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Robust stability analysis in the $*$-norm and Lyapunov-Razumikhin functions for the stability analysis of time-delay systems[†]

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***Abstract*— Lyapunov-Krasovskii functionals have been shown to have connections with input-output techniques considering delay operators mapping L_2 to L_2. It is shown here that Lyapunov-Razumikhin functions can also be connected to the input-output framework by considering operators on L_∞ and the corresponding Small-Gain Theorem. Several important results from the Lyapunov-Razumikhin Theorem are retrieved and extended.**

***Index Terms*— Time-delay systems; Lyapunov-Razumikhin Functions; Robust stability; $*$-norm**

I. Introduction

The stability analysis and control of time-delay is an active research domain and many approaches have been developed along years: frequency domain techniques [1], [2], [3], Lyapunov-Krasovskii and Lyapunov-Razumikhin approaches [4], [5], [6], [7], [2], [8], small-gain-based methods [1], IQCs techniques [9], [10], well-posedness approaches [11], [12] and ISS techniques [13]. The approaches based on Lyapunov-Krasovskii Functionals (LKFs) are one of the most spread since they lead to LMI results, can be applied to a wide range of problems and may provide necessary and sufficient conditions, but very often at the expense of computational complexity and poor scalability. More recently, input-output approaches (IQC, well-posedness) have led to drastic improvements with respect to these drawbacks, at least for the stability analysis problem [14], [15], [16].

There is, although yet not fully proved, a connection between L_2 input-output approaches and LKFs; see e.g. [17], [18] and [8, Section 3.2.1.6]. LKFs may indeed be viewed as robustness analysis tools in the L_2-norm for time-delay systems. However, the existence of an equivalent LKF formulation, given a combination of delay operators in the L_2 input-output framework, is still an open question.

One important particularity of many LKFs is the consideration delay derivative upper bound constraint. Recent works [19], [20], [14], [21] have attempted to get rid of this constraint[1] due to possible applications to aperiodic sampled-data systems [23], where the delay-derivative equals one almost everywhere, and networked control systems (NCSs) for which abrupt changes in the delays values are possible. Such a constraint is however naturally excluded when Lyapunov-Razumikhin Functions (LRFs) are used and this explains their presence in the context of NCCs and network analysis. Nevertheless, their utilization is not so easy since they very often lead tedious and haphazard-looking manipulations, conservative results and quasi-convex conditions. Despite of that, Razumikhin's approach yields structurally simple results, involving a few number of variables and small matrix inequalities, at the difference of some LKFs approaches. For these reasons, the results are more scalable than those obtained from LKFs and the control design made simpler.

We will show here that it is possible, using input/output approaches, to obtain generalized Razumikhin-like conditions, difficult to obtain using a direct approach via the Lyapunov-Razumikhin Theorem. To this aim, a matrix inequality test for the computation of an upper bound on the QL_∞-norm is obtained. This result is extended to incorporate D-scalings in order to prepare its use for the robust stability analysis of uncertain linear systems perturbed by BIBO stable operators. It is shown that the general conditions of the Lyapunov-Razumikhin Theorem in the linear case are the same as the conditions of scaled-small gain result in the QL_∞-norm. By using delay and integral operators combinations, delay-independent and delay-dependent stability conditions, independent of the delay-derivative, are finally obtained. While the delay-independent result is equivalent to the Lyapunov-Razumikhin condition, the delay-dependent results take a slightly more general form. Compared to the original result from the application of Lyapunov-Razumikhin Theorem, the obtained conditions are less complex since they involve fewer nonlinear terms.

The goal of the paper is to bring a new insight on Razumikhin's approach by showing the connection with input/output approaches. Interestingly, the provided approach leads to quick and easy calculations in contrast to the usual Razumikhin approach which needs model transformations, bounding procedures and incorporation of the Razumikhin condition. The operator approach has the potential of leading to a wide diversity of results according to the considered combination of operators. Indeed, as discussed in [24], many operators corresponding to Taylor expansion remainders can be generated and used to describe a time-delay system. Fragmented operators may be considered as well [18].

The paper is structured as follows: Section II discusses about the QL_∞-norm and the corresponding operator norm, the $*$-norm. In Section III, a scaled-small gain in the $*$-norm is developed for robust stability analysis. In Section IV, we apply the developed results in the context of stability analysis

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[1]It is important to mention that, according to several works, when the delay-derivative exceeds one, severe well-posedness problems may occur [22].

of time-delay systems with unconstrained delay-derivative.

