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Constrained control of positive discrete-time periodic systems with delays

N. Bougatef, D. Mehdi, O. Bachelier, M. Chaabane *

Abstract

This paper addresses the control problem of linear periodic discrete-time systems with delays under the positivity constraint, which means that the resulting closed-loop systems are not only stable, but also positive. Linear periodic state feedback controllers are used in the stabilization problem. Necessary and sufficient conditions are established for the existance of such controllers for discrete-time periodic delayed systems. Then, sufficient conditions are proposed under the additional constraint of bounded control, which means that the control inputs and the states of the closed-loop systems are bounded. Finally, the results are extended to uncertain periodic delayed systems with a polytopic uncertainty description. Numerical examples are provided to illustrate the established results.

1. Introduction

A Positive system is very relevant in some continuous-time and discrete-time problems of the common life which cannot be described by negative signals, like, for instance, population dynamics evolutions, prey-predator problems, biological problems, etc. These systems appear in many practical problems when the states represent physical quantities that have an intrinsically constant sign (Absolute temperatures, concentrations, etc) [1]; see, e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10], [11] and [12] and the references therein. Both positive and periodic linear systems have been studied in different application fields ranging from biology and chemistry. Indeed, periodic processes arise very often in nature and engineering and thus application of linear periodic systems may be found in a large spectrum of different fields. Linear periodic systems represent a subclass of linear time-varying systems and are certainly, by far, the simplest. A number of important results on linear periodic systems have been reported in the literature, see, e.g. [13], [14], [15], [16], [17], [18], [19], [20], [21], [22] and the references therein. Moreover, Stability is one of the most important properties of systems, and a massive literature has been concentrated on this issue for positive system [1]. This has impulsed a large number of research results obtained for the stability and robust stability of positive discrete and continuous-time linear systems without delays and with delays; see, e.g. [8], [9], [11], [23], [24], [25], [12], [26], [27], [28], [29], and [7] and the references therein. It turns out in [9], [23], [29] and [24] that the delays have no impact on stability of linear positive systems. Asymptotic stability of the discrete-time positive systems with bounded time-varying delays has been dealt with in [23]. The approach of the linear copositive Lyapunov function captures the nature of positivity; thus, it is widely used in research on positive systems [12]. Some necessary and sufficient stability conditions for positive systems by means of linear composite Lyapunov function are proposed in [7]. Reference [25] tries to adapt the linear copositive Lyapunov function to LTI discrete-time positive systems with delays, focusing on controller design under the positivity constraint and/or with bounds on the input and the states. Authors of reference [29] obtained a necessary and sufficient stability condition for continuous-time positive systems with delays. This result was then improved to design positive observers for delayed continuous-time positive systems in [30]. Then, there is a great number of research results obtained separately for both positive and periodic linear system but not for positive periodic linear system. Then, the idea of [26] is original. In [26] LMI-based conditions for the existence of a desired periodic state feedback controller guaranteeing the resultant closed-loop systems is asymptotically stable and positive are presented. In this context, this article is an extension of works done on the linear positive discrete-time systems for linear periodic positive discrete-time systems. In this paper, the stabilization of linear discrete-time periodic systems with delays under the condition that the closed-loop system is positive is addressed as well as the case of bounded state and or control variables. Our development is based on a Linear Programming approach. The established conditions are in the form of linear equality and linear inequality constraints which can be easily associated to linear objective function resulting in a linear programming technique [12]. The problem of stabilization in the presence of polytopic parameter uncertainty is also dealt with.

Notation: $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space. $\mathbb{R}^n_+$ denotes the non-negative orthant of the n-dimensional real space $\mathbb{R}^n$, $\mathbb{R}^n_{+*} = \mathbb{R}^n_+ - \{0\}$, $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices, I is the $\mathbb{R}^{n\times n}$ identity matrix. Matrix $M \geq 0 (> 0)$

*Miss Bougatef and Mr Chaabane are with Université de Sfax, ENIS, Electrical Engineering Department, Route de Soukra, km 3.5, 3018 Sfax, Tunisia, (e-mail : bougatefnaima@yahoo.fr / chaabane_uca@yahoo.fr). Mr. Bachelier and Mr. Mehdi are with Université de Poitiers, ESIP-LAII, 2, rue Pierre Brousse, 86022 Poitiers Cedex, (e-mails : Olivier.Bachelier / Driss.Mehdi@univ-poitiers.fr)

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means that all its entries are nonnegative (positive), and vectors $x \geq 0 (> 0)$ can be used in the same way. $(.)^\top$ denotes transposition of matrices or vectors. The following notations of matrices are used in this article: $A_l(k) = \left[a^l_{ij}(k)\right]$, $A^r_l(k) = \left[a^{lr}_{ij}(k)\right]$, vector $b^\top_i(k)$ is the ith row vector of matrix $B(k)$, and $b^{r\top}_i(k)$ is the ith row vertor of matrix $B^r(k)$. For simplicity, let $\mathbb{I}_T = [0, 1, \ldots, T-1] \subset \mathbb{N}$ and $\mathbb{I}^*_T = [1, \ldots, T-1]$.

