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Optimal Control of a Class of Stochastic Hybrid Systems with Probabilistic Constraints

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***Abstract*— In this paper, a class of discrete-time stochastic hybrid systems, in which only discrete dynamics are stochastic, is considered. For this system, a solution method for the optimal control problem with probabilistic constraints is proposed. Probabilistic constraints guarantee that the probability that the continuous state reaches a given unsafe region is less than a given constant. In the propose method, first, continuous state regions, from which the state reaches a given unsafe region, are computed by a backward-reachability graph. Next, mixed integer quadratic programming problems with constraints derived from the backward-reachability graph are solved. The proposed method can be applied to model predictive control.**

I. Introduction

Hybrid systems are dynamical systems composed of continuous dynamics such as differential/difference equations and discrete dynamics such as finite automata. Recently, analysis and control of hybrid systems have been extensively studied in the control theory community and the computer science community. Furthermore, the framework on analysis and control of hybrid systems has been extended to stochastic hybrid systems (SHSs) (see e.g., [1], [9]). SHSs are well known as a model of communication networks [8] and biological systems [1], and developing analysis and control methods is one of the significant works from theoretical and practical viewpoints.

Although a general class of SHSs has been proposed in [7], it is difficult to solve control problems, and special classes are frequently considered. One of the typical classes is to assume that continuous dynamics are deterministic. Even if the system is limited to such a class, then there are several applications such as failure-prone systems [5], [6]. In control of such a class of SHSs, an approximate method for solving the finite-time optimal control problem has been proposed so far [10]. In this problem, the cost function is given as the expected value of some non-negative function. In [3], [5], the lower bound of some non-negative function itself is minimized under the constraint that the probability that the optimal mode (discrete state) sequence is realized is larger than a given constant. Especially, in [5], a given system is modeled by a mixed logical dynamical (MLD) model [4], and the above problem is reduced to a mixed integer quadratic programming (MIQP) problem. On the other hand, in [2], probabilistic reachability has been discussed for controlled discrete-time stochastic hybrid systems, and the set of the initial state such that the probability of staying within a given safe set is maximized is computed. In stochastic systems, it is desirable to impose probabilistic constraints, and the above method is useful for analysis of SHSs. However, generating the control input has not been discussed. To our knowledge, the optimal control problem with probabilistic constraints has not been considered so far.

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In this paper, for a class of discrete-time SHSs, in which continuous dynamics are deterministic and only discrete dynamics are stochastic, a solution method for the optimal control problem with probabilistic constraints is proposed, based on backward-reachability analysis. Probabilistic constraints in this paper are given as a constraint that the probability that the continuous state reaches a given unsafe region is less than a given constant. The proposed solution method consists of two steps. First, a backward-reachability graph is computed. In backward-reachability graphs, paths of continuous state regions such that the continuous state reaches a given unsafe region at N discrete-time step are enumerated, where N is the prediction horizon. Backward-reachability graphs can be computed by using a suitable tool to manipulate convex polyhedra, e.g., POLKA [12] and PolyLib [13]. From the backward-reachability graph, the probability that the continuous state reaches a given unsafe region can be computed, and state and input constraints required for satisfying a probabilistic constraint can be derived. Next, the optimal input can be generated by solving MIQP problems with state and input constraints derived from the backward-reachability graph. MIQP problems can be solved by using a suitable solver such as CPLEX [11]. The proposed approach provides us a new framework on optimal control with probabilistic constraints.

Notation: Let $\mathcal{R}$ denote the set of real numbers. Let $\{0,1\}^n$ denote the set of n-dimensional vectors, which consists of elements 0 and 1. For a set M, let 2^M denote the power set. For two events A and B, let $P(A|B)$ denote the conditional probability.

II. Stochastic Hybrid Systems

First, a class of stochastic hybrid systems (SHSs) to be studied here is defined.

Definition 1: A stochastic hybrid system is given by a tuple

$$\mathcal{H} = (\mathcal{X}_c, \mathcal{X}_d, \mathcal{U}_c, f, \mathcal{P}, g) \tag{1}$$

where

- $\mathcal{X}_c \subseteq \mathcal{R}^{n_c}$ represents a continuous state space, and is given as a convex polyhedron. $x_c \in \mathcal{X}_c$ is called here a continuous state.

- $\mathcal{X}_d$ represents a finite set of a discrete state space with $|\mathcal{X}_d| = n_d$. $x_d \in \mathcal{X}_d$ is called here a mode or discrete state. In addition, $(x_d, x_c) \in \mathcal{X}_d \times \mathcal{X}_c$ is called here a hybrid state.
- $\mathcal{U}_c \subseteq \mathcal{R}^{m_c}$ represents a continuous input space, and is given as a convex polyhedron. $u_c \in \mathcal{U}_c$ is called here a continuous input.
- $f : \mathcal{X}_c \times \mathcal{X}_d \times \mathcal{U}_c \to \mathcal{X}_c$ represents a flow map. A flow map expresses continuous dynamics as follows:

