering problem numerically; Section IV shows the simulation results, and conclusions are given in Section V.

## II. MATHEMATICAL FORMULATION AND SOLUTION TO THE EVENT-TRIGGERED FILTERING PROBLEM

In this section, a mathematical model characterizing the problem of event-triggered filtering is presented together with a theoretical solution to this problem.

### A. Notations in this paper

Denote by: $\mathbb{R}^M$, the $M$-dimensional Euclidean space; $\mathbb{R}^+$, the positive real numbers; $\mathbb{Z}$, the integers; $\mathbb{N}$, the non-negative integers; $\|\bullet\|$, the Euclidean norm; $\mathbb{E}[\bullet]$, the (conditional) expectation; $\mathbb{P}$, the generic probability measure; $\Pr\{\bullet\}$, the (conditional) probability of an event; $p(\bullet)$ and $\pi(\bullet)$, the (conditional) probability density function (pdf).

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*B. Mathematical Formulation*

*1) System Dynamics:* Let $\mathbf{X}_t = (X_t^{(1)}\ X_t^{(2)})^T \in \mathbb{R}^2$, $(\forall t \in \mathbb{R}^+)$ is a 2-dimensional Markov stochastic process in which $X_t^{(1)}$ is a dynamic process and $X_t^{(2)}$ is a observation noise process. The stochastic differential equation (sde) governing the evolution of $\mathbf{X}_t$ is given by:

$$d\mathbf{X}_t = \mathbf{f}(\mathbf{X}_t, t)dt + \mathbf{g}(\mathbf{X}_t, t)\mathbf{W}_t \tag{1}$$

where $\mathbf{f} : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2$ is a drift term; $\mathbf{g} : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2 \times \mathbb{R}^2$ is a volatility term; and $\mathbf{W}_t = (W_t^{(1)}\ W_t^{(2)})^T$ is the standard 2-dimensional Brownian motion. Suppose $\mathbf{f}$ and $\mathbf{g}$ are bounded and continuous such that the solutions of (1) exists and is unique [10].

*2) Observation Model:* The dynamics of $X_t^{(1)}$ can be regarded as a particle moving on $\mathbb{R}$ on which the sensors are assumed to be evenly distributed[1] with a separation distance $\gamma > 0$. Then, the $k$-th sensor's location is $k\gamma, (k \in \mathbb{Z})$. When the particle $X_t^{(1)}$ gets close to any sensor, the sensor sends out its position to a centralized estimator. However, due to uncertainties, each sensor's observations are corrupted by some noise which is characterized by $X_t^{(2)}$, an ergodic process with a small magnitude. To describe this mathematically, define:

$$Y_t := X_t^{(1)} + X_t^{(2)} \tag{2}$$

If $Y_t = k\gamma$, the $k$-th sensor sends out its position to a centralized estimator. Thereby, the centralized estimator has the *noisy information* about when the particle passes which sensor. For $k, i \in \mathbb{Z}$, define a series of stopping times and the corresponding observations:

$$\begin{cases} \tau_i := \inf\{t : Y_t = k\gamma\} \\ \hat{Y}_i := \frac{Y_{\tau_i}}{\gamma} \end{cases} \tag{3}$$

with

$$\begin{cases} \tau_0 \equiv 0 \\ \tau_{i_1} < \tau_{i_2} \quad \forall i_1, i_2 \in \mathbb{N} \text{ and } i_1 < i_2 \\ \hat{Y}_{i_1} \neq \hat{Y}_{i_2} \quad \forall i_1, i_2 \in \mathbb{N} \text{ and } |i_1 - i_2| = 1 \end{cases} \tag{4}$$

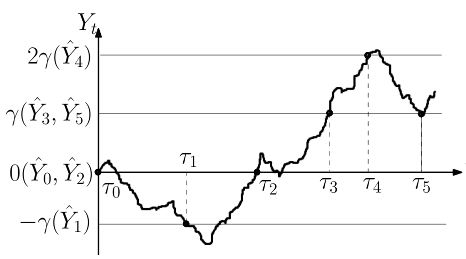


Fig. 2. Illustration of the event-triggered filtering principle.