2. Problem statements and preliminaries

Consider the following antonomous T-periodic system:

$$x(k+1) = A(k)x(k) \tag{1}$$

with $A(T+k) = A(k)$, $\forall k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state vector. The following definition and lemma are needed.

Definition 2.1 *The periodic system* (1) *is said to be positive if and only if for any* $x(0) \geq 0$, *the corresponding trajectory* $x(k) \geq 0$ *holds for all* $k \in \mathbb{N}$.

Systems in which we are interested are positive periodic systems where the state does not vanish during one period, that is $x(k) \in \mathbb{R}^n_{+*}$, for all $k = 0, 1, \ldots, p$. To illustrate our idea let's take the two following cases of system (1):

1. Let $A(2n) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A(2n+1) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, $n \in \mathbb{N}$. For any $x(0) \geq 0$, we have $x(k) \geq 0$ because $x(1) = 0$.
2. Let $p = 3$ and $x(0) = \begin{bmatrix} a \\ b \end{bmatrix} \geq 0$. In addition, $A(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A(1) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $A(2) = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$. Then, $x(k) \geq 0$, because $x(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

In both examples, the system is positive although, at least, one of its state matrix is negative. Therefore, the case where the state vanishs during the first period is not interesting. In the sequel, we define the calss C as the class of systems for which the state does not vanish durig the first period.

Lemma 2.1 *[26] The periodic system* (1) *belonging to C is positive if and only if* $A(k) \geq 0$ *for all* $k \in \mathbb{I}_T$.

By lemma 2.1, in order to determine whether or not a periodic system is positive, one has to check only the positivity of the entries of the dynamic periodic matrix of the periodic system.

Lemma 2.2 *[31] Assume that the periodic system* (1) *is positive. It is stable if and only if there exists* $0 < \lambda(k) \in \mathbb{R}^n$; $k \in \mathbb{I}_T$ *such that:*

$$\begin{aligned} \lambda(i+1) - A(i)\lambda(i) = 0,\ i \in \mathbb{I}_T \\ \lambda(0) - A(T-1)\lambda(T-1) > 0 \end{aligned} \tag{2}$$

Lemma 2.3 *[31] Consider the autonomous positive periodic system* $x(k+1) = A(k)x(k)$, $k \in \mathbb{N}$, *that is* $A_k \geq 0$, $k \in \mathbb{I}_T$. *For any initial condition* $x(0)$ *satisfying* $0 \leq x(0) \leq \bar{x}(0)$, *the two statements are equivalent:*

1. *There exists a* $T-$*periodic vector* $\bar{x}(k) \geq 0$ *such that* $0 \leq x(k) \leq \bar{x}(k)$, $\forall k \in \mathbb{N}$
2. *There exists* $\bar{x}(k) \geq 0$ *given by*

$$\bar{x}(k) = \begin{cases} A(k-1)\bar{x}(k-1), & for\ 1 \leq k < T \\ \bar{x}(j)\ for\ k \geq T;\ k = nT + j;\ n \geq 1;\ j \in \mathbb{I}_T \end{cases} \tag{3}$$

$$\text{and satisfying } \bar{x}(0) - A(T-1)\bar{x}(T-1) \geq 0 \tag{4}$$

3. Main results

This section consists of two subsections. The first subsection is devoted to designing controller for linear discrete-time periodic systems with delays, and the second one is for the periodic systems with delays and uncertainties.

3.1. Stabilization of delayed periodic systems

3.1.1. Unconstrained control. Consider the following linear delayed periodic system:

$$\begin{aligned} x(k+1) = \sum_{l=0}^{h} A_l(k)x(k-l) + B(k)u(k), \\ x(l) \geq 0,\ l = -h, -(h-1), \ldots, 0 \end{aligned} \tag{5}$$

where $A_l(k) \in \mathbb{R}^{n\times n}$, $B(k) \in \mathbb{R}^{n\times m}$ and

$$u(k) = \sum_{l=0}^{h} F_l(k)x(k-l), \tag{6}$$

with $F_l(k) \in \mathbb{R}^{m\times n}$. The closed-loop periodic system is

$$\begin{aligned} x(k+1) = \sum_{l=0}^{h} (A_l(k) + B(k)F_l(k))x(k-l), \\ x(l) \geq 0,\ l = -h, -h+1, \ldots, 0 \end{aligned} \tag{7}$$

The following theorem holds for (7).