$$x_c(k+1) = A_{x_d(k)} x_c(k) + B_{x_d(k)} u_c(k) + a_{x_d(k)}$$

  where $k \in \{0, 1, 2, \ldots\}$ is discrete time, and A_{x_d}, B_{x_d}, and a_{x_d} are certain matrices/vectors given for each $x_d \in \mathcal{X}_d$.
- $\mathcal{P}$ represents a finite set of m_d discrete probabilistic distributions. A distribution $p_r \in \mathcal{P}$, $r = 1, 2, \ldots, m_d$ is given by $\{p_{i(r),j_1}(r), p_{i(r),j_2}(r), \ldots, p_{i(r),j_{q(r)}}(r)\}$, $i(r), j_1, j_2, \ldots, j_{q(r)} \in \mathcal{X}_d$, where

$$p_{i,j}(l) := P(x_d(k+1) = j \mid x_d(k) = i, r = l)$$

  and $\sum_j p_{i(r),j}(r) = 1$ for fixed r. $i(r), j_1, j_2, \ldots, j_{q(r)}$ are given, and $(i(r), j_1), (i(r), j_2), \ldots, (i(r), j_{q(r)})$ corresponds to edges in the directed graph expressing discrete dynamics in SHSs.
- A mapping $g : \mathcal{P} \to 2^{\mathcal{X}_c}$ represents a guard condition. Each guard condition is given as a convex polyhedron. Assume that an intersection of guard conditions is empty except for guard conditions associated with a trap mode. If the discrete state transits to a given trap mode, then the discrete state stay at the trap mode.

In this paper, we consider a discrete time setting, and continuous dynamics are given as linear systems. Basically, the SHS (1) is the same as discrete hybrid stochastic automata proposed in [5], but different notations are used.

We show a simple example.

Example 1: Consider a tank system in Fig. 1, where x_c is the water level, u_c is the volume of water charged to the tank. When the mode is "off", the water level is always decreased. The transition from "on" ("off") to "off" ("on") is probabilistic. If it is repeatedly failed to switch the mode, then the mode transits to "stop". The mode "stop" is a trap mode, that is, if the mode reaches "stop", then the mode stays at "stop".

In this tank system, $n_c = 1$, $\mathcal{X}_d = \{\text{on}, \text{off}, \text{stop}\}$, $m_c = 1$, $m_d = 7$. Continuous dynamics are given as on: $x_c(k+1) = x_c(k) + u_c(k) - a$, off: $x_c(k+1) = x_c(k) - a$, stop: $x_c(k+1) = x_c(k)$. $\mathcal{P} = \{p_1, p_2, \ldots, p_7\}$ is given as follow:

$$p_1 = \{p_{\text{off,off}}\},\ p_{\text{off,off}} = 1.0,$$
$$p_2 = \{p_{\text{off,off}}, p_{\text{off,on}}\},\ p_{\text{off,off}} = 0.1,\ p_{\text{off,on}} = 0.9,$$
$$p_3 = \{p_{\text{off,stop}}\},\ p_{\text{off,stop}} = 1.0,$$
$$p_4 = \{p_{\text{on,on}}\},\ p_{\text{on,on}} = 1.0,$$
$$p_5 = \{p_{\text{on,off}}, p_{\text{on,on}}\},\ p_{\text{on,off}} = 0.9,\ p_{\text{on,on}} = 0.1,$$
$$p_6 = \{p_{\text{on,stop}}\},\ p_{\text{on,stop}} = 1.0,$$
$$p_7 = \{p_{\text{stop,stop}}\},\ p_{\text{stop,stop}} = 1.0.$$

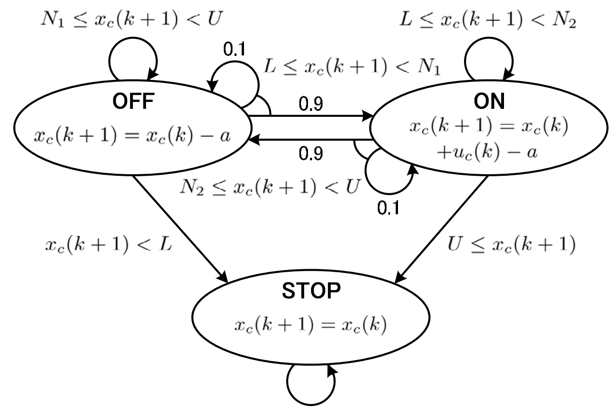

Fig. 1. Simple tank system

The guard condition for p_1 is given by

$$g(p_1) = \{x_c(k+1) \mid N_1 \leq x_c(k+1) < U\}.$$

The guard conditions for $p_2, \ldots, p_7$ are omitted due to limited space (see Fig. 1). Note that in Fig. 1, the intersection of guard conditions except for $g(p_7)$ is empty.

III. Problem Formulation

In this section, the optimal control problem to be studied here is formulated.

First, as preparations, we define one symbol. By $\pi_{x_d(0),x_d(1),\ldots,x_d(N-1)}(0, N-1)$ or $\pi(0, N-1)$ for short, we denote the probability that some mode sequence $x_d(0), x_d(1), \ldots, x_d(N-1)$ is realized. For the example of Fig. 1, suppose that the initial mode is given as "off". Then the probability $\pi_{\text{off,on,off}}(0, 2)$ that the mode sequence off, on, off is realized is 0.81.