*Example 1: Figure 2 illustrates the definitions of $\tau_i$ and $\hat{Y}_i$. The curve shown in Figure 2 is a realization of $Y_t$. The time when $Y_t$ crosses a straight line such that $Y_t = k\gamma$ are characterized as the stopping time $\tau_i$ and the observations are given by (3) with $\hat{Y}_0 = 0, \hat{Y}_1 = -1, \hat{Y}_2 = 0, \hat{Y}_3 = 1, \hat{Y}_4 = 2$ and $\hat{Y}_5 = 1$. Also, Equation (4) states that two consecutive observations must be different. Hence, although $Y_t$ hits the sensor located at 0 twice in the time interval $[\tau_0, \tau_1]$, only the first hit is recorded.*

Define the information generated by $\{\tau_i\}$ and $\{\hat{Y}_i\}$ upto the current time $t$ as:

$$\mathscr{G}_t^\gamma = \sigma(\{\tau_i\}, \{\hat{Y}_i\}) \quad \text{with } i = 1, ..., N \text{ s.t. } \tau_N \leq t < \tau_{N+1} \tag{5}$$

Thus the event-triggered filtering problem can be generalized as a computation of the conditional Markov process:

$$\mathbb{E}[X_t^{(1)} | \mathscr{G}_t^\gamma] \tag{6}$$

*C. Solution to the Event-triggered Filtering Problem*

Let $\mathscr{L}$ be the infinitesimal generator of $\mathbf{X}_t$. Its adjoint is given by:

$$\mathscr{L}^* := -\sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha} \mathbf{f}_\alpha(\mathbf{x}, t) + \frac{1}{2} \sum_{\alpha,\beta=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \mathbf{g}_{\alpha\beta}(\mathbf{x}, t) \tag{7}$$

Let $p_{t,t_0}(\mathbf{x}|\mathbf{x}_0)$ be the transition probability for any $t \geq t_0$. The Fokker-Planck equation governing the evolution of $p_{t,t_0}(\mathbf{x}|\mathbf{x}_0)$ can be written as:

$$\begin{aligned} \frac{\partial}{\partial t} p_{t,t_0}(\mathbf{x}|\mathbf{x}_0) &= \mathscr{L}^* p_{t,t_0}(\mathbf{x}|\mathbf{x}_0) \\ p_{t_0,t_0}(\mathbf{x}|\mathbf{x}_0) &= \delta(\mathbf{x} - \mathbf{x}_0) \end{aligned} \tag{8}$$

where $\delta(\bullet)$ is the Dirac $\delta$-function. By the strong Markov property of $\mathbf{X}_t$, for any bounded and measurable function $f$ and $\mathscr{F}_t$ stopping time $\tau$, we have:

$$\mathbb{E}[f(\mathbf{X}_{\tau+h}) | \mathscr{F}_\tau] = \mathbb{E}[f(\mathbf{X}_{\tau+h}) | \mathbf{X}_\tau] \quad \forall h \in \mathbb{R}^+ \tag{9}$$

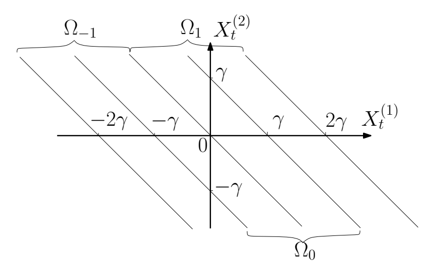


Fig. 3. Domains of $\Omega_m$.

In Figure 3, we define a series of domain $\Omega_m$:

$$\Omega_m := \{(X_t^{(1)}, X_t^{(2)}) : |X_t^{(1)} + X_t^{(2)} - m\gamma| < \gamma\} \quad \forall m \in \mathbb{Z} \tag{10}$$

Let $\partial\Omega_m$ be the boundary of $\Omega_m$. Define:

$$\Gamma_m := \{(X_t^{(1)}, X_t^{(2)}) : X_t^{(1)} + X_t^{(2)} - m\gamma = 0\} \quad \forall m \in \mathbb{Z} \tag{11}$$

Thus $\partial\Omega_m = \Gamma_{m-1} \cup \Gamma_{m+1}$.

Using the definitions given by (10) and (11), the filtering process can be divided into two steps: *Propagation* and *Correction*. Suppose at $\tau_i$, the pdf of $\mathbf{X}_t$, $\pi_{\tau_i}(\mathbf{x}|\mathscr{G}_{\tau_i}^\gamma)$, has already been computed in the last iteration of the algorithm. Thus $\forall t \in (\tau_i, \tau_{i+1})$, the pdf $\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma)$ can be computed by *propagating* the pdf $\pi_{\tau_i}(\mathbf{x}|\mathscr{G}_{\tau_i}^\gamma)$ in the domain $\Omega_{\hat{Y}_i}$ using the Fokker-Planck equation (8), because the information that the

[1]This assumption can be extended to a more general case with arbitrarily distributed sensors.

next stopping time $\tau_{i+1}$ has not arrived can be interpreted as: $\Gamma_{\hat{Y}_i-1} < X_t^{(1)} + X_t^{(2)} < \Gamma_{\hat{Y}_i+1}$. At time $\tau_{i+1}$, the new observation $\hat{Y}_{i+1}$ is used to *correct* the propagated pdf such that it is reduced to $\Gamma_{\hat{Y}_{i+1}}$, because $X_{\tau_{i+1}}^{(1)} + X_{\tau_{i+1}}^{(2)} = \hat{Y}_{i+1}$.