Theorem 3.1 *System* (7) *is stable and positive, if and only if there exist* $T-$*periodic vectors* $z_{lj}(k) \in \mathbb{R}^m$, *and* $0 < \alpha_l(k) = [\alpha_{l1}(k), \alpha_{l2}(k), \ldots, \alpha_{ln}(k)]^\top \in \mathbb{R}^n$, $l \in \mathbb{I}_{h+1}$ *and* $j \in \mathbb{I}^*_{n+1}$ *such that the following conditions hold:*

$$a^l_{ij}(k)\alpha_{lj}(k) + b_i(k)z_{lj}(k) \geq 0;\ i, j \in \mathbb{I}^*_{n+1};\ l \in \mathbb{I}_{h+1}; k \in \mathbb{I}_T \tag{8}$$

$$\alpha_0(k) - \sum_{l=0}^{h} A_l(k-1)\alpha_l(k-1) - B(k-1)\sum_{l=0}^{h}\sum_{j=1}^{n} z_{lj}(k-1) = 0; \quad k \in \mathbb{I}^*_T$$

$$\alpha_l(k) = \alpha_{l-1}(k-1),\ l \in \mathbb{I}^*_{h+1}\ and\ k \in \mathbb{I}^*_T$$

$$\alpha_0(0) - \sum_{l=0}^{h} A_l(T-1)\alpha_l(T-1) - B(T-1)\sum_{l=0}^{h}\sum_{j=1}^{n} z_{lj}(T-1) > 0$$

$$\alpha_l(0) > \alpha_{l-1}(T-1),\quad l \in \mathbb{I}_{h+1}^* \tag{9}$$

Moreover, the state feedback gains are giving by

$$F_l(k) = [z_{l1}(k)/\alpha_{l1}(k), z_{l2}(k)/\alpha_{l2}(k), \ldots, z_{ln}(k)/\alpha_{ln}(k)],\quad l \in \mathbb{I}_{h+1}, k \in \mathbb{I}_T. \tag{10}$$

Proof: Let $x_0(k) = x(k)$, $x_1(k) = x(k-1)$, …, $x_h(k) = x(k-h)$. System (7) is equivalently transformed into

$$y(k+1) = (\bar{A}(k) + \bar{B}(k)F(k))y(k),\ y(0) \geq 0 \tag{11}$$

where $\bar{A}(k)$, $\bar{B}(k)$ and $F(k)$ are defined by

$$\bar{A}(k) = \begin{bmatrix} A_0(k) & A_1(k) & \cdots & A_{h-1}(k) & A_h(k) \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & I & 0 \end{bmatrix};$$

$$\bar{B}(k) = \begin{bmatrix} B(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n(h+1)\times m};\quad F(k) = \begin{bmatrix} F_0(k) & \cdots & F_h(k) \end{bmatrix};$$

$$y(k) = \left[x_0^\top(k), \ldots, x_h^\top(k)\right]^\top \tag{12}$$

It suffices to show that system (11) is stable and positive if and only if (8) and (9) hold. The remaining of the proof is divided into two parts: sufficiency and necessity.

Sufficiency: First, because $\alpha_{lj}(k) > 0$ for all $k \in \mathbb{I}_T$, (8) implies $a_{ij}^l(k) + b_i(k)z_{lj}(k)/\alpha_{lj}(k) \geq 0$; $i, j \in \mathbb{I}_{n+1}^*$; $l \in \mathbb{I}_{h+1}$; $k \in \mathbb{I}_T$, which in turn is equivalent to the fact that $A_l(k) + B(k)F_l(k) \geq 0$ and therefore $\bar{A}(k) + \bar{B}(k)F(k) \geq 0$ for all $k \in \mathbb{I}_T$. According to lemma 2.1, the periodic system (11) is positive.
Second, since

$$F_l(k)\alpha_l(k) = [z_{l1}(k)/\alpha_{l1}(k), z_{l2}(k)/\alpha_{l2}(k), \ldots, z_{ln}(k)/\alpha_{ln}(k)] = \sum_{j=1}^{n} z_{lj}(k) \tag{13}$$

It follows from (12) and (13) that (9) is equivalent to

$$\begin{aligned} \alpha(i+1) - \bar{A}^c(i)\alpha(i) = 0,\ i \in \mathbb{I}_T \\ \alpha(0) - \bar{A}^c(T-1)\alpha(T-1) > 0 \end{aligned} \tag{14}$$

where $\bar{A}^c(j) = (\bar{A}(j) + \bar{B}(j)F(j))$, for $j \in \mathbb{I}_T$.

$$\alpha(k) = \left[\alpha_0^\top(k), \alpha_1^\top(k), \ldots, \alpha_h^\top(k)\right]^\top > 0,\quad k \in \mathbb{I}_T \tag{15}$$

Using lemma 2.2 together with the fact that $\alpha(k) > 0$ and $\bar{A}(k) + \bar{B}(k)F(k) \geq 0$ for all $k \in \mathbb{I}_T$, implies that (11) is stable. To sum up, (11) is both stable and positive.

Necessity: According to lemma 2.1 and lemma 2.2, the positivity and stability of (11) indicate that $\bar{A}(k) + \bar{B}(k)F(k) \geq 0$ and guarantee the existence of $0 < \alpha(k)$, defined by (15), such that (14) holds.

On the other hand, $\bar{A}^c(k) \geq 0$ is nothing but (8), and (14) implies (9). The proof is thus completed.