Next, an unsafe mode is given for the system. In Fig. 1, "stop" corresponds to an unsafe mode. In addition, an unsafe state region assigned to an unsafe mode is given. In Fig. 1, an unsafe state region is $x_c < L$ and $x_c \geq U$. The following assumptions are made for the unsafe mode and the unsafe state region.

Assumption 1: The unsafe mode is a trap, i.e., if $x_d(k)$ is the unsafe mode, then for all $\bar{k} > k$, $x_d(\bar{k})$ is also the unsafe mode.

Assumption 2: The unsafe state region is given as one convex polyhedron.

Assumption 1 implies that if the mode reaches the unsafe mode, then the system stays at the unsafe mode. The tank system in Fig. 1 satisfies Assumption 1, but does not satisfy Assumption 2. If the probabilistic distribution p_2 is given as $p_2 = \{p_{\text{off,on}}\}$, $p_{\text{off,on}} = 1.0$, then this system satisfies Assumption 2. As one of the other methods, either $x_c < L$ or $x_c \geq U$ may be ignored. Although Assumption 2 is imposed for simplicity of discussion, the proposed method in Section IV can be extended to the case that the unsafe state region is given as more than two convex polyhedra.

Then, for the SHS (1), consider the following optimal control problem.

Problem 1: Suppose that for the SHS (1), the initial hybrid state $x_c(0) = x_{c0}$, $x_d(0) = x_{d0}$, the unsafe mode

$d \in \mathcal{X}_d$, the unsafe state region $D \subseteq \mathcal{X}_c$, the constants $\varepsilon, \rho \in [0, 1]$, the control time N, the weighting matrices $Q, Q_f \in \mathcal{R}^{n_c \times n_c}$, $R \in \mathcal{R}^{m_c \times m_c}$, and the offset vector $\bar{x} \in \mathcal{X}_c$ are given. Then find a continuous input $u_c(0), u_c(1), \ldots, u_c(N-1)$ satisfying the following conditions:
(i) the probability that the continuous state reaches a given unsafe state region is equal to or less than ε,
(ii) for all mode sequences satisfying

$$\pi(0, N-1) \geq \rho, \tag{2}$$

the lower bound of the following cost function

$$J = \sum_{i=0}^{N-1} \left\{ \hat{x}_c^T(i) Q \hat{x}_c(i) + u_c^T(i) R u_c(i) \right\} + \hat{x}_c^T(N) Q_f \hat{x}_c(N) \tag{3}$$

is minimized, where $\hat{x}_c(i) := x_c(i) - \bar{x}$.

First, we discuss the condition (i). In the SHS (1), the input constraints can be deterministically imposed. However, since the behavior of the SHS (1) is stochastic, it is not appropriate to impose the state constraints deterministically. If the system is not satisfied state constraints, then the system stops in many situations. So we impose the probabilistic constraint.

Next, we discuss the condition (ii). In standard control methods of stochastic systems, the expected value of some non-negative function is minimized. However, for the SHS (1), it is difficult to evaluate the expected value, because all combinations of mode sequences must be enumerated. In this paper, instead of the expected value, the lower bound of a given cost function is minimized and evaluated. Then, if the constraint (2) is not imposed in Problem 1, i.e., $\rho = 0$, then the behaviors of the SHS (1) are regarded as uncertain behaviors, and the best performance is derived in Problem 1. However, since combinations of mode sequences selected with low probability are included, the derived performance index may not be appropriate. So in order to exclude such combinations, we impose the constraint (2). One of the methods for deciding ρ is to give ρ as the mean probability that some mode sequence is selected.

Problem 1 without the condition (i) has been discussed in [5], and can be reduced to a mixed integer quadratic programming (MIQP) problem. However, the probabilistic constraint has not been considered.

IV. Proposed Solution Method

In this section, first, the outline of the proposed solution method of Problem 1 is explained by using a simple example. Next, a general case is explained.

A. Outline

The proposed solution method consists of two steps.

Procedure for solving Problem 1:
Step 1: Compute a backward-reachability graph from a given unsafe state region.

Step 2: Solve MIQP problems with constraints derived from the computed backward-reachability graph.

First, Step 1 is explained by using the tank system in Fig. 1. Suppose that the unsafe state region is given as $91 \leq x_c \leq 120$. In addition, suppose that the input constraint is given as $20 \leq u(k) \leq 30$, and a is given as $a = 6$. Consider finding the continuous state region such that the continuous state reaches $91 \leq x_c \leq 120$. Then we can obtain the backward-reachability graph in Fig. 2. Each node corresponds to a hybrid state. Note that the continuous state is given as some region. The label on each edge is a transition probability at one discrete-time step, and a set of linear inequalities with respect to the continuous state and input is assigned to each edge. For example, the transition probability from $(\mathrm{on}, [77, 91))$ to $(\mathrm{stop}, [91, 120])$ is given as 0.1 under the constraint $20 \leq u_c \leq 30$. Next, there are two transitions in $(\mathrm{on}, [67, 77))$. In the transition to $(\mathrm{on}, [77, 91))$, the transition probability is given as 0.1 under the constraints