The aforementioned filtering algorithm is formally solved by the following two steps:

- **Propagation:** By the strong Markov property (9) and the law of total probability, $\forall t \in (\tau_i, \tau_{i+1})$, the pdf for $\mathbf{X}_t$ conditioned on the previous observations can be written as:
$$\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma) = \int_{\mathbf{y}\in\Gamma_{\hat{Y}_i}} p_{t,\tau_i}(\mathbf{x}|\mathbf{y})\pi_{\tau_i}(\mathbf{y}|\mathscr{G}_{\tau_i}^\gamma)d\mathbf{y} \qquad (12)$$
where $p_{t,\tau_i}(\mathbf{x}|\mathbf{y})$ is a solution to the Fokker-Planck equation (8) within the domain $\Omega_{\hat{Y}_i}$.
- **Correction:** At time $\tau_{i+1}$, when the new observation $\hat{Y}_{i+1}$ arrives, the domain of the pdf $\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma)$ is reduced to $\Gamma_{\hat{Y}_{i+1}}$ from $\Omega_{\hat{Y}_i}$. Define a probability flow vector $\mathbf{J}(\mathbf{x},t) = [J_1\ J_2]^T \in \mathbb{R}^2$ by [11]:
$$J_\alpha(\mathbf{x},t) = \mathbf{f}_\alpha(\mathbf{x},t)\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma) - \frac{1}{2}\sum_{\beta=1}^{2}\frac{\partial}{\partial x_\beta}\mathbf{g}_{\alpha\beta}(\mathbf{x},t)\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma) \qquad (13)$$
with $\alpha = 1,2$. An associated flux operator $\hat{\mathbf{F}}$ is defined such that
$$\mathbf{J}(\mathbf{x},t) = \hat{\mathbf{F}}\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma) \qquad (14)$$
By Gauss's Law, the pdf $p_{\tau_{i+1}}(\mathbf{x}|\mathscr{G}_{\tau_{i+1}}^\gamma)$ can be written as [11], [9]:
$$\pi_{\tau_{i+1}}(\mathbf{x}|\mathscr{G}_{\tau_{i+1}}^\gamma) \propto \mathbf{n}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x},\tau_{i+1}^-) = \mathbf{n}(\mathbf{x})\cdot\hat{\mathbf{F}}\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma) \qquad (15)$$
where $\mathbf{n}(\mathbf{x})$ is a unit vector oriented at $\mathbf{x}$ which is normal to $\Gamma_{\hat{Y}_{i+1}}$ and pointing outwards $\Omega_{\hat{Y}_i}$; $\tau_{i+1}^-$ is the left limit of time $\tau_{i+1}$.

The proposed algorithm is summarized in Table I:

TABLE I

EVENT-TRIGGERED FILTERING ALGORITHM

**1. Initialization:**
  Initial state pdf: $\pi_0(\mathbf{x}|\mathscr{G}_0^\gamma)$ and initial observation $\mathscr{G}_0^\gamma$
**2. Iteration:**
  **for** $(i = 1, \cdots, N)$ **do**:
    **if** $t \in (\tau_i, \tau_{i+1})$ **do**:
      Solve Fokker-Plank equation (8) within the domain $\Omega_{\hat{Y}_i}$.
      Compute the pdf $\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma)$ using (12).
    **else if** $t = \tau_{i+1}$ **do**:
      Reduce the pdf $\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma)$ to the boundary $\Gamma_{\hat{Y}_{i+1}}$ using (15).
    **end if**
    $i = i + 1$
  **end for**

## III. STOCHASTIC APPROXIMATION ALGORITHM

To numerically solve the event-triggered filtering problem (see also Table I), (8), (12) and (15) need to be computed numerically. In this section, a stochastic approximation approach is applied to solve the event-triggered filtering problem by using a Markov chain to approximate the system's evolution.