3.1.2. Constrained control. The remainder of this subsection deals with the following constrained periodic system:

$$x(k+1) = \sum_{l=0}^{h} A_l(k)x(k-l) + B(k)u(k),\ 0 \leq u(k) \leq \bar{u}(k)$$
$$x(l) \geq 0,\ \ l = -h, -h+1, \ldots, 0 \tag{16}$$

where $\bar{u}(k)$ is a $T-$periodic constant vector serving as the upper bound of the input $u(k)$ defined by (6). The closed-loop periodic system of (16) is

$$x(k+1) = \sum_{l=0}^{h}(A_l(k) + B(k)F_l(k))x(k-l),\ 0 \leq u(k) \leq \bar{u}(k)$$
$$x(l) \geq 0,\ \ l = -h, -h+1, \ldots, 0 \tag{17}$$

In this situation, the goal is to find out the $T-$periodic vector $0 < \bar{y}_l(k) \in \mathbb{R}^n$ for all $l \in \mathbb{I}_{h+1}$, such that there exists a state-feedback control law $u(k) = \sum_{l=0}^{h} F_l(k)x(k-l) \leq \bar{u}(k)$ for all $k \in \mathbb{N}$ under which the following two constraints are satisfied.
$*$ the closed-loop periodic system is positive and stable.
$*$ $x(k) \leq \bar{y}_0(k)$, $k \in \mathbb{N}$, if and only $0 \leq x(-l) \leq \bar{y}_l(0)$, $l \in \mathbb{I}_{h+1}$.
The result is stated in the following theorem.

Theorem 3.2 *For an arbitrary $T-$periodic vector $0 \leq \bar{u}(k) \in \mathbb{R}^m$, suppose that there exist a $T-$periodic vectors $0 < \bar{y}_l(k) = [\bar{y}_{l1}(k), \bar{y}_{l2}(k), \ldots, \bar{y}_{ln}(k)]^\top \in \mathbb{R}^n$ and $0 \leq z_{lj}(k) \in \mathbb{R}^m$, $l \in \mathbb{I}_{h+1}$ and $j \in \mathbb{I}_{n+1}^*$ such that the following conditions hold:*

$$\sum_{l=0}^{h}\sum_{j=1}^{n} z_{lj}(k) \leq \bar{u}(k),\quad a_{ij}^l(k)\bar{y}_{lj}(k) + b_i(k)z_{lj}(k) \geq 0;$$
$$i, j \in \mathbb{I}_{n+1}^*;\ l \in \mathbb{I}_{h+1};\ k \in \mathbb{I}_T$$

$$\bar{y}_0(k) - \sum_{l=0}^{h} A_l(k-1)\bar{y}_l(k-1) - B(k-1)\sum_{l=0}^{h}\sum_{j=1}^{n} z_{lj}(k-1) = 0;\quad k \in \mathbb{I}_T^*;$$

$$\bar{y}_l(k) = \bar{y}_{l-1}(k-1),\ \ l \in \mathbb{I}_{h+1}^* \ \text{ and } \ k \in \mathbb{I}_T^*;$$

$$\bar{y}_0(0) - \sum_{l=0}^{h} A_l(T-1)\bar{y}_l(T-1) - B(T-1)\sum_{l=0}^{h}\sum_{j=1}^{n} k_{lj}(T-1) > 0;$$

$$\bar{y}_l(0) > \bar{y}_{l-1}(T-1),\ \ l \in \mathbb{I}_{h+1}^*$$

Then, the periodic system (17) is stable and positive, and $0 \leq x(k) \leq \bar{y}_0(k)$ and $0 \leq u(k) \leq \bar{u}(k)$ for all $k \in \mathbb{N}$ whenever $0 \leq x(-l) \leq \bar{y}_l(0)$, with $F_l(k) = [z_{l1}(k)/\bar{y}_{l1}(k), z_{l2}(k)/\bar{y}_{l2}(k), \ldots, z_{ln}(k)/\bar{y}_{ln}(k)]$, $l \in \mathbb{I}_{h+1}$

and $k \in \mathbb{I}_T$.

Proof: The proof is similar to that of theorem 3.1. Note that (17) is equivalent to:

$$\begin{aligned} y(k+1) &= (\bar{A}(k) + \bar{B}(k)F(k))y(k), \\ y(0) &\geq 0, \;\; 0 \leq F(k)y(k) \leq \bar{u}(k), \end{aligned} \quad (18)$$

where $y(k)$, $\bar{A}(k)$, $\bar{B}(k)$ and $F(k)$ are as in (12).
Just as in the sufficiency part of the proof of theorem 3.1, one can show that $\bar{A}(k) + \bar{B}(k)F(k) \geq 0$ and

$$\begin{aligned} \bar{y}(i+1) - \bar{A}^c(i)\bar{y}(i) &= 0, i \in \mathbb{I}_T \\ \bar{y}(0) - \bar{A}^c(T-1)\bar{y}(T-1) &> 0 \end{aligned} \quad (19)$$

with $\bar{y}(k) = \left[\bar{y}_0^\top(k), \bar{y}_1^\top(k), \ldots, \bar{y}_h^\top(k)\right]^\top > 0$, $k \in \mathbb{I}_T$. Then (18) is positive and stable according to lemma 2.1 and lemma 2.2. Lemma 2.3 shows that $0 \leq y(k) \leq \bar{y}(k)$ and therefore $0 \leq x(k) \leq \bar{y}_0(k)$ for all $k \in \mathbb{N}$. By definition, $F(k) \geq 0$, $k \in \mathbb{I}_T$. Hence, $u(k) = F(k)y(k) \leq F(k)\bar{y}(k) = \sum_{l=0}^h F_l(k)\bar{y}_l(k) = \sum_{l=0}^h \sum_{j=1}^n z_{lj}(k) \leq \bar{u}(k)$, i.e., $0 \leq u(k) \leq \bar{u}(k)$ for all $k \in \mathbb{N}$. The proof is completed.