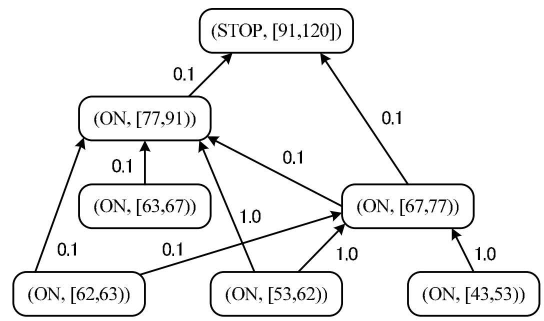

Fig. 2. Example of backward-reachability graphs

$$x_c + u_c < 97, \; u_c \geq 20, \; x_c \geq 67. \tag{4}$$

In the transition to $(\mathrm{stop}, [91, 120])$, the transition probability is given as 0.1 under the constraints

$$x_c + u_c \geq 97, \; u_c \leq 30, \; x_c < 77. \tag{5}$$

See Section IV-B for further details. In $(\mathrm{on}, [67, 77))$, the continuous state can transit to either $(\mathrm{on}, [77, 91))$ or $(\mathrm{stop}, [91, 120])$ with the probability 0.1 by appropriately selecting the value of the continuous input. That is, we can select either 0.1 or 0.01 as the transition probability from $(\mathrm{on}, [67, 77))$ to $(\mathrm{stop}, [91, 120])$. In other words, if the transition probability is selected as 0.01, then the probability that the continuous state avoids the unsafe state region is 0.99. Thus by using a backward-reachability graph, the probability that the continuous state reaches a given unsafe state region can be computed, and the constraints for avoiding the unsafe state region can be derived.

Next, Step 2 is explained. By assigning a binary variable to each edge, the SHS (1) can be modeled by a mixed logical dynamical (MLD) model [4]. Then Problem 1 without the condition (i) can be reduced to an MIQP problem [5]. Furthermore, by imposing the constraints derived from the backward-reachability graph, Problem 1 including the condition (i) can be reduced to MIQP problems.

The above is the outline of the proposed solution method of Problem 1. In Section IV-B, a method to compute a backward-reachability graph is proposed. In Section IV-C, we explain details of the procedure for reducing Problem 1 to MIQP problems.

B. Computation of Backward-Reachability Graphs

First, some symbols used in this subsection are defined. By $Pre(X)$, denote a set of a pair of the hybrid state and input such that the hybrid state reaches a given set $X = S \times T \subset \mathcal{X}_d \times \mathcal{X}_c$ at one discrete-time step. More precisely

$$Pre(X) := \{(x_d, x_c, u_c) \in \mathcal{X}_d \times \mathcal{X}_c \times \mathcal{U}_c \mid \exists x_c' \in T,\ x_c' = A_{x_d} x_c + B_{x_d} u_c + a_{x_d}\}.$$

By $Pro_X(Y)$, denote a projection of $Y \subseteq \mathcal{X}_d \times \mathcal{X}_c \times \mathcal{U}_c$ to the hybrid state space $\mathcal{X}_d \times \mathcal{X}_c$. $Pre(X)$ and $Pro_X(Y)$ can be computed by using a suitable tool to manipulate convex polyhedra, e.g., POLKA [12] and PolyLib [13]. In addition, for given unsafe mode $d \in \mathcal{X}_d$ and state region $D \subseteq \mathcal{X}_c$, define the enlarged unsafe state region $D_u := \{d\} \times D \times \mathcal{U}_c$.

Then the following procedure for deriving a backward-reachability graph ($N \geq 2$) is proposed.

Procedure for deriving a backward-reachability graph:
Step 1: Set $k = 1$, and compute

$$Y_1 := Pre(D) - D_u,\ \ Y_i := Pre(Pro_X(Y_{i-1})) - D_u$$

and $W_i := Pro_X(Y_i)$, $i = 1, 2, \ldots, \bar{p}$, where $\bar{p} = N$.

Step 2:
Step 2-1: Split W_i, $i = 1, 2, \ldots, \bar{p}$ to the following two sets

$$W_i - \cup_{j \neq i} W_j,\ \ W_i \cap (\cup_{j \neq i} W_j)$$

where $j \in \{1, 2, \ldots, \bar{p}\}$.
Step 2-2: If obtained convex polyhedra are included in multiple guard conditions, then split corresponding convex polyhedra. By $X_1, X_2, \ldots, X_p$, denote the unsafe region $\{d\} \times D$ and obtained convex polyhedra.
Step 2-3: For each X_i, assign the minimum number of discrete-time steps such that the hybrid state included in X_i reaches $\{d\} \times D$.

Step 3: If a new split does not occur in Step 2, or $k = N$, then go to Step 5. Otherwise, set $k = k + 1$ and go to Step 4.

Step 4: Except X_i such that the minimum step is N, and denote remaining X_i by $X_1, X_2, \ldots, X_{\bar{p}}$, $\bar{p} \leq p$. Compute

$$W_i := Pro_X(Pre(X_i) - D_u),\ \ i = 1, 2, \ldots, \bar{p}$$

and return to Step 2.

