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Stabilizing switching rule design for affine switched systems

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***Abstract*— We propose a method for designing switching rules that can drive the state of the switched dynamic system to a desired equilibrium point. The method deals with the class of switched systems where each sub-system has an affine vector field. The results are given in terms of linear matrix inequalities and they guarantee global asymptotic stability of the tracking error dynamics even if sliding motion occurs along a switching surface of the system. The switching rules are based on complete and partial state measurements. Two examples are used to illustrate the approach.**

I. INTRODUCTION

The problem of designing switching rules for switched systems have been given considerable attention and several results are now available in the literature (see for instance the surveys [1], [2].)

In continuous-time switched systems, sliding motions are a well understood phenomenon that plays an important theoretical role as they can represent complex dynamics found in many practical applications [3]. In controlled switched systems, it is possible to handle sliding modes at the expense of much complication, by considering the sliding motions and their associated dynamics as additional sub-systems to which the system can switch [1], [3]. For this reason, it is rare to find control design methodologies that can handle sliding modes.

Furthermore, control strategies based on sliding motions cannot be implemented because, in practice, real actuators cannot operate under the arbitrarily fast switching frequencies of a sliding mode, a phenomenon commonly referred to as *chattering*. Most results in the literature *avoid* chattering by introducing minimum dwell time constraints or structural state dependent constraints during the switching rule design [2], [4].

The results in the present paper allow one to design a stabilizing switching law that allows for sliding motions among any number of sub-systems. The work generalizes and extend the results of [5], [6]. The results are based on a Lyapunov function of the type $\max_i\{v_i(x)\}$ where x is the system state and $\{v_i(x)\}$ is a set of auxiliary functions to be determined. This particular type of Lyapunov function was also considered in [5], [7], [8]. In [5], [7] each sub-system is associated with one auxiliary function $v_i(x)$ while in [8] each sub-system is associated with the whole set of functions $\{v_i(x)\}$. In the latter, the number of auxiliary functions may be greater than the number of sub-systems. The function $\max_i\{v_i(x)\}$ has interesting properties but some technical difficulties appear when dealing with sliding motion. See for instance [8], [7] for details. To the author's knowledge there is no switching rule design method in the literature for this type of Lyapunov function that can handle sliding motions involving any number of sub-systems. The main contribution of this paper is to present an LMI solution to this problem. The conditions guarantee global asymptotic stability of the switched system for the class of affine sub-systems.

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Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices; $\|\cdot\|$ stands for the Euclidean norm of vectors and its induced spectral norm of matrices; $\bullet$ represents block matrix terms that can be deduced from symmetry; 0_n and $0_{m\times n}$ are the $n\times n$ and $m\times n$ matrices of zeros, I_n is the $n\times n$ identity matrix. For a real matrix S, S' denotes its transpose and $S>0$ ($S<0$) means that S is symmetric and positive-definite (negative-definite). The symbol $\otimes$ denotes the Kronecker product and $\vartheta(\Theta)$ represents the set of all vertices of the unit simplex $\Theta := \{\theta=[\theta_1\ldots\theta_m]' : \sum_{i=1}^m \theta_i = 1 \;\; , \;\; \theta_i \geq 0\}$.

II. PRELIMINARIES

Consider a switched dynamical system composed of m affine sub-systems

$$\dot{x}(t) = A_i x(t) + b_i \quad , \quad i \in \mathscr{M} := \{1,\ldots,m\} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, which, at this point, is assumed to be available from measurements. The case of partial state measurements will be treated later on. Matrices A_i, and vectors b_i are real with compatible dimensions.

Out goal is to design a switching rule that asymptotically drives the system state to a given constant equilibrium point x_{eq}. We do so by asking that x_{eq} be the only globally asymptotically stable equilibrium point in closed loop.

Given the desired equilibrium point x_{eq} we can represent the tracking error dynamics as the switched system

$$\dot{e}(t) = A_i e(t) + k_i \quad , \quad k_i = b_i + A_i x_{eq} \quad , \quad e(t) := x(t) - x_{eq} \tag{2}$$

where $i \in \mathscr{M} := \{1,\ldots,m\}$. Using the above error system we seek to converge to the origin. With this idea in mind,

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consider the switching rule given by

$$\sigma(e(t)) := \arg\max_{i\in\mathscr{M}}\{v_i(e(t))\}, v_i(e(t)) = e(t)'P_i e(t) + 2e(t)'S_i \quad (3)$$

where $P_i = P_i' \in \mathbb{R}^{n\times n}$ and $S_i \in \mathbb{R}^n$ are matrices to be determined. At each instant of time $\sigma(e(t))$ is an index set corresponding to the set of sub-systems having 'maximum energy'. For instance, $\sigma(e(t_0)) = \{i,j,k\}$ means that at instant $t = t_0$ the error trajectory is at the switching surface defined from the sub-systems $\{i,j,k\}$ because $v_i(e(t_0)) = v_j(e(t_0)) = v_k(e(t_0)) = \max_{i\in\mathscr{M}}\{v_i(e(t_0))\}$.