### A. Discretization of the state space

To work with a Markov chain defined on a discrete state space, the domain of the event-triggered filtering problem needs to be discretized. Consider a bounded domain $\mathscr{D}$ which is large enough such that the pdf outside $\mathscr{D}$ can be ignored. Suppose that $\mathscr{D}$ is discretized into square grids with size $\varepsilon > 0$. Without loss of generality, assume that $\gamma$ is a multiple of $\varepsilon$, i.e., $\gamma/\varepsilon \in \mathbb{Z}$. Figure 4 shows an example of a discretized domain with $\gamma/\varepsilon = 4$. Let $\mathscr{Q}$ be the set of coordinates of the grids within $\mathscr{D}$:

$$\mathscr{Q} := \{(n_1\ n_2) : (n_1\ n_2) \in \mathbb{Z}^2 \text{ and } (n_1\varepsilon\ n_2\varepsilon) \in \mathscr{D}\}$$

Thus, $\mathscr{Q}$ is the discrete state space of the Markov chain. $\forall m \in \mathbb{Z}$, define

$$\mathscr{Q}_m^\Omega := \{(n_1\ n_2) : (n_1\ n_2) \in \mathbb{Z}^2 \text{ and } (n_1\varepsilon\ n_2\varepsilon) \in \Omega_m\} \subset \mathbb{Z}^2$$
$$\mathscr{Q}_m^\Gamma := \{(n_1\ n_2) : (n_1\ n_2) \in \mathbb{Z}^2 \text{ and } (n_1\varepsilon\ n_2\varepsilon) \in \Gamma_m\} \subset \mathbb{Z}^2$$

Thus, $\mathscr{Q}_m^\Omega$ and $\mathscr{Q}_m^\Gamma$ represent approximations of $\Omega_m$ and $\Gamma_m$ in the discrete state space $\mathscr{Q}$. Suppose $q := (q_1\ q_2) \in \mathscr{Q}$, we denote the neighborhood of $q$ as:

$$\mathscr{N}_q := \{q + (j_1\ j_2) : j_1, j_2 \in \{-1,0,1\} \text{ and } (j_1\ j_2) \neq (0\ 0)\}$$

Figure 5 illustrates a state and its neighborhood. Let $\mathbf{x}_q$ be the coordinate of $q$ in the $X_t^{(1)} - X_t^{(2)}$ plane, i.e.,

$$\mathbf{x}_q = \varepsilon q \qquad (16)$$

Let $G(q) \subset \mathbb{R}^2$ be the area covered by a square of size $\varepsilon$ and centered at $\mathbf{x}_q$, i.e.,

$$G(q) := \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_q\|_\infty \leq \frac{1}{2}\varepsilon\} \qquad (17)$$

where $\|\bullet\|_\infty$ is the $\infty$-norm. Then we use $\hat{\Omega}_m$ and $\hat{\Gamma}_m$ to approximate $\Omega_m$ and $\Gamma_m$ in $\mathbb{R}^2$ with:

$$\hat{\Omega}_m := \bigcup_{q\in\mathscr{Q}_m^\Omega} G(q) \subset \mathbb{R}^2$$
$$\hat{\Gamma}_m := \bigcup_{q\in\mathscr{Q}_m^\Gamma} G(q) \subset \mathbb{R}^2 \qquad (18)$$

Figure 4 shows an example in which $\hat{\Gamma}_m$ are marked by the grey squares.

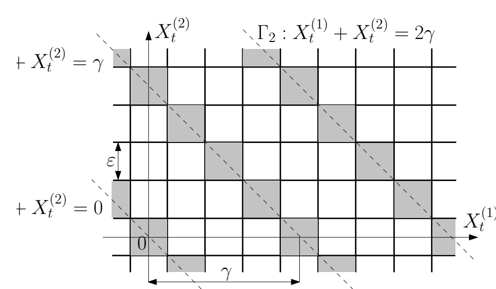


Fig. 4. Illustration of a discretized domain.

### B. Propagation Based on Markov Chain Approximation

The solution to (1) is a Markov process $\mathbb{E}[\mathbf{X}_t|\mathscr{G}_t^\gamma]$. The proposed stochastic approximation algorithm uses a Markov chain to approximate the evolution of $\mathbb{E}[\mathbf{X}_t|\mathscr{G}_t^\gamma]$. By carefully choosing the transition probabilities of the Markov chain, the distribution of the Markov chain converges weakly to the distribution of $\mathbb{E}[\mathbf{X}_t|\mathscr{G}_t^\gamma]$ when the grid size and time step approach to zero. Thus, by simulating the evolution of the Markov chain, we can approximate the pdf $\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma)$.

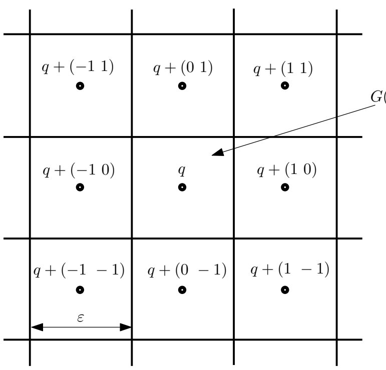


Fig. 5. State $q$ and its neighborhood $\mathscr{N}_q$.