Step 5: Compute a backward-reachability graph, where vertices correspond to the region $X_1, X_2, \ldots, X_p$. Edges can be derived from the minimum number of steps such that the hybrid state included in X_i reaches $\{d\} \times D$. The transition probability and the constraints assigned to each edge can be derived from data computed in the above step.

Note here that in the proposed procedure, $X_1, X_2, \ldots, X_p$ are in general non-convex. If X_i is non-convex, then X_i is expressed by a set of convex polyhedra.

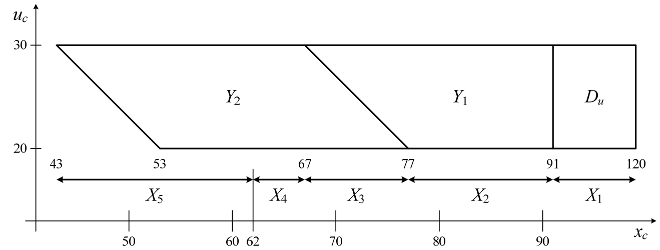

Fig. 3. Step 1 and Step 2. It is omitted to illustrate the space $\mathcal{X}_d$.

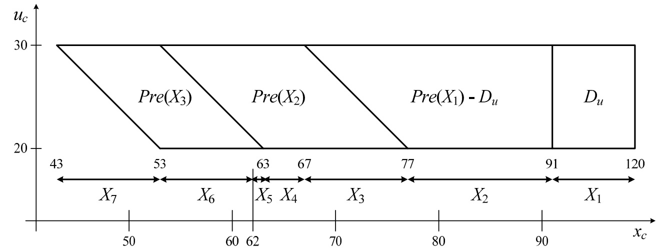

Fig. 4. Step 3 and Step 4. It is omitted to illustrate the space $\mathcal{X}_d$.

Using the tank system in Fig. 1, the proposed procedure is explained. Consider the case of $a = 6$, $N_1 = 41$, and $N_2 = 62$. Suppose that the input constraint and the prediction horizon are given as $20 \leq u_c \leq 30$ and $N = 2$, respectively. The enlarged unsafe state region is given as $\{\text{stop}\} \times [91, 120] \times [20, 30]$. Since the dimension of $\mathcal{X}_c \times \mathcal{U}_c$ is two, we can illustrate the space $\mathcal{X}_c \times \mathcal{U}_c$ such as Fig. 3 and Fig. 4. It is omitted to illustrate the space $\mathcal{X}_d$.

In Step 1, we can obtain Y_1 and Y_2 shown in Fig. 3, where the mode for each region is "on". In addition, $W_1 = \{\text{on}\} \times [67, 91)$, $W_2 = \{\text{on}\} \times [43, 77)$ can be derived.

In Step 2-1, from obtained W_1, W_2, W_1 is split to $W_1 - W_2 = \{\text{on}\} \times [77, 91)$ and $W_1 \cap W_2 = \{\text{on}\} \times [67, 77)$. In a similar way, W_2 is split to $W_2 - W_1 = \{\text{on}\} \times [43, 67)$ and $W_2 \cap W_1 = \{\text{on}\} \times [67, 77)$. In Step 2-2, noting $N_2 = 62$, the set $\{\text{on}\} \times [43, 67)$ is split to $\{\text{on}\} \times [43, 62)$ and $\{\text{on}\} \times [62, 77)$. Thus we can obtain $X_1, X_2, \ldots, X_5$ (i.e., $p = 5$) shown in Fig. 3. In Step 2-3, the minimum numbers of steps such that the hybrid state included in X_i, $i = 1, 2, \ldots, 5$ reaches $\{d\} \times D$ are 0, 1, 1, 2, and 2, respectively.

In Step 3, $k = 1$ is updated to $k = 2$, and go to Step 4.

In Step 4, X_4 and X_5 are excepted, because the minimum step is $2 (= N)$. That is, $\bar{p} = 3$ holds. In addition, $Pre(X_1) - D_u, Pre(X_2) - D_u (= Pre(X_2)), Pre(X_3) - D_u (= Pre(X_3))$ shown in Fig. 4 can be derived, where the mode for each region is "on". Thus $W_1 = \{\text{on}\} \times [67, 91)$, $W_2 = \{\text{on}\} \times [53, 77)$, and $W_3 = \{\text{on}\} \times [43, 63)$ can be derived, and return to Step 2.

In Step 2-1, $W_1 = \{\text{on}\} \times [67, 91)$ is split to $\{\text{on}\} \times [67, 77)$ and $\{\text{on}\} \times [77, 91)$. $W_2 = \{\text{on}\} \times [53, 77)$ is split to $\{\text{on}\} \times [63, 67)$ and a pair of $\{\text{on}\} \times [53, 63)$ and $\{\text{on}\} \times [67, 77)$, which is a non-convex polyhedron and is expressed as two convex polyhedra. $W_3 = \{\text{on}\} \times [43, 63)$ is split to $\{\text{on}\} \times [43, 53)$ and $\{\text{on}\} \times [53, 63)$. In Step 2-2, the region $\{\text{on}\} \times [53, 63)$ is split to $\{\text{on}\} \times [53, 62)$ and $\{\text{on}\} \times [62, 63)$. Thus we can obtain $X_1, X_2, \ldots, X_7$ shown in Fig. 4.