Assuming the sliding mode dynamics of the system can be represented as convex combinations of the sub-systems [3], the global switched system, including the sub-system dynamics and the sliding mode dynamics that may occur in any switching surface, is represented by

$$\dot{e}(t) = \sum_{i=1}^{m}\theta_i(e(t))\,(A_i e(t) + k_i)\ ,\ \theta(e(t)) \in \Theta \quad (4)$$

where $\theta(e(t))$ is the vector with entries $\theta_i(e(t))$, Θ is the unitary simplex and $\theta_i(e(t)) = 0$ if $i \notin \sigma(e(t))$ and $\{\theta_i(e(t)),\ \forall i \in \sigma(e(t))\}$ are defined according to Filippov [3, p.50]. Recall that a sliding motion may occur at a point $e(t)$ only if it is possible to find a convex combination of the sub-system vector fields such that $\dot{e}(t)$ is a vector on the hyperplane tangent to the switching surface at $e(t)$.

In order to have $e(t) = 0$ as equilibrium point of (4) we must have $\sum_{i=1}^{m}\theta_i(0)k_i = 0$ where $\{\theta_i(0),\ i \in \mathscr{M}\}$ are constants that can be chosen as design parameters. This leads to the following assumption.

Assumption 1: There exist constant scalars $\bar{\theta}_i$ such that

$$\sum_{i=1}^{m}\bar{\theta}_i k_i = 0\ \ ,\ \ \sum_{i=1}^{m}\bar{\theta}_i = 1\ \ ,\ \ \bar{\theta}_i \geq 0\ \ ,\ \ k_i = b_i + A_i x_{eq}. \quad (5)$$

See Remark 1 for a comment on the matrix $\sum_{i=1}^{m}\bar{\theta}_i A_i$.

In the sequel we present some preliminary results and definitions.

Lemma 1 (Finsler's Lemma): Let $\mathscr{W} \subseteq \mathbb{R}^s$ be a given polytopic set, $M(.) : \mathscr{W} \mapsto \mathbb{R}^{q\times q}$, $G(.) : \mathscr{W} \mapsto \mathbb{R}^{r\times q}$ be given matrix functions, with $M(.)$ symmetric. Let $Q(w)$ be a basis for the null space of $G(w)$. Then the following are equivalent:

- (i) $z'M(w)z > 0, \forall z \in \mathbb{R}^q, \forall w \in \mathscr{W}$ such that $G(w)z = 0$.
- (ii) $\exists L(.) : \mathscr{W} \mapsto \mathbb{R}^{q\times r}$ such that $M(w) + L(w)G(w) + G(w)'L(w)' > 0$, $\forall w \in \mathscr{W}$.
- (iii) $Q(w)'M(w)Q(w) > 0$, $\forall w \in \mathscr{W}$. □

Two cases are of particular interest to this paper. The first is when $M(.), G(.)$ are affine functions and L is constrained to be constant. In this situation **(i)**,**(ii)** are no longer equivalent, but **(ii)** is clearly a sufficient polytopic LMI condition for **(i)**. The second case is when $M(.)$ is affine function and G is constrained to be constant, leading Q to be constant as well. In this case **(i)**,**(iii)** are yet equivalent and **(iii)** is a polytopic LMI with a smaller number of decision variable when compared to **(ii)**. The interest of these two polytopic LMI problems is that they are numerically efficient alternatives to the condition **(i)**, which is an infinite dimensional problem. See for instance [9] for more details on the Finsler's Lemma.

Another definition of interest is as follows.

Definition 1 (Linear Annihilator): Given a vector function $f(.) : \mathbb{R}^q \mapsto \mathbb{R}^s$, a matrix function $\aleph_f(.) : \mathbb{R}^q \mapsto \mathbb{R}^{r\times s}$ will be called a *Linear Annihilator of f* if it satisfies the following two requirements **(i)** $\aleph_f(.)$ is linear and **(ii)** $\aleph_f(z)f(z) = 0\ ,\ \forall z \in \mathbb{R}^q$ of interest. □

Observe that the matrix representation of a Linear Annihilator is not unique. Suppose that $z = \begin{bmatrix} z_1 & \dots & z_q \end{bmatrix}' \in \mathbb{R}^q$. Taking into account all possible pairs z_i, z_j for $i \neq j$ without repetition, i.e. for $\forall i,j \in \{1\dots q\}$ with $j > i$, we get an annihilator given by

$$\aleph_z(z) = \begin{bmatrix} \phi_1(z) & Y_1(z) \\ \vdots & \vdots \\ \phi_{(q-1)}(z) & Y_{(q-1)}(z) \end{bmatrix} \in \mathbb{R}^{r\times q}\ ,\ r = \sum_{j=1}^{q-1} j \quad (6)$$

$$Y_i(z) = -z_i I_{(q-i)},\ i \in \{1\dots q-1\}, \phi_1(z) = \begin{bmatrix} z_2 & \dots & z_q \end{bmatrix}'$$

$$\phi_i(z) = \begin{bmatrix} & z_{(i+1)} \\ 0_{(q-i)\times(i-1)} & \vdots \\ & z_q \end{bmatrix},\ i \in \{2\dots q-1\}$$

Linear Annihilators will be used jointly with the Finsler's Lemma to reduce the conservativeness of parameter dependent LMIs.