The notations are standard. Given two real symmetric matrices A, B, $A \prec (\preceq) B$ means that $A - B$ is negative (semi)definite. The Kronecker product is denoted by $\otimes$.

II. Preliminaries

A. Signal Norms

We consider in this paper the space of bounded functions also taking bounded values on measure zero sets. Given a bounded function $w : \mathbb{R}_+ \to \mathbb{R}^n$, the L_∞-norm and the QL_∞*-norm* (Q for quadratic) defined as

$$\begin{array}{rcl} ||w||_{L_\infty} & = & \sup_{t\geq 0} ||w(t)||_\infty \\ ||w||_{QL_\infty} & = & \sup_{t\geq 0} ||w(t)||_2. \end{array} \tag{1}$$

both induce the same space of bounded functions, i.e. $w \in L_\infty \Leftrightarrow w \in QL_\infty$. They indeed define the same topology on the space of bounded functions since

$$||w||_{L_\infty} \leq ||w||_{QL_\infty} \leq \sqrt{n}||w||_{L_\infty} \tag{2}$$

for any function $L_\infty \ni w : \mathbb{R} \to \mathbb{R}^n$. This norm has been first introduced in [25] in order to provide tractable conditions for peak-to-peak gain minimization. Until now, there is unfortunately no efficient way of minimizing the L_∞-norm or using it for robustness analysis by simple means [26].

Note also that it is voluntary here to consider the supremum rather than the essential supremum in the norm definition since the norm based on the latter does not always characterize the pure time-varying delay operator as a bounded operator. To see this, consider the time-varying delay operator

$$\mathscr{D}_h \quad : \quad w(t) \quad \to \quad w(t - h(t)) \tag{3}$$

defined for any bounded input function $w : \mathbb{R} \to \mathbb{R}^n$ and delay $h : \mathbb{R} \to \mathbb{R}_+$. Choosing the input

$$w(t) = \left\{ \begin{array}{ll} 1 & \text{if } t = t_0 \\ 0 & \text{otherwise} \end{array} \right. \tag{4}$$

and the delay

$$h(t) = \left\{ \begin{array}{ll} 0 & \text{if } t \in [0, t_0] \\ t - t_0 & \text{otherwise} \end{array} \right. \tag{5}$$

we obtain

$$w(t - h(t)) = \left\{ \begin{array}{ll} 1 & \text{if } t \geq t_0 \\ 0 & \text{otherwise.} \end{array} \right. \tag{6}$$

Using the above signals, we obtain the results of Table I. We can see that the use of the 'ess sup'-based norm leads to an unbounded operator which is not desirable. Note that this may also occur with bounded delays. It will be shown in Section IV, that the QL_∞-norm indeed defines bounded delay operators for any delay trajectory.

norm	$\|\|w\|\|$	$\|\|\mathscr{D}_h(w)\|\|$	$\|\|\mathscr{D}_h\|\|$
ess sup $\|\|\cdot\|\|_\infty$	0	1	$+\infty$
QL_∞	1	1	1

TABLE I

Comparison of norms and induced-norms for input signal (4) and delay (5)

B. Systems Norms

It is important to discuss briefly about the operator-norm induced by the QL_∞-norm which we refer to as the $*$*-norm*. This norm induces the same topology as the L_∞-induced norm (the L_1-norm) on the space of asymptotically stable linear systems:

Proposition 1: For any given bounded operator H (finite L_1- and $*$-norms) mapping p inputs to q outputs, we have

$$p^{-1/2}||H||_{L_1} \leq ||H||_* \leq q^{1/2}||H||_{L_1}. \tag{7}$$

Proof: The proof follows from inequality (2). ∎

This shows that in the SISO case, the two norms coincide. Moreover, when the number of output is one, then the $*$-norm is always smaller than the L_1-norm. In such circumstances, when considering the stability of an interconnection of a system H with an uncertain term Δ verifying $||\Delta||_* = ||\Delta||_{L_1}$, the use of the $*$-norm may be beneficial since $||H||_* \leq ||H||_{L_1}$, authorizing then a larger set of uncertainties. It is however difficult to conclude on anything in the general MIMO case.