3.2. Robust stabilization of periodic uncertain delayed systems

This subsection will generalize the results in the subsection 3.1 to periodic uncertain delayed system. The proof are omitted, because they are derived in the same way as section 3.1.
Consider the following periodic uncertain system with delays:

$$\begin{aligned} x(k+1) &= \sum_{l=0}^h \hat{A}_l(k)x(k-l) + \hat{B}(k)u(k), \\ x(l) &\geq 0, \;\; l = -h, -h+1, \ldots, 0 \end{aligned} \quad (20)$$

where the periodic matrices $\hat{A}_l(k) \in \mathbb{R}^{n\times n}$ and $\hat{B}(k) \in \mathbb{R}^{n\times p}$ are not exactly determined. the $(h+1)-$tuple $\left(\hat{A}_0(k), \ldots, \hat{A}_h(k), \hat{B}(k)\right)$ is assumed to belong to the following convex set:

$$\mathbb{S}_k := \left\{ \sum_{r=1}^m \beta_r \left(A_0^r(k), \ldots, A_h^r(k), B^r(k)\right); \; \sum_{r=1}^m \beta_r = 1; \beta_r \geq 0 \right\}; \quad (21)$$

where $A_0^r(k), \ldots, A_h^r(k), B^r(k)$, $r \in \mathbb{I}_{h+1}$ and $k \in \mathbb{I}_T$, are known $T-$periodic matrices.
Suppose that the periodic state-feedback control law of (20) is chosen as (6). The closed-loop periodic system is

$$\begin{aligned} x(k+1) &= \sum_{l=0}^h (\hat{A}_l(k) + \hat{B}(k)F_l(k))x(k-l), \\ x(l) &\geq 0, \;\; l = -h, -h+1, \ldots, 0 \end{aligned} \quad (22)$$

A convex-combination argument based on theorem 3.1 results in the following theorem.

Theorem 3.3 *Assume that there exist $T-$periodic vectors $z_{lj}(k) \in \mathbb{R}^m$, and $0 < \alpha_l(k) = [\alpha_{l1}(k), \alpha_{l2}(k), \ldots, \alpha_{ln}(k)]^\top \in \mathbb{R}^n$, $l \in \mathbb{I}_{h+1}$ and $j \in \mathbb{I}_{n+1}^*$ such that the following conditions hold:*

$$a_{ij}^{lr}(k)\alpha_{lj}(k) + b_i^r(k)z_{lj}(k) \geq 0; \; i, j \in \mathbb{I}_{n+1}^*; \; l \in \mathbb{I}_{h+1}; \; r \in \mathbb{I}_{m+1}^*; \; k \in \mathbb{I}_T \quad (23)$$

$$\alpha_0(k) - \sum_{l=0}^h A_l^r(k-1)\alpha_l(k-1) - B^r(k-1)\sum_{l=0}^h\sum_{j=1}^n z_{lj}(k-1) = 0; \; r \in \mathbb{I}_{m+1}^*; \; k \in \mathbb{I}_T^*;$$

$$\alpha_l(k) = \alpha_{l-1}(k-1), \;\; l \in \mathbb{I}_{h+1}^* \;\; \text{and} \;\; k \in \mathbb{I}_T^*;$$

$$\alpha_0(0) - \sum_{l=0}^h A_l^r(T-1)\alpha_l(T-1) - B^r(T-1) \times \sum_{l=0}^h\sum_{j=1}^n z_{lj}(T-1) > 0, \;\; r \in \mathbb{I}_{m+1}^*;$$

$$\alpha_l(0) > \alpha_{l-1}(T-1), \;\; l \in \mathbb{I}_{h+1}^*. \quad (24)$$

Then (22) is stable and positive for every $(h+1)-$tuple $\left(\hat{A}_0(k), \ldots, \hat{A}_h(k), \hat{B}(k)\right) \in \mathbb{S}_k$, with $F_l(k) = [z_{l1}(k)/\alpha_{l1}(k), z_{l2}(k)/\alpha_{l2}(k), \ldots, z_{ln}(k)/\alpha_{ln}(k)]$, $l \in \mathbb{I}_{h+1}$; $k \in \mathbb{I}_T$.