For $t \in (\tau_i, \tau_{i+1})$, define a Markov chain $\{Q_k, k \in \mathbb{N}\}$ with the state space $\mathscr{Q}_{\hat{Y}_i}^\Omega$. Our purpose is to use $Q_k$ to approximate the distribution of $\pi_t(\mathbf{x}|\mathscr{G}_t^\gamma)$ for all $t \in (\tau_i, \tau_{i+1})$. Let $\Delta$ be the time length between two successive steps of the Markov chain $\{Q_k, k \geq 0\}$ such that $\Delta = \lambda \varepsilon^2$ for some fixed positive constant $\lambda$.

The Markov chain $\{Q_k, k \geq 0\}$ is designed to have the following two properties:

- Each state on $\partial \mathscr{Q}_{\hat{Y}_i}^\Omega$ is absorbing:

$$\Pr\{Q_{k+1} = q'|Q_k = q\} = \begin{cases} 1, & q' = q \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

- For the state in the interior of $\mathscr{Q}_{\hat{Y}_i}^\Omega$, the one step transition probability is given by:

$$\Pr\{Q_{k+1} = q'|Q_k = q\} = \begin{cases} p_{q'}^k(q), & q' \in \mathscr{N}_q \cup q \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

  where $p_{q'}^k(q)$ is a transition probability which is a function of $q$ and $k\Delta$

There are many ways to parameterize $p_{q'}^k(q)$ if only the following weak convergence theorem of the Markov chain $\{Q_k, k \geq 0\}$ is satisfied.Suppose for $\varepsilon \to 0$:

$$\begin{aligned} &\frac{1}{\Delta}\mathbb{E}[Q_{k+1} - Q_k|Q_k = q] \to \mathbf{f}(\mathbf{x}, k\Delta) \\ &\frac{1}{\Delta}\mathbb{E}[(Q_{k+1} - Q_k)(Q_{k+1} - Q_k)^T|Q_k = q] \to \mathbf{g}(\mathbf{x}, k\Delta)\mathbf{g}(\mathbf{x}, k\Delta)^T \end{aligned} \tag{21}$$

$\forall \mathbf{x} \in \Omega_{\hat{Y}_i} \cap \mathscr{D}$ and $q \in \mathscr{Q}_{\hat{Y}_i}^\Omega$ is the point closest to $\mathbf{x}$.

*Theorem 1:* Suppose a Markov chain $\{Q_k,\ k \geq 0\}$ with the transition probabilities given by (19) and (20) satisfies Condition (21). Define a continuous time Markov process $\{Q_t,\ t \geq 0\}$ given by $Q_t = Q_k,\ \forall t \in [k\Delta, (k+1)\Delta)$. Suppose that $Q_t$ starts from the same initial distribution $\pi_0(\mathbf{x})$ as $\mathbf{X}_t$. Then $Q_t$ converges weakly to $\mathbf{X}_t$ as $\varepsilon$ and $\Delta$ approach to zero.

*Proof:* The proof of the theorem is a modification of the proof of Theorem 8.7.1 of [12]. ■

To compute the evolution of $Q_k$, we need the initial condition $\pi_{\tau_i}(q|\mathscr{G}_{\tau_i}^\gamma)$ which is defined on the state space of $Q_k$. Suppose the transition probabilities (19) and (20) satisfy (21). The Fokker-Planck equation (8) can be approximated by its discretized version given by the Chepman-Kolmogorov forward equation which computes the pdf $\pi_t(q|\mathscr{G}_{\tau_i}^\gamma)$ iteratively at any time $t = \tau_i + k\Delta \in (\tau_i, \tau_{i+1})$:

$$\pi_{\tau_i+(k+1)\Delta}(q'|\mathscr{G}_{\tau_i}^\gamma) = \sum_{q \in \mathscr{Q}_{\hat{Y}_i}^\Omega} \Pr\{Q_{k+1} = q'|Q_k = q\}\pi_{\tau_i+k\Delta}(q|\mathscr{G}_{\tau_i}^\gamma) \tag{22}$$

and $\forall h \geq 0$ with $\tau_i + h < \tau_{i+1}$:

$$\pi_{\tau_i+h}(q'|\mathscr{G}_{\tau_i}^\gamma) = \pi_{\tau_i+\lfloor\frac{h}{\Delta}\rfloor\Delta}(q'|\mathscr{G}_{\tau_i}^\gamma) \tag{23}$$