III. Main Results

Before presenting our main result, we introduce some auxiliary notation. Consider the vectors $\theta, \bar{\theta} \in \mathbb{R}^m$ with entries $\theta_i, \bar{\theta}_i$ defined in (4),(5) respectively. For convenience the dependence of θ and its entries with respect to $e(t)$ will be omitted. Let $\aleph_\theta, \aleph_{\bar{\theta}}$ be linear annihilators of $\theta, \bar{\theta}$ as in Definition 1, and define the auxiliary matrices

$$\begin{aligned} A &= \begin{bmatrix} A_1 & \dots & A_m \end{bmatrix}\ ,\ P = \begin{bmatrix} P_1 & \dots & P_m \end{bmatrix} \\ K &= \begin{bmatrix} k_1 & \dots & k_m \end{bmatrix}\ ,\ S = \begin{bmatrix} S_1 & \dots & S_m \end{bmatrix} \\ \alpha &= \begin{bmatrix} \alpha_1 I_n & \dots & \alpha_m I_n \end{bmatrix}\ ,\ \mathbf{1}_m = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{1\times m} \end{aligned} \quad (7)$$

$$C_a = \begin{bmatrix} 0_{(1\times mn)} & \mathbf{1}_m \end{bmatrix},\ C_b(\theta) = \begin{bmatrix} \aleph_\theta \otimes I_n & 0_{(rn\times m)} \\ 0_{(r\times nm)} & \aleph_\theta - \aleph_{\bar{\theta}} \end{bmatrix} \quad (8)$$

$$I_a = \mathbf{1}_m \otimes I_n\ \ ,\ \ \aleph_\theta, \aleph_{\bar{\theta}} \in \mathbb{R}^{r\times m}\ \ ,\ \ \bar{P} = \sum_{i=1}^{m}\bar{\theta}_i P_i \quad (9)$$

$$\Psi = \begin{bmatrix} (A+\alpha)'P + P'(A+\alpha) - \alpha'\bar{P}I_a - I_a'\bar{P}\alpha & \bullet \\ K'P + S'A + 2S'\alpha & K'S + S'K \end{bmatrix} \quad (10)$$

Theorem 1: Let x_{eq} be a given constant vector representing the desired equilibrium point of the system (1) and suppose the state $x(t)$ is available from measurements. Consider the system (4) under Assumption 1. With the auxiliary notation (7)-(10), let Q_a be a given basis for the null space of C_a and L be a matrix to be determined with the dimensions of $C_b(\theta)'$.

Suppose $\exists P, S, L$ solving the following LMI problem

$$\sum_{i=1}^{m}\bar{\theta}_i P_i > 0\ \ ,\ \ \sum_{i=1}^{m}\bar{\theta}_i S_i = 0 \quad (11)$$

$$Q_a'(\Psi + LC_b(\theta) + C_b(\theta)'L')Q_a < 0\ ,\ \forall\theta \in \vartheta(\Theta) \quad (12)$$

Then the system (4) is globally asymptotically stable with the switching rule (3) and

$$V(e(t)) := \max_{i\in\mathscr{M}}\{v_i(e(t))\}\ ,\ v_i(e(t)) = e(t)'P_i e(t) + 2e(t)'S_i$$

is a Lyapunov function for the switched system. □

Proof: As $\theta_i(e(t)) = 0$ for $i \notin \sigma(e(t))$ and $V(e(t)) = v_i(e(t))$, $\forall i \in \sigma(e(t))$ we get the identities

$$\sum_{i=1}^{m}\theta_i(e(t)) = \sum_{i\in\sigma(e(t))}\theta_i(e(t)) = 1 \tag{13}$$

and

$$\sum_{i=1}^{m}\theta_i(e(t))\,v_i(e(t)) = \sum_{i\in\sigma(e(t))}\theta_i(e(t))\,v_i(e(t)) = \left(\sum_{i\in\sigma(e(t))}\theta_i(e(t))\right)V(e(t)) = V(e(t)) \tag{14}$$

Thus the following holds

$$V(e(t)) := \max_{i\in\mathscr{M}}\{v_i(e(t))\} = \sum_{i=1}^{m}\theta_i(e(t))\,v_i(e(t)) \tag{15}$$

From (11) it follows that

$$\sum_{i=1}^{m}\bar{\theta}_i v_i(e(t)) = e(t)'(\sum_{i=1}^{m}\bar{\theta}_i P_i)e(t) > 0\quad ,\quad \forall e(t)\neq 0 \tag{16}$$