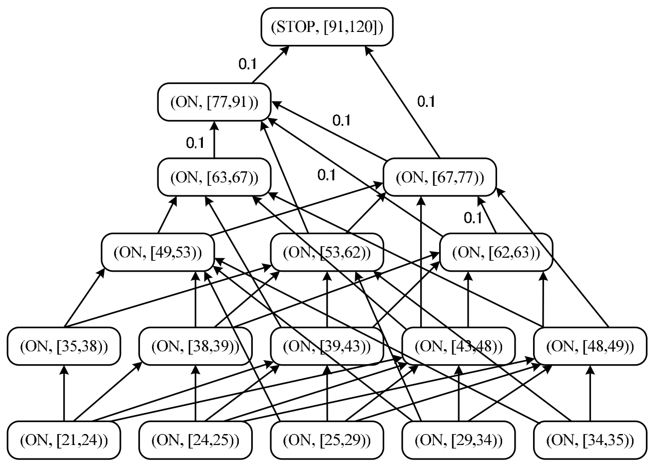

Fig. 5. Example of backward-reachability graphs ($N = 5$). If the probability assigned to the edge is equal to 1, then this is not denoted.

In Step 3, since $N = 2$, go to Step 5.

In Step 5, a backward-reachability graph is computed. Thus we can obtain the graph shown in Fig. 2.

In the case of $N = 5$, we can obtain the backward-reachability graph in Fig. 5.

C. Reduction to MIQP Problems

In this subsection, the outline of a method to model the SHS (1) as the MLD model is explained by the tank system in Fig. 1 at first. See [4], [5] for a general case.

First, a binary variable is assigned to each edge in Fig. 1 (see Fig. 6). Assume that the following equality constraint

$$\begin{aligned}&\delta_{11}^1(k) + \delta_{11}^2(k) + \delta_{12}(k) + \delta_{13}(k)\\&+\delta_{22}^1(k) + \delta_{22}^2(k) + \delta_{21}(k) + \delta_{23}(k) + \delta_{33}(k) = 1\end{aligned}$$

holds. In addition, a binary variable $\delta_i(k)$ is assigned to each node in Fig. 1. "off", "on" and "stop" correspond to mode 1, 2 and 3, respectively (see also Fig. 6). If the mode at time k is i, then $\delta_i(k) = 1$ and $\delta_j(k) = 0$, $j \neq i$ hold. In this case, the following relations: $\delta_1(k) = \delta_{11}^1(k) + \delta_{11}^2(k) + \delta_{12}(k) + \delta_{13}(k)$, $\delta_2(k) = \delta_{22}^1(k) + \delta_{22}^2(k) + \delta_{21}(k) + \delta_{23}(k)$, and $\delta_3(k) = \delta_{33}(k)$ hold. A binary variable assigned to each edge is associated with a given guard condition. For example, for the guard condition $g(p_2)$, we can obtain $[\delta_{11}^1(k) = 1] \vee [\delta_{12} = 1] \rightarrow [L \leq x_c(k+1) < N_1]$, which can be expressed as a set of linear inequalities [4].

Continuous dynamics can be expressed as

$$\begin{aligned}x_c(k+1) = {}&\delta_1(k)\,(x_c(k) - a) + \delta_2(k)\,(x_c(k) + u_c(k) - a)\\&+\delta_3(k)x_c(k).\end{aligned}$$

Although this expression is nonlinear, this can be transformed into a linear form by using a set of linear inequalities. Discrete dynamics can be expressed as $\delta_1(k+1) = \delta_{11}^1(k) + \delta_{11}^2(k) + \delta_{21}(k)$, $\delta_2(k+1) = \delta_{22}^1(k) + \delta_{22}^2(k) + \delta_{12}(k)$, and $\delta_3(k+1) = \delta_{33}(k) + \delta_{13}(k) + \delta_{23}(k)$.

By using the above expressions, the SHS (1) can be expressed as the following MLD model:

$$\begin{cases} x(k+1) = Ax(k) + Bv(k), \\ Cx(k) + Dv(k) \leq E \end{cases} \tag{6}$$

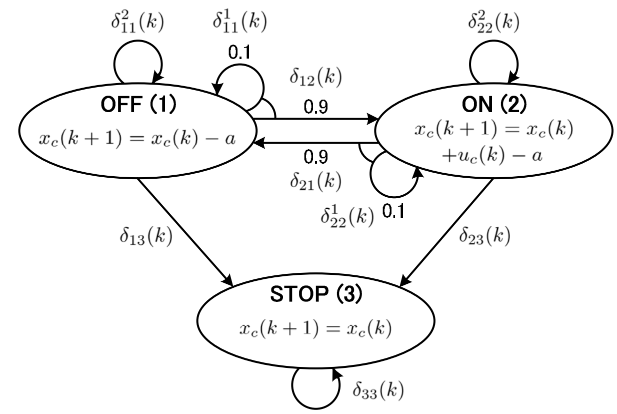

Fig. 6. Assignment of binary variables

where $x(k) \in \mathcal{R}^{n_1} \times \{0,1\}^{n_2}$ is the state, $v(k)$ is given by $v(k) = [\ u^T(k)\ \ z^T(k)\ \ \delta^T(k)\ ]^T$, $u(k) \in \mathcal{R}^{m_{1c}} \times \{0,1\}^{m_{1d}}$ is the input, and $z(k) \in \mathcal{R}^{m_2}$ and $\delta(k) \in \{0,1\}^{m_3}$ are auxiliary continuous and binary variables, respectively.