C. Computational Results

Let us consider here, an LTI system H whose state-space representation is given by

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & Cx(t) + Fw(t) \\ x(0) & = & x_0 \end{array} \tag{8}$$

where $x, x_0 \in \mathbb{R}^n$, $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$ are respectively the system state, the initial condition, the exogenous input and the controlled output.

There is, at this time, no efficient way of computing/optimizing the exact $*$-norm. A Riccati inequality approach was proposed in [25] to compute an upper-bound on the $*$-norm. Later, a quasi-LMI (qLMI) solution was proposed in [27]. We continue in the same vein and consider the matrix inequality framework. We, however, slightly improve the qLMI result by providing a smaller one involving (possibly) fewer decision variables:

Lemma 1 ($*$-Bounded Real Lemma): The LTI system H with state-space representation (8) is asymptotically stable if there exist a symmetric matrix $P \succ 0$ and scalars $\xi, \zeta, \varepsilon > 0$ such that the matrix inequality

$$\begin{bmatrix} A^TP + PA + \xi P + \varepsilon I & PE & 0 & 0 \\ \star & -\zeta I & 0 & F^T \\ \star & \star & -\xi P & C^T \\ \star & \star & \star & -\zeta I \end{bmatrix} \preceq 0 \tag{9}$$

holds. Moreover, in such a case, we have $||H||_* \leq \zeta$. △

Proof: The proof is inspired from [27] but with different subtleties allowing to derive a more compact and flexible result. The proof is divided in two parts. First, we provide two matrix inequality conditions that characterize an upper-bound on the $*$-norm of a given system. The second part is devoted to the merging of these conditions into a single one by variable elimination.

Part 1. We start from the following matrix inequalities

$$\begin{bmatrix} A^TP+PA+\xi P+\varepsilon I & PE \\ \star & -Q \end{bmatrix} \prec 0 \tag{10}$$

and

$$\begin{bmatrix} \xi P & 0 & C^T \\ \star & \zeta I - Q & F^T \\ \star & \star & \zeta I \end{bmatrix} \succeq 0 \tag{11}$$

defined for some matrices $P = P^T \succ 0$, $Q = Q^T \succ 0$ and scalars $\varepsilon, \xi, \zeta > 0$. We prove now that the feasibility of the above conditions implies that the $*$-norm is bounded from above by ζ. The first inequality is equivalent to $\dot{V}(t) + \xi V(t) - w(t)^T Q w(t) \leq -\varepsilon x(t)^T x(t)$, for all $x(t) \in \mathbb{R}^n, w(t) \in \mathbb{R}^p$ and where $V(t) = x(t)^T P x(t)$. Hence the quadratic function $V(t)$ cannot exceed the value $\xi^{-1} w(t)^T Q w(t)$. From the second inequality, we have that

$$\zeta^{-1} \begin{bmatrix} C^T \\ F^T \end{bmatrix} \begin{bmatrix} C^T \\ F^T \end{bmatrix}^T - \begin{bmatrix} \xi P & 0 \\ 0 & \zeta I - Q \end{bmatrix} \succeq 0. \tag{12}$$

Thus we get

$$\begin{array}{rcl} z(t)^T z(t) & \leq & \zeta\xi x(t)^T P x(t) + \zeta w(t)^T (\zeta I - Q) w(t) \\ & \leq & \zeta w(t)^T Q w(t) + \zeta w(t)^T (\zeta I - Q) w(t) \\ & \leq & \zeta^2 w(t)^T w(t) \end{array} \tag{13}$$

and hence $||z||_{QL_\infty} \leq \zeta ||w||_{QL_\infty}$.

Part 2. Note that, using a Schur complement, the second inequality is equivalent to

$$Q \preceq \zeta I - \begin{bmatrix} 0 & F^T \end{bmatrix} \begin{bmatrix} \xi P & C^T \\ C & \zeta I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ F \end{bmatrix}. \tag{14}$$

Hence, letting Q to be equal to the RHS, substituting it into the first matrix inequality, we get a matrix inequality which is identical to (9) modulo a Schur complement. This concludes the proof. ∎

It has been possible to merge the two inequalities due to the full symmetric term Q in the inequalities. In the initial formulation, the change of variables is not possible due to the presence of a diagonal term (μI) instead of a full symmetric one in the present case. Closed inequalities have also been considered in order to equate some variables. This will appear to be useful in the sequel.