Keeping in mind that the maximum element of a finite set of real numbers is always greater than or equal to any convex combination of the elements of the set we conclude from (15),(16) that $\forall e(t)\neq 0$ we get

$$V(e(t)) \geq e(t)'(\sum_{i=1}^{m}\bar{\theta}_i P_i)e(t) = e(t)'\bar{P}e(t) > 0 \tag{17}$$

Thus $V(e(t))$ is positive definite and radially unbounded as the right hand side of (17) is a positive definite quadratic form in view of (11). Moreover, $v_i(e(t)) \leq \beta_i(\|e(t)\|)$ where $\beta_i(\|e(t)\|) := \|P_i\|\|e(t)\|^2 + 2\|S_i\|\|e(t)\|$. This shows that

$$\lambda_{min}(\bar{P})\|e(t)\|^2 \leq V(e(t)) \leq \max_{i\in\mathscr{M}}\{\beta_i(\|e(t)\|)\} \tag{18}$$

where the upper and lower bounds are class $\mathscr{K}_\infty$ functions. Next we show that $V(e(t))$ is strictly decreasing. With this purpose note that for any point e and direction h the directional derivative of $V(e)$ exists and is given by [10, p.420]

$$D_h V(e) = \max_{i\in\sigma(e)}\nabla v_i(e)\,h \tag{19}$$

where $\nabla v_i(e) = 2(e'P_i + S_i')$ denotes the gradient of $v_i(e)$.

As the system state $e(t)$ is a continuous function of time and $V(e(t))$ is a continuous function of $e(t)$, for the points $e(t)$ in the regions of continuity of the vector field of (4), namely $f(e(t)) := \sum_{i=1}^{m}\theta_i(e(t))\,(A_i e(t) + k_i)$, the time derivative of $V(e(t))$ exists and is given by the directional derivative in the direction $h = \dot{e}(t)$ [3, p.155]. Note that in each region of continuity of the vector field $f(e(t))$ the index set $\sigma(e(t))$ is constant. Thus if $\sigma(e(t))$ has more than one element at a point $e(t)$ and does not change on an infinitesimal increment of time $t^+ - t > 0$ we must have

$$v_i(e(t)) = v_j(e(t)) = V(e(t))\ ,\ \forall i,j\in\sigma(e(t)) = \sigma(e(t^+)) \tag{20}$$

$$\nabla v_i(e(t))\,f(e(t)) = \nabla v_j(e(t))\,f(e(t))\ ,\ \forall i,j\in\sigma(e(t)) \tag{21}$$

While (20) represents the maximum energy property of $v_i(e(t))$ associated with $\sigma(e(t))$, the condition (21) implies $\sigma(e(t))$ does not change on an incremental step in the direction $f(e(t))$ and thus a sliding motion involving the sub-systems indexed by $\sigma(e(t))$ is occurring. Observe (21) may be rewritten as $\nabla v_{ij}(e(t))\,f(e(t)) = 0$, where $v_{ij}(e(t)) = v_i(e(t)) - v_j(e(t))$, that shows the vector field $f(e(t))$ belongs to the tangent hyperplane of the switching surface given by $v_{ij}(e(t)) = 0$. From (21) and (19) with $h = \dot{e}(t) = f(e(t))$ it follows that the time derivative of $V(e(t))$ for the regions of continuity of the vector field $f(e(t))$ is given by

$$\dot{V}(e(t)) = \nabla v_i(e(t))\,f(e(t))\ ,\ \forall i\in\sigma(e(t)) \tag{22}$$

Next, recall from (4) that $\theta_i(e(t)) = 0$, $\forall i\notin\sigma(e(t))$ and $\sum_{i=1}^{m}\theta_i(e(t)) = 1$. Then it follows from (21) that (22) can be rewritten as

$$\dot{V}(e(t)) = \sum_{i=1}^{m}\theta_i(e(t))\,\nabla v_i(e(t))\,f(e(t)) \tag{23}$$

If $\dot{V}(e(t))$ given above is negative definite for all points in the regions of continuity of the vector field of (4) then it is still negative definite if the points of discontinuity are included by using the differential inclusion associated with (4). As shown in [3, p.155] this operation does not change the upper boundary of $\dot{V}(e(t))$. For global stability it is required $\dot{V}(e(t))$ to be negative definite $\forall e(t)\neq 0$, $\forall\theta(e(t))\in\Theta$, with $\theta(e(t))\neq\bar{\theta}$. Observe $e(t) = 0$, $\theta(e(t)) = \bar{\theta}$ is the desired equilibrium and $\dot{V}(0) = 0$ as $f(0) = 0$ in view of (5).