### C. Correction of the Discretized Propagation Results

At time $\tau_{i+1}$ when a new observation $\hat{Y}_{i+1}$ arrives, Equation (15) is used to correct $\pi_{\tau_{i+1}^-}(\mathbf{x}|\mathscr{G}_{\tau_{i+1}^-}^\gamma)$ to get the corrected pdf $\pi_{\tau_{i+1}}(\mathbf{x}|\mathscr{G}_{\tau_{i+1}}^\gamma)$. Combining (7) and (13) yields:

$$\mathscr{L}^* = \nabla \cdot \mathbf{J} \tag{24}$$

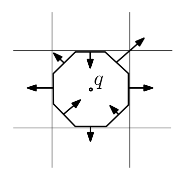


Fig. 6. Probability flow defined on the boundary of an octagon contained in $G(q)$.

Consider an octagon $\hat{G}(q)$ contained in $G(q)$ and denote each side of the boundary of $\hat{G}(q)$ by $s_{q'}, q' \in \mathscr{N}_q$ (see also Figure 6 for an illustration). By Gauss's Law,

$$\iint_{\hat{G}_q} \nabla \cdot \mathbf{J}(\mathbf{x}, t) d\mathbf{x} = \int_{\partial \hat{G}_q} \mathbf{J}(\mathbf{x}, t) d\mathbf{s} \propto \sum_{q' \in \mathscr{N}_q} \mathbf{n}_{q'} \cdot \mathbf{J}_{q'}(q, t) \tag{25}$$

where $\mathbf{J}_{q'}$ is the probability flow through the boundary $s_{q'}$ which separates $q$ and $q' \in \mathscr{N}_q$; $\mathbf{n}_{q'}$ is an outward unit normal vector of $s_{q'}$. In Figure 6, the probability flows defined on each side of the octagon are shown by bold black arrows. Suppose at time $k$ we have $\Pr\{Q_k = q\} = 1$. Thus, the probability flow on boundary $s_{q'}$ can be computed by:

$$\mathbf{J}_{q'} \propto \Pr\{Q_{k+1} = q'|Q_k = q\} \quad \forall q' \in \mathscr{N}_q \tag{26}$$

Now, the problem of reducing the pdf on $\hat{\Omega}_{\hat{Y}_i}$ ($\mathscr{Q}_{\hat{Y}_i}^\Omega$) to $\hat{\Gamma}_{\hat{Y}_{i+1}}$ ($\mathscr{Q}_{\hat{Y}_{i+1}}^\Gamma$) at $\tau_i$ can be transformed to the computation of the probability flow. Specifically at $t = \tau_{i+1}$, the last term in (25) can be written in the following form which is used to compute the probability defined on $\hat{\Gamma}_{\hat{Y}_{i+1}}$ or $\mathscr{Q}^{\Gamma}_{\hat{Y}_{i+1}}$, $\forall q \in \mathscr{Q}^{\Gamma}_{\hat{Y}_{i+1}}$:

$$\mathbf{n}(q)\cdot\mathbf{J}(q,\tau_{i+1}) \propto \sum_{q'\in\mathscr{Q}^{\Omega}_{\hat{Y}_{i+1}}\cap\mathscr{N}_q} \mathbf{n}(q)\cdot\mathbf{J}_q(q',\tau^{-}_{i+1}) \tag{27}$$

The numerical algorithm for the event-triggered filtering problem is summarized in Table II.

TABLE II

NUMERICAL ALGORITHM FOR EVENT-TRIGGERED FILTERING

**1. Initialization:**

Design a Markov chain $\{Q_k, k\in\mathbb{N}\}$ that satisfies properties in (19), (20) and (21).

Discretize initial state pdf: $\pi_0(\mathbf{x}|\mathscr{G}^{\gamma}_0)$ to get a pdf $\pi_0(q|\mathscr{G}^{\gamma}_0)$ defined on the initial state space: $\mathscr{Q}^{\Omega}_{\hat{Y}_0}$.