Next, consider how to express the constraint (2) as a linear form. In the case of the tank system in Fig. 1 and Fig. 6, we can obtain $\ln\pi(0, N-1) = \sum_{k=0}^{N-1} L\delta_L(k)$, where $L := [\ \ln 0.1\ \ \ln 0.9\ \ \ln 0.1\ \ \ln 0.9\ ]$, $\delta_L(k) := [\ \delta_{11}^1(k)\ \ \delta_{12}(k)\ \ \delta_{22}^1(k)\ \ \delta_{21}(k)\ ]^T$. So the constraint (2) can be transformed into a linear inequality constraint by using the natural logarithm.

Finally, paths such that the condition (i) in Problem 1 is satisfied are enumerated among the computed backward-reachability graph. By $Z(i) \subseteq (\mathcal{X}_d \times \mathcal{X}_c)^{N+1}$, $i = 1, 2, \ldots, \bar{\gamma}$, denote paths from some hybrid state to an unsafe state region. By $P_Z(i)$, $i = 1, 2, \ldots, \bar{\gamma}$, denote the probability that each path is realized. In addition, linear inequality constraints corresponding to each path are denoted by

$$F(k,i)x(k) + G(k,i)v(k) \leq H(k,i) \tag{7}$$

where $k = 0, 1, \ldots, N-1$ and $i = 1, 2, \ldots, \bar{\gamma}$. Note here that linear inequality constraints are in general time-varying.

Consider the backward-reachability graph in Fig. 2 as an example. The initial hybrid state is given as $(\text{on}, 55)$. Suppose $N = 2$. Then we obtain $\bar{\gamma} = 2$,

$$\begin{aligned}Z(1) &= ((\text{on}, [53, 62)), (\text{on}, [77, 91)), (\text{stop}, [91, 120])),\\ Z(2) &= ((\text{on}, [53, 62)), (\text{on}, [67, 77)), (\text{stop}, [91, 120]))\end{aligned}$$

and $P_Z(1) = 0.1$, $P_Z(2) = 0.1$. Linear inequality constraints can be derived from Fig. 4.

After paths are enumerated according to the initial hybrid state, we select paths such that the condition (i) in Problem 1 is satisfied. By $\gamma(\leq \bar{\gamma})$, denote the number of selected paths. By imposing linear inequality constraints (7) to the system (6), the condition (i) is satisfied, and γ time-varying MLD models are derived. Note here that even if any model is selected, then the condition (i) is satisfied. So the optimal path is selected according to the condition (ii) in Problem 1.

Based on the above discussion, let us consider solving

Problem 1. Consider the following γ MIQP problems.

i-th MIQP problem ($i = 1, 2, \ldots, \gamma$):

$$\begin{array}{rl} \text{find} & v(k), \quad k = 0, 1, \ldots, N-1 \\ \min & \text{Cost function (3)} \\ \text{subject to} & \text{System (6)}, \ x_c(0) = x_{c0}, \ x_d(0) = x_{d0}, \\ & F(k,i)x(k) + G(k,i)v(k) \leq H(k,i), \\ & \ln\pi(0, N-1) \geq \ln\rho. \end{array}$$

By J_i^*, $i = 1, 2, \ldots, \gamma$, denote the optimal value of a given cost function in the i-th MIQP problem. Thus we obtain the following theorem immediately.

Theorem 1: The optimal value J^* of a given cost function in Problem 1 is derived as

$$J^* = \min \left\{ J_1^*, J_2^*, \ldots, J_\gamma^* \right\}.$$

The optimal input sequence is derived as the input sequence corresponding to J^*.

From this theorem, we see that the optimal input sequence in Problem 1 can be derived by solving γ MIQP problems.

V. Numerical Example

In this section, we show a numerical example. Consider solving Problem 1 for the tank system in Fig. 1.

Parameters in the tank system are given as $a = 6$, $L = 13$, $N_1 = 41$, $N_2 = 62$, and $U = 91$. The continuous state space and the continuous input space are given as $\mathcal{X}_c = [0, 120]$ and $\mathcal{U}_c = [20, 30]$, respectively. The initial hybrid state is given as $(\text{off}, 20)$. The unsafe state region is given as $[91, 120]$. In the cost function (3), N, Q, Q_f, R, and $\bar{x}$ are given as $N = 5$, $Q = Q_f = 1$, $R = 1$, and $\bar{x} = 50$, respectively. ε in the condition (i) of Problem 1 is given as $\varepsilon = 0.002$. ρ in the constraint (2) is given as $\rho = 0.1 \times 0.9^3$.