III. Robust stability analysis in the $*$-norm

Let us consider here the uncertain LTI system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw_0(t) \\ z_0(t) & = & Cx(t) + Fw_0(t) \\ w_0(t) & = & \Delta(z_0)(t) \\ x(0) & = & x_0 \end{array} \tag{15}$$

where $x, x_0 \in \mathbb{R}^n$, $w_0 \in \mathbb{R}^{n_0}$ and $z_0 \in \mathbb{R}^{n_0}$ are the system state, the initial condition, the robustness-channel input and output respectively. The uncertain operator Δ is assumed to be bounded, i.e. $||\Delta||_* \leq \eta^{-1}$, $\eta > 0$. Similarly as in [28], the set of D-scalings is defined as

$$\mathcal{D}_\Delta^k := \left\{ U \in \mathbb{R}^{k\times k} : \ U = U^T \succ 0, U\Delta = \Delta U \right\} \tag{16}$$

and captures the structure of the operator Δ through a commutation property. We are now able to state the following result:

Theorem 1 (Scaled Small $*$-Gain Theorem): The uncertain system (15) is asymptotically stable if there exist symmetric matrices $P \succ 0$, $S \in \mathcal{D}_\Delta^{n_0}$ and scalars $\varepsilon, \xi > 0$ such that the matrix inequality

$$\begin{bmatrix} A^TP+PA+\xi P+\varepsilon I & PE & 0 & 0 \\ \star & -\eta S & 0 & F^TS \\ \star & \star & -\xi P & C^TS \\ \star & \star & \star & -\eta S \end{bmatrix} \preceq 0 \tag{17}$$

holds. △

Proof: Following [28], introduce a nonsingular matrix L such that $\Delta L = L\Delta$, thus $\Delta = L\Delta L^{-1}$. Incorporating the scalings in the system (15), we obtain the 'scaled' system

$$\begin{array}{rcl} \dot{\tilde{x}}(t) & = & A\tilde{x}(t) + BL^{-1}\tilde{w}(t) \\ \tilde{z}(t) & = & LC\tilde{x}(t) + LDL^{-1}\tilde{w}(t) \end{array} \tag{18}$$

where $\tilde{w}(t) = Lw(t)$ and $\tilde{z}(t) = Lz(t)$. Substituting then the above system into (9) with $\zeta = \eta$, performing a congruence transformation with respect to $\mathrm{diag}(I, L, I, L)$ and defining $S := L^T L \in \mathcal{D}_\Delta^{n_0}$ yield the result. ∎

IV. Input/output Interpretation of the Lyapunov-Razumikhin Theorem

We use here the developed results to study the stability of delay-systems with a particular emphasis on connections with the Lyapunov-Razumikhin Theorem [4]. Let us consider the following linear time-delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bx(t-h(t)) \\ x(s) & = & \phi(s), \ s \leq 0 \end{array} \tag{19}$$

with time-varying delay $h : \mathbb{R} \to \mathbb{R}_+$ and functional initial condition $\phi \in L_\infty((-\infty, 0], \mathbb{R}^n)$. We make no assumption on the delay $h(t)$ for the moment.

A. General Connection to the Lyapunov-Razumikhin Theorem

Let us recall the simplified Lyapunov-Razumikhin Theorem for global asymptotic stability of linear time-delay systems of the form (19) for which we assume for simplicity that the delay is constant and bounded, i.e. $h \in [0, \bar{h}]$:

Theorem 2 ([2, Proposition 5.1]): Assume there exists a bounded quadratic function W satisfying

$$W(x) \geq \varepsilon ||x||_2^2 \tag{20}$$

for some $\varepsilon > 0$ and whose derivative along the system trajectory $\dot{W}(x(t))$ verifies

$$\dot{W}(x(t)) \leq -\varepsilon ||x(t)||_2^2, \tag{21}$$

whenever

$$W(x(t+\theta)) \leq pW(x(t)),\ \theta \in [-\bar{h}, 0] \tag{22}$$

for some constant $p > 1$. Then the time-delay system (19) with constant delay $h \in [0, \bar{h}]$ is globally uniformly asymptotically stable. △