Now applying the S-Procedure to the condition $\dot{V}(e(t)) < 0$ and taking into account the constraint (17) that represents the relation $V(e(t)) = \max_{i\in\mathscr{M}}\{v_i(e(t))\}$ we get

$$\dot{V}(e(t)) + 2\alpha_\theta\left(V(e(t)) - e(t)'\bar{P}e(t)\right) < 0 \tag{24}$$

$\forall e(t)\neq 0$, $\forall\theta(e(t))\in\Theta$, with $\theta(e(t))\neq\bar{\theta}$ and $\alpha_\theta := \sum_{i=1}^{m}\alpha_i\theta_i(e(t)) > 0$ is a scaling factor with positive constants α_i chosen according to the Remark 1.

As the dependence of $\theta(e(t))$ with respect to $e(t)$ is difficult to take into account we will use a more conservative condition where $\theta(e(t))$ is replaced with an arbitrary time-varying parameter, namely θ, free to take values in the unitary simplex Θ. Now with the notation

$$P_\theta =: \sum_{i=1}^{m}\theta_i P_i,\ A_\theta =: \sum_{i=1}^{m}\theta_i A_i,\ K_\theta =: \sum_{i=1}^{m}\theta_i k_i,\ S_\theta =: \sum_{i=1}^{m}\theta_i S_i$$

$\dot{V}(e(t))$ from (23), $V(e(t))$ from (15) we can rewrite (24) as

$$\begin{bmatrix} e(t)\\ 1\end{bmatrix}'\begin{bmatrix} A_\theta' P_\theta + P_\theta A_\theta + 2\alpha_\theta(P_\theta - \bar{P}) & \bullet\\ K_\theta' P_\theta + S_\theta' A_\theta + 2S_\theta'\alpha_\theta & 2K_\theta' S_\theta\end{bmatrix}\begin{bmatrix} e(t)\\ 1\end{bmatrix} < 0 \tag{25}$$

Next rewrite (25) with the notation (7)-(10) as

$$\begin{bmatrix} e_\theta \\ \theta \end{bmatrix}' \Psi \begin{bmatrix} e_\theta \\ \theta \end{bmatrix} < 0 \tag{26}$$
$$e_\theta = \begin{bmatrix} \theta_1 e(t)' & \ldots & \theta_m e(t)' \end{bmatrix}' \in \mathbb{R}^{mn}$$

Now observe from (5) and (11) that $K\bar{\theta} = S\bar{\theta} = 0$ and with Ψ given from (10) it follows that $\begin{bmatrix} 0 & \bar{\theta}' \end{bmatrix} \Psi = 0$. Thus we can rewrite (26) as

$$\begin{bmatrix} e_\theta \\ \theta \end{bmatrix}' \Psi \begin{bmatrix} e_\theta \\ \theta \end{bmatrix} = \begin{bmatrix} e_\theta \\ \theta - \bar{\theta} \end{bmatrix}' \Psi \begin{bmatrix} e_\theta \\ \theta - \bar{\theta} \end{bmatrix} < 0 \tag{27}$$

With $C_a, C_b(\theta)$ from (8) it follows that

$$C_a \begin{bmatrix} e_\theta \\ \theta - \bar{\theta} \end{bmatrix} = 0 \quad , \quad C_b(\theta) \begin{bmatrix} e_\theta \\ \theta - \bar{\theta} \end{bmatrix} = 0 \tag{28}$$

Then, for any matrix L of suitable dimension we can rewrite (27) as

$$\begin{bmatrix} e_\theta \\ \theta - \bar{\theta} \end{bmatrix}' (\Psi + LC_b(\theta) + C_b(\theta)'L') \begin{bmatrix} e_\theta \\ \theta - \bar{\theta} \end{bmatrix} < 0 \tag{29}$$

Taking into account the null space of C_a through the Finsler's Lemma we get the LMI in (12) as a sufficient condition for $\dot{V}(e(t))$ in (23) to satisfy $\dot{V}(e(t)) < 0$, $\forall e(t) \neq 0 \in \mathbb{R}^n$ and $\forall \theta(e(t)) \neq \bar{\theta} \in \Theta$ where $e(t) = 0$, $\theta(e(t)) = \bar{\theta}$ is the desired equilibrium.

In summary, $V(e(t))$ is continuous, positive definite and satisfies the bounds (18) globally. Moreover, $V(e(t))$ is globally strictly decreasing for the dynamics of the system (4) that includes the sub-system dynamics and the sliding mode dynamics that eventually may occur at any switching surface associated with the switching rule (3). By considering the sliding motions on each sliding surface as the dynamics of an additional sub-system of (4) it follows that a finite number of changes will occur in any finite time and global asymptotic stability follows from the results in [3]. □