Now consider

$$\begin{aligned} x(k+1) &= \sum_{l=0}^h \hat{A}_l(k)x(k-l) + \hat{B}(k)u(k), \\ &\qquad 0 \leq u(k) \leq \bar{u}(k) \\ x(l) &\geq 0, \;\; l = -h, -h+1, \ldots, 0 \end{aligned} \quad (25)$$

where $\bar{u}(k)$ and $u(k)$ are as in (16), and $\hat{A}_l(k)$, $\hat{B}(k)$ are as in (20). Based on theorem 3.2, theorem 3.4 below can be obtained.

Theorem 3.4 *For an arbitrary $T-$periodic vector $0 \leq \bar{u}(k) \in \mathbb{R}^m$, suppose that there exist $T-$periodic vectors $0 < \bar{y}(k) = [\bar{y}_{l1}(k), \bar{y}_{l2}(k), \ldots, \bar{y}_{ln}(k)]^\top \in \mathbb{R}^n$ and $0 \leq z_{lj}(k) \in \mathbb{R}^m$, $l \in \mathbb{I}_{h+1}$ and $j \in \mathbb{I}_{n+1}^*$ such that the following conditions hold:*

$$\sum_{l=0}^h\sum_{j=1}^n z_{lj}(k) \leq \bar{u}(k), \quad a_{ij}^{rl}(k)\bar{y}_{lj}(k) + b_i^r(k)z_{lj}(k) \geq 0;$$

$$i, j \in \mathbb{I}_{n+1}^*; \; l \in \mathbb{I}_{h+1}; \; k \in \mathbb{I}_T; \; r \in \mathbb{I}_{m+1}^*$$

$$\bar{y}_0(k) - \sum_{l=0}^h A_l^r(k-1)\bar{y}_l(k-1) - B^r(k-1) \times \sum_{l=0}^h\sum_{j=1}^n z_{lj}(k-1) = 0; k \in \mathbb{I}_T^*; \; r \in \mathbb{I}_{m+1}^*;$$

$$\bar{y}_l(k) = \bar{y}_{l-1}(k-1), \;\; l \in \mathbb{I}^*_{h+1} \;\; and \;\; k \in \mathbb{I}^*_T;$$

$$\bar{y}_0(0) - \sum_{l=0}^{h} A^r_l(T-1)\bar{y}_l(T-1) - B^r(T-1) \times \sum_{l=0}^{h}\sum_{j=1}^{n} z_{lj}(T-1) > 0, r \in \mathbb{I}^*_{m+1};$$

$$\bar{y}_l(0) > \bar{y}_{l-1}(T-1), \;\; l \in \mathbb{I}^*_{h+1}. \tag{26}$$

Then for every $(h+1)$–tuple $\left(\hat{A}_0(k), \ldots, \hat{A}_h(k), \hat{B}(k)\right) \in \mathbb{S}_k$, $k \in \mathbb{I}_T$, (20) is stable and positive, and $0 \le x(k) \le \bar{y}_0(k)$ and $0 \le u(k) \le \bar{u}(k)$ for all $k \in \mathbb{N}$, whenever $0 \le x(-l) \le \bar{y}_l(0)$, with $F_l(k) = [z_{l1}(k)/\bar{y}_{l1}(k), k_{l2}(k)/\bar{y}_{l2}(k), \ldots, k_{ln}(k)/\bar{y}_{ln}(k)]$, $l \in \mathbb{I}_{h+1}$ and $k \in \mathbb{I}_T$.

4. Illustrative example

Consider the following 2−periodic system (16) with:

$$\begin{bmatrix} A_0(0) & A_1(0) & B(0) \end{bmatrix} = \left[\begin{array}{cc|cc|c} -0.2 & 0.4 & -0.3 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.2 & 0.22 & 0.1 \end{array}\right];$$

$$\begin{bmatrix} A_0(1) & A_1(1) & B(1) \end{bmatrix} = \left[\begin{array}{cc|cc|c} -0.2 & 0.3 & -0.3 & 0.35 & 0.42 \\ 0.1 & 0.2 & 0.1 & 0.25 & 0.15 \end{array}\right]$$

It is easy to check that this 2−periodic system is neither stable nor positive.
We are looking for periodic state-feedback control law that stabilizes the 2−periodic delayed system and enforces the state to be nonnegative with respect to the following constraint on the control signal: $0 \le u(k) \le \min(u(0), u(1)) = 0.3$.
Let u_k as in (6) where $F_l(k)$ is described as (10).
$0 < \bar{y}_l(k) = [\bar{y}_{l1}(k), \bar{y}_{l2}(k)]^\top \in \mathbb{R}^2$, and $z_{l1}(k), z_{l2}(k) \in \mathbb{R}_+$, $l \in \mathbb{I}_{h+1}$, $k \in \mathbb{I}_T$, and $h = 1$ are the solutions of the following linear programming (LP) optimization problem

$$\min_{(\bar{y}_{li}(0), \bar{y}_{li}(1), z_{li}(0), z_{li}(1))} \sum_{i=1}^{2} (\bar{y}_{li}(0) + \bar{y}_{li}(1)) \tag{27}$$