Note that the condition (21)-(22) is equivalent to the inequality

$$W(x(t)) < \frac{1}{p} W(x(t+\theta)),\ \theta \in [-\bar{h}, 0] \tag{23}$$

since $W(x(t))$ decreases when condition (22) holds. This hence defines, for each $t \geq 0$, an invariant subset for $W(x(t))$ depending on $W(x(t+\theta))$. Moreover, since $p > 1$, then $W(x(t))$ contracts to 0 as time goes. We show now that this condition is naturally enforced using Theorem 1. To this aim, substitute in (17): $z(t) \leftarrow x(t)$, $w(t) \leftarrow x(t-h(t))$, $C \leftarrow I$, $D \leftarrow 0$ and $E \leftarrow B$. Then, following the same arguments as in the proof of Lemma 1, this implies that

$$x(t)^T S x(t) \leq \eta^2 x(t-h(t))^T S x(t-h(t)). \tag{24}$$

Hence, picking the Lyapunov-Razumikhin function $W(x) = x^T S x$ and $p = \eta^{-1/2}$, we can see that Theorem 1 implies the stability condition of the Lyapunov-Razumikhin Theorem. Therefore, asymptotic stability is ensured provided that $\eta < 1$, emphasizing the small-gain interpretation of the Razumikhin condition.

Note that in the proposed formulation the storage function $V = x^T P x$ generally differs from the the Lyapunov-Razumikhin function $W = x^T S x$. We will however see in Section IV-C that they may coincide.

B. Norms of Delay Operators

The following discussion addresses the problems of $*$-norm computation of some delay operators.

Lemma 2: The operator $\mathscr{D}_h$ defined in (3) satisfies

$$||\mathscr{D}_h||_* = 1 \tag{25}$$

for any delay $h : \mathbb{R}_+ \to \mathbb{R}_+$. △

Proof: Under the standard assumption of zero initial conditions, it is clear that

$$\sup_{t \geq 0} ||w(t-h(t))||_2^2 = \sup_{s \geq 0} ||w(s)||_2^2$$

for any $h : \mathbb{R}_+ \to \mathbb{R}_+$ and any $w \in L_\infty$. This bound is indeed attained as shown in Section II-A. ■

Lemma 3: The operator[2] $\mathscr{S}_h := (I - \mathscr{D}_h) \circ \mathcal{I}$ satisfies

$$||\mathscr{S}_h||_* = \bar{h} \tag{26}$$

for any $h : \mathbb{R}_+ \to [0, \bar{h}]$. △

Proof: Considering again zero initial conditions, we have

$$\begin{array}{rcl} \left|\left| \displaystyle\int_{t-h(t)}^{t} w(s) ds \right|\right|_2^2 & \leq & h(t) \displaystyle\int_{t-h(t)}^{t} ||w(s)||_2^2 ds \\ & \leq & h(t)^2 \displaystyle\sup_{t \geq 0} ||w(t)||_2^2 \end{array} \tag{27}$$

[2] $\mathcal{I}$ and $\circ$ are the integration and the composition operators respectively.

where the first inequality has been obtained using the Jensen's inequality. Hence, we have $||\mathscr{S}_h||_* \leq \bar{h}$. To see that this bound is attained, it is enough to choose the constant input signal $w \equiv 1$ and the constant delay $h \equiv \bar{h}$. ■

In the following, we use the above delay-operators and the Scaled Small $*$-Gain Theorem to derive stability results for time-delay systems. It is also shown that these results can be interpreted as generalized Razumikhin's criteria.

C. Delay-Independent Stability

Using the $\mathscr{D}_h$ operator defined in (3), the system (19) can be equivalently rewritten as

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & x(t) \\ w(t) & = & \mathscr{D}_h(z)(t) \end{array} \tag{28}$$

where the operator $\mathscr{D}_h$ is considered as a norm-bounded uncertainty with $*$-norm equal to 1. Applying the Scaled Small $*$-Gain Theorem obtained in Section IV-B, we get the following theorem for delay-independent stability:

Theorem 3: The system (19) is asymptotically stable independently of the delay if there exist a matrix $P = P^T \succ 0$ and a scalar $\xi > 0$ such that the matrix inequality

$$\begin{bmatrix} A^T P + PA + \xi P & PB \\ \star & -\xi P \end{bmatrix} \prec 0 \tag{29}$$

holds.