Then we obtain the backward-reachability graph in Fig. 5. From Fig. 5, we can obtain the following three paths such that the condition (i) of Problem 1 is satisfied:

Path 1: $(\text{on}, [19, 24)) \to (\text{on}, [43, 48)) \to (\text{on}, [62, 63)) \to (\text{on}, [67, 77)) \to (\text{on}, [77, 91)) \to (\text{stop}, [91, 120])$,
Path 2: $(\text{on}, [19, 24)) \to (\text{on}, [39, 43)) \to (\text{on}, [62, 63)) \to (\text{on}, [67, 77)) \to (\text{on}, [77, 91)) \to (\text{stop}, [91, 120])$,
Path 3: $(\text{on}, [19, 24)) \to (\text{on}, [38, 39)) \to (\text{on}, [62, 63)) \to (\text{on}, [67, 77)) \to (\text{on}, [77, 91)) \to (\text{stop}, [91, 120])$.

In addition, $P_Z(1) = P_Z(2) = P_Z(3) = 0.001$.

By solving the MIQP problem for each path, we can obtain

$$J_1^* = 3715, \quad J_2^* = 3795, \quad J_3^* = 3830.$$

So we see that Path 1 ($Z(1)$) is optimal. In addition, the optimal continuous input sequence is derived as $u_c(0) = 28$, $u_c(1) = 25$, $u_c(2) = 20$, $u_c(3) = 20$, and $u_c(4) = 20$. Sample trajectories of the continuous state in the closed-loop system are shown in Fig. 7. In addition, the number of trajectories such that the continuous state reaches the unsafe state region $[91, 120]$ is 127 for 100000 sample trajectories of the continuous state. So we see that the probabilistic constraint in the condition (i) of Problem 1 is satisfied.

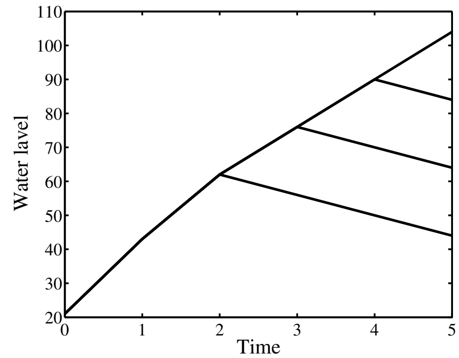

Fig. 7. Sample trajectories of the continuous state

VI. Conclusion and Future Works

In this paper, we have considered the optimal control problem of a class of stochastic hybrid systems. In particular, probabilistic constraints have been focused. In the proposed solution method, first, a backward-reachability graph is computed, and next, MIQP problems are solved. The obtained result is useful as a method to solve the optimal control problem with probabilistic constraints.

The backward-reachability graph in this paper is closely related to discrete abstraction techniques in hybrid systems. It is one of future works to clarify the relation. In addition, it is also significant to apply the proposed approach to several applications.

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References

[1] A. Abate, J. Lygeros, and S. Sastry Eds., *47th IEEE Conf. on Decision and Control/Workshop on Stochastic Hybrid Systems: Theory and Applications*, 2008.
[2] A. Abate, M. Prandini, J. Lygeros, and S. Sastry, Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems, *Automatica*, vol. 44, pp. 2724–2734, 2008.
[3] F. Adamek, M. Sobotka, and O. Stursberg, Stochastic optimal control for hybrid systems with uncertain discrete dynamics, *Proc. 4th IEEE Conf. on Automation Science and Engineering*, pp. 23–28, 2008.
[4] A. Bemporad and M. Morari, Control of systems integrating logic, dynamics, and constraints, *Automatica*, vol. 35, pp. 407–427, 1999.
[5] A. Bemporad and S. Di Cairano, Model-predictive control of discrete hybrid stochastic automata, *IEEE Trans. on Automatica Control*, vol. 56, no. 6, pp. 1307–1321, 2011.
[6] P. Bolzern, P. Colaneri, and G. De Nicolao, Markov Jump Linear Systems with switching transition rates: Mean square stability with dwell-time, *Automatica*, vol. 46, pp. 1081–1088, 2010.
[7] M. L. Bujorianu and J. Lygeros, General stochastic hybrid systems: Modelling and optimal control, *43rd IEEE Conf. on Decision and Control*, pp. 1872–1877, 2004.
[8] J. P. Hespanha, A model for stochastic hybrid systems with application to communication networks, *Nonlinear Analysis*, vol. 62, pp. 1353–1383, 2005.
[9] J. P. Hespanha, Why should I care about stochastic hybrid systems?, *49th IEEE Conf. on Decision and Control*, Semi-plenary talk, 2010.
[10] L. Shi, A. Abate, and S. Sastry, Optimal control of a class of stochastic hybrid systems, *Proc. 43rd IEEE Conf. on Decision and Control*, pp. 1842–1847, 2004.
[11] http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/
[12] http://www.inrialpes.fr/pop-art/people/bjeannet/newpolka/html/polka.html
[13] http://icps.u-strasbg.fr/polylib/