Subjet to:

$$\bar{y}_0(k) - \sum_{l=0}^{1} A_l(k-1)\bar{y}_l(k-1) - B(k-1)\sum_{l=0}^{1}\sum_{j=1}^{2} z_{lj}(k-1) = 0; \quad k \in \mathbb{I}^*_T$$

$$\bar{y}_l(k) = \bar{y}_{l-1}(k-1), \; l \in \mathbb{I}^*_{h+1}, \; and \; k \in \mathbb{I}^*_T$$

$$\bar{y}_0(0) - \sum_{l=0}^{1} A_l(1)\bar{y}_l(1) - B(1)\sum_{l=0}^{1}\sum_{j=1}^{2} z_{lj}(1) > 0$$

$$\bar{y}_l(0) > \bar{y}_{l-1}(1), \; l \in \mathbb{I}^*_{h+1}$$

$$\sum_{l=0}^{1}\sum_{j=1}^{2} z_{lj}(k) \le \bar{u}(k); \; a^l_{ij}(k)\bar{y}_{lj}(k) + b_i(k)z_{lj}(k) \ge 0 \,, j \in \mathbb{I}^*_{n+1}; \quad l \in \mathbb{I}_{h+1} \; and \; k \in \mathbb{I}_T.$$

Using linear programming (the function linprog from the optimisation toolbox is used to determine u_k), and applying theorem 3.2, we obtain the following matrices:

$$\left[\, F_0(0)^\top \;\middle|\; F_1(0)^\top \,\right] = \left[\begin{array}{c|c} 0.5046 & 0.7874 \\ 0.0481 & 0.0290 \end{array}\right];$$

$$\left[\, F_0(1)^\top \;\middle|\; F_1(1)^\top \,\right] = \left[\begin{array}{c|c} 0.4963 & 0.7328 \\ 0.0194 & 0.0240 \end{array}\right]$$

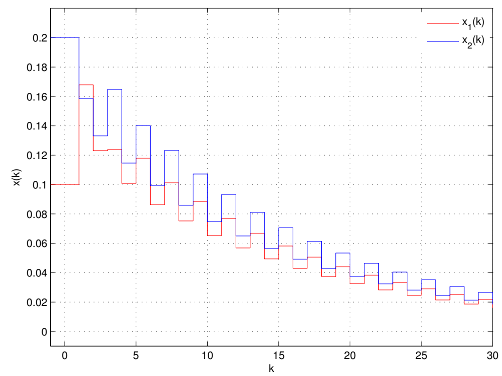

Figure 1. State trajectory $x(k)$ from random positive initial value

Figure 1 shows the evolution of the state variables of the 2−periodic system stating that the considered periodic system is asymptotically stable and positive.

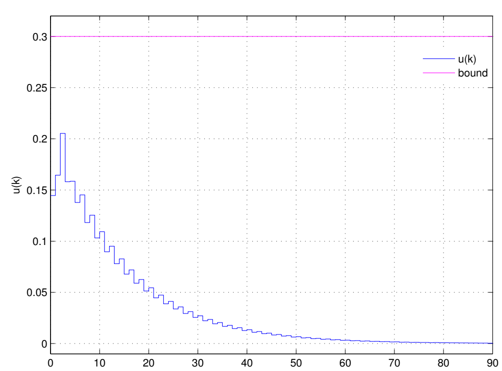

Figure 2. Evolutions of the control signal

Figure 2 shows that the control signal respect the constraints bounds.

5. Conclusion

In this paper, new necessary and sufficient conditions in the form of equality and inequality constraints to stabilizing linear discrete-time periodic system with delays under the positivity constraint are proposed. The problem is also considered for bounded control. The solution to the case where the control vector is positive and upper-bounded is also provided. The case of uncertain discrete-time system with delays, is also addressed and robust state space feedback controller is designed in order to ensure the stability and the positivity of the uncertain closed-loop system. All the proposed conditions are solvable in terms of Linear Programming (LP). The future direction is to generalize the obtained results to positive discrete-time periodic systems with unknown time-delay.

References

[1] B. Roszak and E. Davison, "Necessary and sufficient conditions for stabilizability of positive LTI systems," *Systems and Control Letters*, vol. 58, pp. 474–481, 2009.

[2] H. Gao, J. Lam, C. Wang, and S. Xu, "Control for stability and positivity: Equivalent conditions and computation," *IEEE Transactions on circuits and systems*, vol. 52, no. 9, pp. 540–544, 2005.

[3] A. Hmamed, M. Ait-Rami, F. Tadeo, and A. Benzaouia, "Memoryless control of delayed systems in continuous time,to impose nonnegative closed-loop states," in *IFAC World Congress*, Seoul,Korea, 2009.

[4] M. Ait-Rami and F. Tadeo, "Controller synthesis for positive linear systems," in *IFAC World Congress*, Prague, 2005.

[5] M. Ait-Rami, "Stability analysis and synthesis for linear positive systems with time-varying delays," in *Positive Systems Conference,POSTA 09*, Valencia,Spain, 2009.

[6] A. Benzaouia and F. Tadeo, "Output feedback stabilization of positive switching linear discrete-time systems," in *16th Mediterranean Conference on Control and Automation*, Ajaccio,France, 2008, pp. 119–124.