**2. Iteration:**

**for** $(i=1,\cdots,N)$ **do**:
  $k=0$
  **for** $(k\Delta+\tau_i<\tau_{i+1})$ **do**:
    Compute the pdf $\pi_{\tau_i+(k+1)\Delta}(q|\mathscr{G}^{\gamma}_{\tau_i})$ using (22) for all $q\in\mathscr{Q}^{\Omega}_{\hat{Y}_i}$.
    $k=k+1$
  **end for**
  Reduce the pdf $\pi_t(q|\mathscr{G}^{\gamma}_t)$ to the boundary $\mathscr{Q}^{\Gamma}_{\hat{Y}_{i+1}}$ using (26) and (27).
  $i=i+1$
**end for**

## IV. SIMULATIONS

In this section, an example is presented to demonstrate the performance of the proposed algorithm. Assume that the dynamics of $X^{(1)}_t$ is given by a geometric Brownian motion which is widely used to model the change of a stock price in a financial market (the Black-Scholes' model) [13]. The sde governing the evolution of $X^{(1)}_t$ is given by:

$$dX^{(1)}_t = aX^{(1)}_t dt + bX^{(1)}_t dW^{(1)}_t \tag{28}$$

where $a$ and $b$ are positive constants indicating the drift and volatility of a stock price. We assume that $\gamma=1$, i.e., when $X^{(1)}_t$ gets close to any integer, the corresponding sensor sends out a signal to a centralized estimator. The observation noise can be modeled by a Ornstein-Uhlenbeck (O-U) process which is an ergodic process. In financial engineering, the O-U process is often used to model the volatility of exchange rates of different currencies. The dynamics of the OU process is given by:

$$dX^{(2)}_t = -\kappa X^{(2)}_t dt + u dW^{(2)}_t \tag{29}$$

where $\kappa$ and $u$ are positive constants. Combining (28) and (29) yields the general form of the sde given in (1). Since $W^{(1)}_t$ and $W^{(2)}_t$ are independent, $X^{(1)}_t$ and $X^{(2)}_t$ are decoupled. In the simulation, we choose $a=0.3$, $b=0.1$, $\kappa=1.2$ and $u=1$. Figure 7 shows one realization of $X^{(1)}_t$, $X^{(2)}_t$ and $Y_t$. Note that at time $t=0$, $X^{(2)}_t=0$ and $X^{(1)}_t=Y_t=1$. The same realization and boundaries $\Gamma_0-\Gamma_4$ are shown in Figure 8 in the $X^{(1)}_t-X^{(2)}_t$ coordinate. In Figure 8, the trajectory of $\mathbf{X}_t$

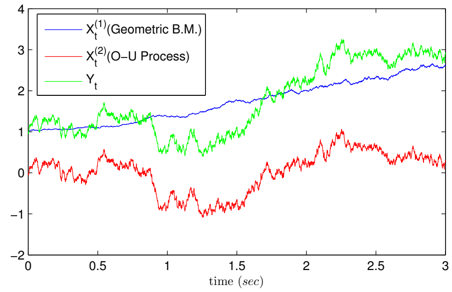


Fig. 7. Realization of $X^{(1)}_t$, $X^{(2)}_t$ and $Y_t$ vs. time.

starts from $(1,0)$ which is located in $\Omega_1$. The sensor output $\hat{Y}_i$ is plotted in Figure 9. Note that $\hat{Y}_0\equiv 1$ and $\tau_0\equiv 0$. Now, we use the stochastic approximation algorithm proposed in Section III to solve this event-triggered filtering problem. Since $X^{(1)}_t$ and $X^{(2)}_t$ are independent, we can design two Markov chains $\{Q^{(1)}_k\}$ and $\{Q^{(2)}_k\}$ to approximate $X^{(1)}_t$ and $X^{(2)}_t$ respectively. Then, $\{Q^{(1)}_k\}$ and $\{Q^{(2)}_k\}$ are combined to get $\{Q(k)\}$ which approximates $\mathbf{X}_t$ conditioned on the observations. To discretize the time and space, we choose $\Delta=0.01$ and $\varepsilon=0.2$. Suppose a Markov chain $\{Q_k\}$

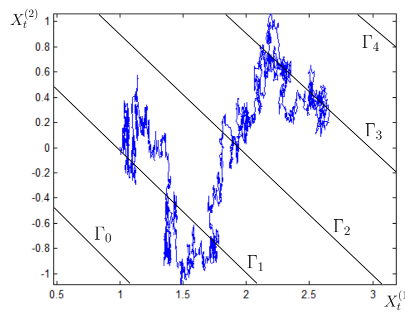


Fig. 8. Realization of $X^{(1)}_t$ vs. $X^{(2)}_t$

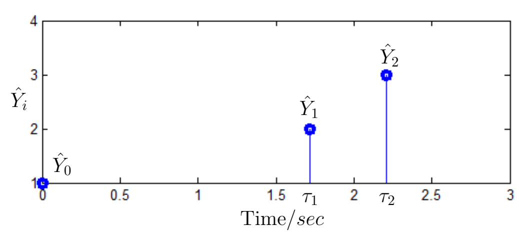


Fig. 9. $\hat{Y}_i$ vs. time

is constructed such that it satisfies the weak convergence condition (21). Figure 11 shows the evolution of $\mathbb{E}[\mathbf{X}_t|\mathscr{G}^{\gamma}_t]$ which is computed by the proposed numerical algorithm recursively. In this realization, the first hitting time $\tau_1$ is around $1.72s$. Thus, during the interval $t\in(0,\tau_1)$, the Markov chain approximates the solution of the Fokker-Planck equation in the region $\hat{\Omega}_{\hat{Y}_0}$, which is illustrated by Figure 10(a). At time $\tau_1$, the sensor located at 1 gives an output