Remark 1: Notice that for the global stability problem considered in this paper, a necessary condition for (25) to be satisfied is $A'_\theta P_\theta + P_\theta A_\theta + 2\alpha_\theta(P_\theta - \bar{P}) < 0$. As $\bar{\theta} \in \Theta$ this condition implies, for $\theta = \bar{\theta}$, that $A'_\theta(\bar{\theta})P_\theta(\bar{\theta}) + P_\theta(\bar{\theta})A_\theta(\bar{\theta}) < 0$ which in turn implies $A_\theta(\bar{\theta})$ must be Hurwitz stable because $P_\theta(\bar{\theta}) = \bar{P} > 0$. The requirement of $A_\theta(\bar{\theta})$ being Hurwitz stable is removed if θ is not allowed to take values in the whole simplex Θ so that $\theta = \bar{\theta}$ cannot occur, i.e. there is no sliding mode at the equilibrium point. If there exists a suitable region of the simplex Θ that contains the equilibrium $\bar{\theta}$ and that is known to be free of sliding motions then it is possible to consider problems in which $A_\theta(\bar{\theta})$ is not Hurwitz stable after minor changes in the Theorem 1. Due to space limitation this point will be addressed in a future work. Observe in addition that we can rewrite the above inequality as $(A_\theta + \alpha_\theta I_n)'P_\theta + P_\theta(A_\theta + \alpha_\theta I_n) - 2\alpha_\theta \bar{P} < 0$. As $\alpha_\theta \bar{P} > 0$ this condition suggests the constants α_i can be chosen as in [5] in the interval $0 < \alpha_i < |\underline{\lambda}_i|$ where $\underline{\lambda}_i$ denotes the real part of the stable eigenvalue of A_i nearest to the imaginary axis and $|\underline{\lambda}_i|$ its absolute value. The idea is to get exponential decreasing of $V(e(t))$ in the directions where the negative term $-2\alpha_\theta e(t)'\bar{P}e(t)$ in (24) can be neglected. In this case (24) becomes the exponential performance requirement of [5]. □

A. Partial state measurement

The results of the Theorem 1 are essentially state feedback: the complete state of the system is necessary to determine the active mode according to the switching rule (3). In practice, however, the whole state is often not available. In the sequel, we introduce a switching rule that uses output feedback, that is partial state measurements. Consider the system (1) with measurement vector $y(t) = C_i x(t) \in \mathbb{R}^{g_i}$, and $C_i \in \mathbb{R}^{g_i \times n}$ for $i \in \mathscr{M}$ given matrices. Define the output tracking error

$$\varepsilon(t) = y(t) - C_i x_{eq} = C_i e(t) \tag{30}$$

Assume that the auxiliary functions $v_i(e(t))$, $i \in \mathscr{M}$ have the following structure

$$P_i =: P_0 + C_i' Q_i C_i \quad , \quad S_i =: S_0 + C_i' R_i \quad , \quad \forall i \in \mathscr{M} \tag{31}$$

where $P_0 = P_0' \in \mathbb{R}^{n \times n}$, $S_0 \in \mathbb{R}^n$, $R_i \in \mathbb{R}^{g_i}$, $Q_i = Q_i' \in \mathbb{R}^{g_i \times g_i}$. In this case the functions $v_i(e(t))$ can be rewritten as

$$v_i(e(t)) = e(t)'P_0 e(t) + 2e(t)'S_0 + \mu_i(\varepsilon(t))$$

where $\mu_i(\varepsilon(t)) =: \varepsilon(t)'Q_i\varepsilon(t) + 2\varepsilon(t)'R_i$. Therefore

$$\max_{i \in \mathscr{M}} \{v_i(e(t))\} = e(t)'P_0 e(t) + 2e(t)'S_0 + \max_{i \in \mathscr{M}} \{\mu_i(\varepsilon(t))\}$$

and from (3) the switching rule becomes now a function of the output tracking error as

$$\arg \max_{i \in \mathscr{M}} \{v_i(e(t))\} = \arg \max_{i \in \mathscr{M}} \{\mu_i(\varepsilon(t))\}. \tag{32}$$

This shows that the Theorem 1 can be directly applied to cope with the case of partial state information by introducing the constraints (31) on the structure of the matrices P_i, S_i.

IV. Numerical Examples

We have used SeDuMi with the Yalmip [11] interface to solve the LMIs and Simulink to get the trajectories of the switched system in the next examples.

Example 1: Consider a buck-boost converter with a linear (resistor) load in the state space representation (1) with two sub-systems, $\mathscr{M} = \{1, 2\}$, where $b_2 = 0_{2x1}$ and

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \; b_1 = \begin{bmatrix} \frac{E_{in}}{L} \\ 0 \end{bmatrix}$$

The system states are the inductor current (x_1) and the output capacitor voltage (x_2). The constants $E_{in} = 15\,[volt], L = 10^{-3}\,[Henry], C = 10^{-6}\,[farad]$ are, respectively, the external source voltage, the inductance of the input circuit and the capacitance of the output filter. The nominal load resistance is $R = 30\,[ohm]$. The eigenvalues of A_1 and A_2 are, respectively, $\{-33333.34, 0\}$ and $\{-16666.7 \pm j26874.1\}$. According to Remark 1, the design parameters were chosen as $\alpha_1 = 333$ and $\alpha_2 = 166$.