[7] M. Ait-Rami and F. Tadeo, "Controller synthesis for positive linear systems with bounded controls," *IEEE Transactions on Circuits and Systems II*, vol. 54, pp. 151–155, 2007.

[8] X. Liu, "Constrained control of positive systems with delays," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, 2009.

[9] ——, "Stability and stabilization of positive systems with delays," in *International Conference on Communications, Circuits and Systems*, Milpitas, CA, 2009.

[10] E. De Santis and G. Pola, "Positive switching systems," in *positive systems*, ser. Lecture Notes in Control and Information Science. Springer, 2006, vol. 341, pp. 49–56.

[11] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, ser. Pure and Applied Mathematics. New York, USA: Wiley-Interscience, 2000.

[12] M. Ait-Rami, F. Tadeo, and A. Benzaouia, "Control of constrained positive discrete systems," in *IEEE American Control Conference*, New York,USA, 2007.

[13] S. Bittanti, "Deterministic and stochastic linear periodic systems," in *Time Series and Linear Systems*, S. Bittanti, Ed. Berlin: Springer-Verlag, 1986, pp. 141–182.

[14] S. Bittanti, P. Bolzern, and P. Colaneri, "Stability analysis of linear periodic systems via the Lyapunov equation," in *10th IFAC World Congress*, Budapest, Hungary, 1984, pp. 169–172.

[15] S. Bittanti and P. Bolzern, "Discrete-time linear periodic systems: Gramian and modal criteria for reachability and controllability," *International Journal of Control*, vol. 41, pp. 909–928, 1985.

[16] ——, "Stabilizability and detectability of linear periodic systems," *Systems and Control Letters*, vol. 6, pp. 141–145, 1985.

[17] D. Arzelier., D. Peaucelle., and C. Farges., "Robust analysis and synthesis of linear polytopic discrete-time periodic systems via LMIs," in *Decision and control, 2005 and 2005 European control conference. CDC-ECC'05. 44th IEEE conference on*, 2005, pp. 5734–5739.

[18] S. Bittanti, P. Colaneri, and G. D. Nicolao, "The difference periodic Riccati equation for the periodic prediction problem," *IEEE Transactions on Automatic Control*, vol. 33, pp. 706–712, 1988.

[19] P. Bolzern and P. Colaneri, "The periodic Lyapunov equation," *SIAM Journal of Matrix Analysis Application*, vol. 9, pp. 499–512, 1988.

[20] D. Mehdi, O. Bachelier, and K. Galkowski, "State feedback stabilization of a class of discrete-time, linear, time variant systems," in *Mathematical Theory of Networks and Systems (MTNS)*, Leuven, Belgium, 2004.

[21] C. D. Souza and A. Trofino, "An LMI approach to stabilization of linear discrete-time periodic systems," *Int. J. Control*, vol. 73, no. 8, pp. 696–703, 2000.

[22] S. Bittanti and P. Colaneri, "Stability analysis of linear periodic systems via the Lyapunov equation," in *The 10th IFAC World Congress*, Budapest,Hungary, 1984, pp. 169–172.

[23] X. Liu, W. Yu, and L.Wang, "Stability analysis of positive systems with bounded time-varying delays," *IEEE Transactions on Circuits and Systems*, vol. 56, no. 7, 2009.

[24] T. Kaczorek, "Simple stability conditions for linear positive discrete-time systems with delays," *Bulletin of the Polish Academy of Sciences: Technical Sciences*, vol. 56, no. 4, 2008.

[25] X. Liu, L. Wang, W. Yu, and S. Zhong, "Constrained control of positive discrete-time systems with delays," *IEEE Transactions on Circuits and Systems*, vol. 55, no. 2, 2008.

[26] N. Bougatef, M. Chaabane, O. Bachelier, D. Mehdi, and S. Cauet, "On the stabilization of a class of periodic positive discrete time systems," in *The 49th Conference on Decision and Control*, Atlanta, Georgia USA, 2010.

[27] F. Mesquine, F. Tadeo, and A. Benzaouia, "Regulator problem for linear systems with constraints on control and its increment or rate," *Automatica*, vol. 40, no. 8, p. 1387Ű1395, 2004.

[28] A. Benzaouia and F. Mesquine, "Regulator problem for uncertain linear discrete-time systems with constrained control," *International Journal of Robust and Nonlinear Control*, vol. 4, no. 3, pp. 387–395, 1994.

[29] T. Kaczorek, "Stability of positive continuous-time linear systems with delays," *Bulletin of the Polish Academy of Sciences: Technical Sciences*, vol. 57, no. 4, 2009.

[30] M. A. Rami, U. Helmke, and F. Tadeo, "Positive observation problem for linear time-delay positive systems," in *Mediterranean Conference on Control and Automatic*, Athens, Greece, 2007.

[31] N. Bougatef, M. Chaabane, O. Bachelier, and D. Mehdi, "Stability and stabilization of constrained positive discrete-time periodic systems," in *The 8th International Multi-Conference on Systems, Signals and Devices*, Sousse, Tunisia, 2010.