Proof: Substituting the system (28) into the matrix inequality (17) with $\eta = 1$ yields

$$\begin{bmatrix} A^T P + PA + \xi P + \varepsilon I & PB & 0 & 0 \\ \star & -S & 0 & 0 \\ \star & \star & -\xi P & S \\ \star & \star & \star & -S \end{bmatrix} \preceq 0. \tag{30}$$

Equivalently we have

$$\begin{bmatrix} A^T P + PA + \xi P + \varepsilon I & PB \\ \star & -S \end{bmatrix} \preceq 0 \tag{31}$$

and $-\xi P + S \preceq 0$ which is, in turn, equivalent to (29). The proof is complete. ■

We can recognize in the above result the matrix inequality condition for delay-independent stability obtained using the Lyapunov-Razumikhin Theorem [4], [6], [2]. It is interesting to note that, using the same operator but in the L_2-framework, i.e. using the usual Scaled Small-Gain Theorem, equivalence is shown with the Lyapunov-Krasovskii Functional [8, Section 3.2.1.6]:

$$V(x_t) = x(t)^T P x(t) + \int_{t-h(t)}^{t} x(s)^T Q x(s) ds$$

with symmetric matrices $P, Q \succ 0$.

D. A First Result on Delay-Dependent Stability

To study delay-dependent stability, let us consider the following nonequivalent comparison system for (19):

$$\begin{array}{rcl}\left[\begin{array}{c}\dot{\tilde{x}}(t)\\ \hline z(t)\end{array}\right] &=& \left[\begin{array}{c|ccc}A+B & 0 & -\bar{h}BA & -\bar{h}B^2\\ \hline I & 0 & 0 & 0\\ I & 0 & 0 & 0\\ 0 & I & 0 & 0\end{array}\right]\left[\begin{array}{c}\tilde{x}(t)\\ \hline w(t)\end{array}\right]\\ w &=& \operatorname{diag}(\mathscr{D}_h,\mathscr{S}_h,\mathscr{S}_h)(z).\end{array} \tag{32}$$

for which we assume that $h(t)\in[0,\bar{h}]$ for all $t\geq 0$ and some $\bar{h}>0$. Using the Scaled Small $*$-Gain Theorem, we get the following result for delay-dependent stability:

Theorem 4: The system (19) is asymptotically stable for all $h(t)\in[0,\bar{h}]$, $\bar{h}>0$ if there exist symmetric matrices $P,S_1\succ 0$ and a scalar $\xi>0$ such that the matrix inequality

$$\begin{bmatrix}\mathcal{M}+\xi P & -\bar{h}PBA & -\bar{h}PB^2\\ \star & -\xi P+S_1 & 0\\ \star & \star & -S_1\end{bmatrix}\prec 0 \tag{33}$$

holds where $\mathcal{M}=(A+B)^TP+P(A+B)$.

Proof: The D-scaling corresponding to the uncertainty structure is given by

$$\operatorname{diag}\left(S_1,\begin{bmatrix}S_2 & Q\\ Q^T & S_3\end{bmatrix}\right)\succ 0.$$

After substitution of the comparison system (32) into the Scaled Small $*$-Gain Theorem. A Schur complement and a row/colmun reorganization yield

$$\begin{bmatrix}\mathcal{M}+\xi P+\varepsilon I & -\bar{h}PBA & -\bar{h}PB^2\\ \star & -S_2 & -Q\\ \star & \star & -S_3\end{bmatrix}\preceq 0 \tag{34}$$

and

$$\begin{bmatrix}-S_1+S_3 & Q^T\\ \star & -\xi P+S_1+S_2\end{bmatrix}\preceq 0. \tag{35}$$

Note that (35) is equivalent to

$$\begin{bmatrix}S_2 & Q\\ Q^T & S_3\end{bmatrix}\preceq\begin{bmatrix}-S_1+\xi P & 0\\ \star & S_1\end{bmatrix}. \tag{36}$$

Finally, equating the matrices and substituting into (34) yield the result. ■

We show below that the above theorem reduces to the Lyapunov-Razumikhin condition of [2, Corollary 5.8] and can then be viewed as a more general Razumikhin condition.