The desired equilibrium point x_{eq} and the constants k_i of (2) are as follows

$$x_{eq} = \begin{bmatrix} \frac{E_{out}^2 - E_{out}E_{in}}{E_{in}R} \\ E_{out} \end{bmatrix}, k_1 = \begin{bmatrix} \frac{E_{in}}{L} \\ -\frac{E_{out}}{RC} \end{bmatrix}, k_2 = \begin{bmatrix} \frac{E_{out}}{L} \\ -\frac{E_{out}^2}{E_{in}RC} \end{bmatrix}$$

where E_{out} is the desired value of the regulated output. The output voltage has opposite polarity if compared to the input. The following relationship $\frac{E_{out}}{E_{in}} = -\frac{\bar{\theta}_1}{\bar{\theta}_2} = -\frac{\bar{\theta}_1}{1-\bar{\theta}_1}$ can be established. It shows that the converter operates as a buck if $\bar{\theta}_1 < 0.5$ and as a boost if $\bar{\theta}_1 > 0.5$. Note that the sub-system 1 is not Hurwitz stable, however any convex combination of these sub-systems is stable.

Assume now that $E_{out} = -9\,[volt]$, which means the converter operates as a buck. Solving the LMIs of the Theorem 1, we get the matrices $\{P_1, S_1, P_2, S_2\}$ from which the switching rule (3) can be computed. The numerical values are omitted due to space limitation. The switched system response to the zero initial state is shown in Figure 1. Observe that the output voltage is correctly regulated. The phase plane of the tracking error is also shown in Figure 1. Note that when the trajectory touches the switching surface for the second time, a sliding motion occurs driving the error towards the origin. The case where $E_{out} = -21\,[volt]$,

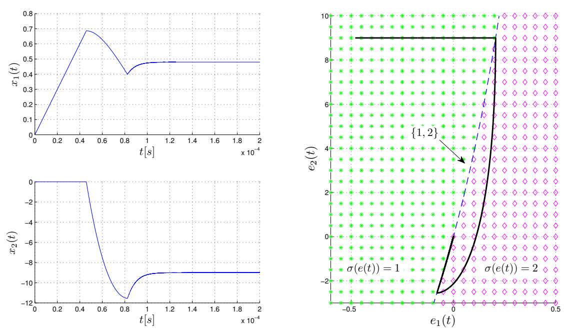

Fig. 1. Buck-boost converter in buck type operation with $E_{out} = -9\,[volt]$.

which means the converter operates as a boost, was also considered. The system response and the phase plane are shown in Figure 2; observe that the output voltage is also correctly regulated. There are several switchings in finite time before the sliding motion starts driving the error to the origin. The oscillations of the regulated output could be attenuated by including a performance requirement to the problem. Theorem 1 deals only with the regulation problem. However, according to Remark 1 it is possible to improve the transient response with a suitable choice of the parameters α_i. It is important to emphasize that feasibility of the conditions in Theorem 1 typically occurs for a wide range of these parameters. For this converter, in particular, the range is approximately $\alpha_i \in \{20|\underline{\lambda_i}|, |\underline{\lambda_i}|/1000\}$ where $\underline{\lambda_i}$ denotes the real part of the stable eigenvalue of A_i nearest to the imaginary axis. Typically the response is fast, often oscillatory, for small values, and slow, often damped, for large values of α_i. Figure 3 shows the system response in buck and boost operation for a switching rule designed with $\alpha_1 = 24.975 \times 10^3$ and $\alpha_2 = 12.450 \times 10^3$, which corresponds to $\alpha_i = 0.75\,|\lambda_i|$. $\square$

Example 2: In this example we consider a system in the state space representation (2) with three sub-systems, $\mathscr{M} = \{1,2,3\}$, where A_1, A_2, A_3, k_1, k_2 are respectively:

$$\begin{bmatrix} 0 & 1 \\ -1 & -\beta \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2\beta & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $k_3 = \begin{bmatrix} -2 & -1 \end{bmatrix}'$. For this system the desired equilibrium is the origin and $\bar{\theta}_1 = \bar{\theta}_2 = \bar{\theta}_3 = 1/3$ satisfy (5). As $\bar{\theta}$ represents the duty cycle at the equilibrium, we must have a sliding mode among the three sub-systems at equilibrium. Start with the case where $\beta = 1$, where all sub-systems are Hurwitz stable, but the desired equilibrium point (origin) is not an equilibrium of any sub-system. The eigenvalues of A_1, A_2 and A_3 are, respectively, $\{-0.5 \pm j0.866\}$, $\{-1 \pm j\}$, and $\{-1.5 \pm j0.866\}$. According to Remark 1, observe that the matrix $A_\theta(\bar{\theta}) = \sum_{i=1}^{3} A_i \bar{\theta}_i$ is Hurwitz stable and the design parameters $\{\alpha_i\}$ were chosen as $\alpha_1 = 0.25$, $\alpha_2 = 0.50$, and $\alpha_3 = 0.75$. The Theorem 1 was applied to get the matrices $\{P_i, S_i,\ i \in \mathscr{M}\}$ from which the switching rule (3) is computed. Simulation results for different initial conditions are shown in the phase plane of Figure 4. It can be seen that in all cases the error system states converge to the origin. When the trajectory reaches the origin, a sliding mode involving the three sub-systems occurs, as expected. Sliding motions outside the origin also occur in the switching surfaces of the sub-systems $\{2,3\}$ and $\{3,1\}$. Next consider the case where $\beta = -1$. In this case the system has two unstable sub-systems, A_1 and A_2 with eigenvalues $\{0.5 \pm j0.866\}$ and $\{0.73, -2.73\}$ respectively, and one Hurwitz stable sub-