$\hat{Y}_1 = 2$ (see also Figure 9) and the pdf $\mathbb{E}[\mathbf{X}_t|\mathscr{G}_t^{\gamma}]$ is reduced to the boundary $\hat{\Gamma}_{\hat{Y}_1}$, which is shown in Figure 10(b). Then, the Fokker-Plank equation is solved again in region $\hat{\Omega}_{\hat{Y}_1}$ using the aforementioned Markov Chain again during the time $t \in (\tau_1, \tau_2)$ (Figure 10(c)). At the time $\tau_2 \approx 2.21s$, the pdf is reduced to $\hat{\Gamma}_{\hat{Y}_2}$ with $\hat{Y}_2 = 3$ (Figure 10(d)) and after that time, before new observation arrives, the Fokker-Planck equation is solved again by the Markov chain in region $\hat{\Omega}_{\hat{Y}_2}$.

## V. CONCLUSIONS

In this paper, we have presented a model of the event-triggered filtering problem. The event-triggered filtering problem can be solved in two steps: (1) solving a Fokker-Planck equation during the interval between two consecutive stopping times when no sensor is triggered; and (2) reducing the pdf to a boundary when the corresponding sensor is triggered. A numerical algorithm has been proposed to compute the pdf of the conditional Markov process by using a Markov chain to approximate the Markov process in continuous time. The performance of the proposed algorithm is validated through numerical simulations.

## VI. ACKNOWLEDGEMENT

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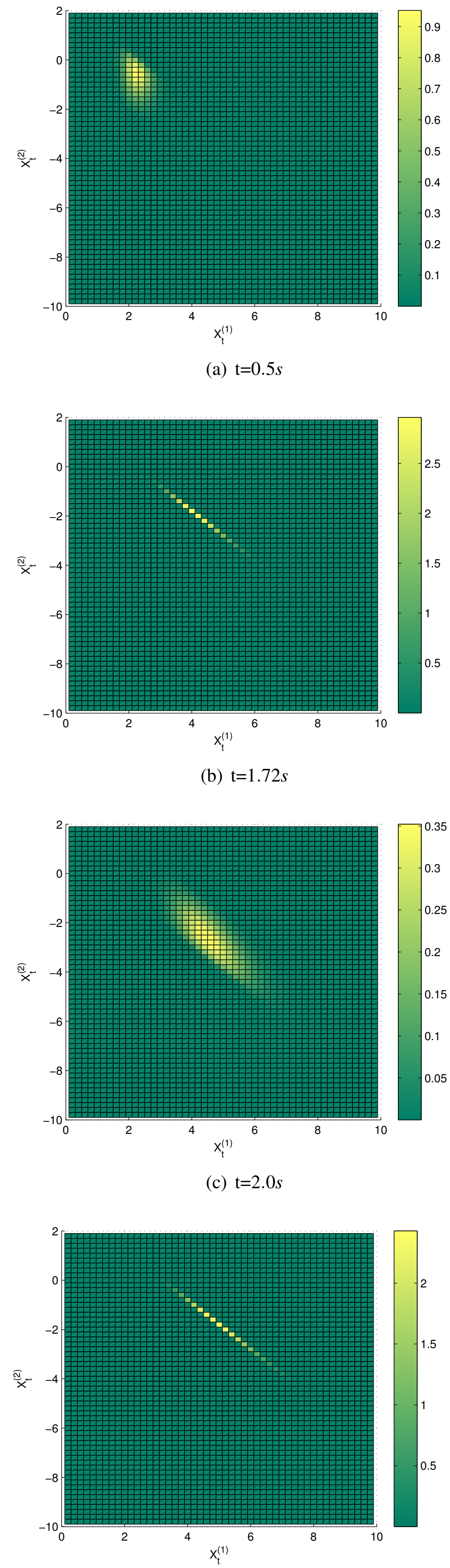


(a) t=0.5$s$

(b) t=1.72$s$

(c) t=2.0$s$

(d) t=2.21$s$

Fig. 10. Evolution of the pdf of $\mathbb{E}[\mathbf{X}_t|\mathscr{G}_t^{\gamma}]$