Corollary 1: The system (19) is asymptotically stable for all $h(t)\in[0,\bar{h}]$, $\bar{h}>0$ if there exist a symmetric matrix $P\succ 0$ and scalars $\xi,\epsilon_1,\epsilon_2>0$ such that the matrix inequality:

$$\begin{bmatrix}\bar{h}^{-1}\mathcal{M}+(\epsilon_1+\epsilon_2)P & -PBA & -PB^2\\ \star & -\epsilon_2 P & 0\\ \star & \star & \epsilon_1 P\end{bmatrix}\prec 0 \tag{37}$$

holds.

Proof: Setting $S_1=\mu_1 P$, $\xi=\mu_1+\mu_2$, $\mu_1,\mu_2>0$ and multiplying then the obtained matrix inequality by $\bar{h}^{-1}$, we get the result where $\epsilon_i=\mu_i\bar{h}^{-1}$, $i=1,2$. ■

Note that the condition (33) obtained from the Scaled Small $*$-Gain Theorem is easier to solve since it involves a smaller number of nonlinear terms.

Example 1: Let us consider the time-delay system (19) with matrices

$$A=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix}\text{ and }B=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}.$$

Using Theorem 4, the maximal delay $\bar{h}=0.98$ is obtained as roughly determined in [19], [14].

Example 2: Let us consider now the matrices [29]

$$A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\text{ and }B=\begin{bmatrix}0 & 0\\ -1 & 1\end{bmatrix}.$$

Using Theorem 4, the maximal delay $\bar{h}=0.1739$ is obtained.

E. A Second Result on Delay-Dependent Stability

Consider now the comparison system

$$\begin{array}{rcl}\left[\begin{array}{c}\dot{\tilde{x}}(t)\\ \hline z(t)\end{array}\right] &=& \left[\begin{array}{c|cc}A+B & 0 & -\bar{h}B\\ \hline I & 0 & 0\\ A & B & 0\end{array}\right]\left[\begin{array}{c}\tilde{x}(t)\\ \hline w(t)\end{array}\right]\\ w &=& \operatorname{diag}(\mathscr{D}_h,\mathscr{S}_h)(z).\end{array} \tag{38}$$

where $h(t)\in[0,\bar{h}]$ for some $\bar{h}>0$. We obtain the following theorem:

Theorem 5: The system (19) is asymptotically stable for all $h(t)\in[0,\bar{h}]$, $\bar{h}>0$ if there exist symmetric matrices $P,S_1,S_2\succ 0$ and a scalar $\xi>0$ such that the matrix inequalities

$$\begin{bmatrix}\mathcal{M}+\xi P & -\bar{h}PB\\ \star & -S_2\end{bmatrix}\prec 0 \tag{39}$$

and

$$\begin{bmatrix}-S_1+B^TS_2B & B^TS_2A\\ \star & -\xi P+S_1+A^TS_2A\end{bmatrix}\preceq 0 \tag{40}$$

hold where $\mathcal{M}=(A+B)^TP+P(A+B)$.

Proof: The proof is similar to the one of Theorem 4 and is omitted. ■

It is also interesting to stress that using the same operator, but in the L_2-framework, equivalence[3] is shown with the Lyapunov-Krasovskii Functional:

$$\begin{array}{rcl}V(x_t) &=& x(t)^TPx(t)+\displaystyle\int_{t-h(t)}^{t}x(s)^TQx(s)ds\\ && +\displaystyle\int_{-\bar{h}}^{0}\int_{t+s}^{t}\dot{x}(\theta)^TR\dot{x}(\theta)d\theta ds\end{array} \tag{41}$$

for some symmetric matrices $P,Q,R\succ 0$. Note however, that is functional would lead to a delay-derivative-dependent result.

[3]in terms of the resulting LMI conditions.

V. Conclusion

Razumikhin-like results have been obtained by considering an input/output approach in the QL_∞-norm. It has been shown that the Razumikhin condition has a small-gain interpretation using the $*$-norm, corresponding to the norm induced by the QL_∞-norm. Several standard delay operators have been considered and have led to generalized Razumikhin conditions. Although, drastic improvements of Razumikhin idea have not been obtained, this approach brings a new insight on Razumikhin's approach and opens the door of input/ouput techniques for the derivation of stability results independent of the delay derivative. Future works will be devoted to the study of possible improvements of the Razumikhin's approach using this input-output framework.

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