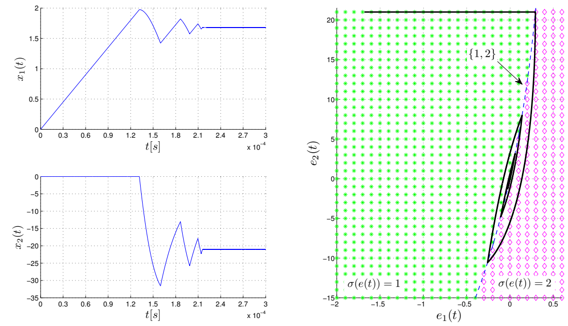

Fig. 2. Buck-boost converter in boost type operation with $E_{out} = -21\,[volt]$.

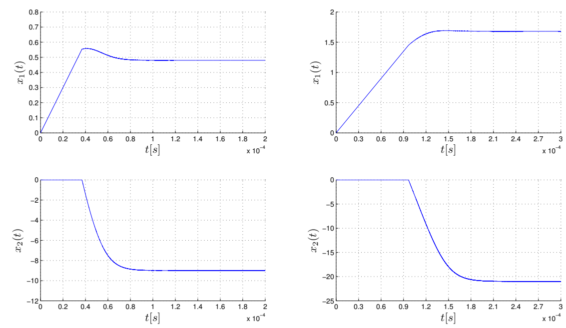

Fig. 3. Buck-boost converter in buck (left curves) and boost (right curves) operation for a switching rule designed with a suitable choice of the parameters α_i .

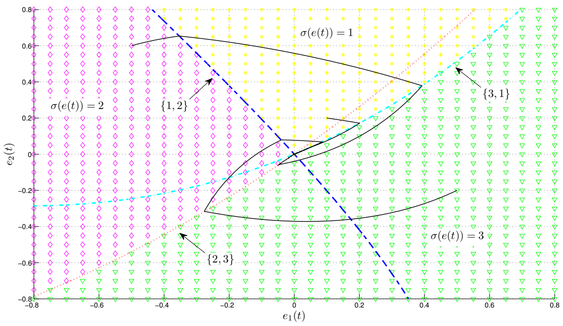

Fig. 4. Stable sub-systems ($\beta = 1$). Solid (black) lines are error trajectories; Dashed (color) lines are switching surfaces.

system, A_3 with eigenvalues $\{-1.5 \pm j0.866\}$. The design parameters $\{\alpha_i,\ i \in \mathscr{M}\}$ and $\bar{\theta}$ have the same values used in the previous case and $A_\theta(\bar{\theta})$ is also Hurwitz stable in this case. Figure 5 presents the simulation results in a phase plane for one specific initial condition. As in the previous case, at the origin we observe a sliding motion among the three sub-systems and outside the origin two sliding motions occur for this trajectory in the switching surfaces of the sub-systems $\{1,2\}$ and $\{3,1\}$. $\square$

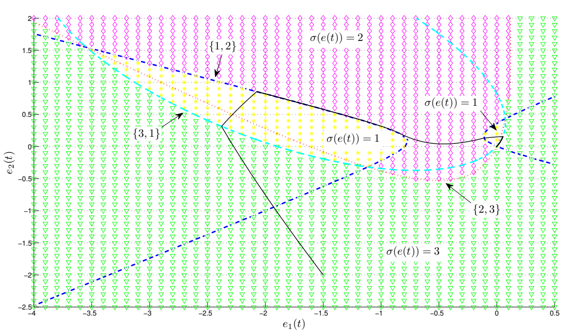

Fig. 5. Unstable sub-systems ($\beta = -1$). Solid (black) line is the error trajectory; Dashed (color) lines are switching surfaces.

V. Concluding Remarks

The switching rule design proposed in the present paper can be extended in several directions. For instance, it can include H_∞ and guaranteed cost performance. The case of uncertain affine sub-systems is easy from [12] if the equilibrium point ($\bar{\theta}$) is not uncertain. The switching rules obtained in the present paper may contain ideal sliding modes. Extension to the case where dwell time restrictions are applied to avoid chattering are currently being investigated and will be reported in future works.

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