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Identifying a Wiener system using a variant of the Wiener G-Functionals

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***Abstract*— This paper concerns the identification of nonlinear systems using a variant of the Wiener G-Functionals. The system is modeled by a cascade of a single input multiple output (SIMO) linear dynamic system, followed by a multiple input single output (MISO) static nonlinear system. The dynamic system is described using orthonormal basis functions. The original ideas date back to the Wiener G-functionals of Lee and Schetzen. Whereas the Wiener G-Functionals use Laguerre orthonormal basis functions, in this work Takenaka-Malmquist orthonormal basis functions are used. The poles that these basis functions contain, are estimated using the best linear approximation of the system. The approach is illustrated on the identification of a Wiener system.**

I. INTRODUCTION

A Wiener model can describe a large class of nonlinear systems [1]. The dynamics of the system being modeled are described by a set of orthonormal basis functions. If the pole locations of these basis functions match the system poles closely, then highly accurate models are obtained with only a relatively small number of parameters to be estimated [2]. In the original settings, Laguerre orthonormal basis functions were used [1].

The Laguerre orthonormal basis functions are characterized by a real valued pole, which makes them suitable for describing well-damped systems with dominant first order dynamics. For describing moderately damped systems with dominant second order dynamics, it is more appropriate to use Kautz basis functions [3], [4]. In this paper the Takenaka-Malmquist basis functions are used, since they can deal with multiple real and complex valued poles [2].

In order to obtain good estimates of the system poles, the best linear approximation [5], [6] of the system is used. These estimates are then used to construct the Takenaka-Malmquist basis functions.

Numerical simulations illustrate the proposed approach on the identification of a Wiener system.

II. WIENER MODEL

The Wiener model of a nonlinear system, with a nonlinearity up to degree Q, basically consists of two components connected in tandem (see Fig. 1): a single input multiple output (SIMO) linear dynamic system and a multiple input single output (MISO) nonlinear static system. In that way the dynamics of the system are completely modeled in the first section and the nonlinearities are completely modeled in the second section.

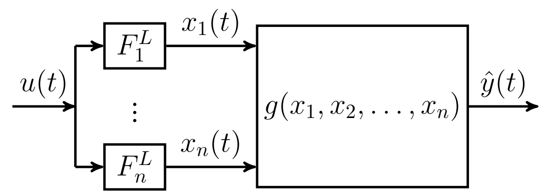

Fig. 1. Wiener model

The first section of the model consists of a set of orthonormal basis functions F_k^L, which filter the input signal $u(t)$. In the original settings Laguerre filters were used [1]. Expressed in the z-domain the Laguerre filters are given by [2]

$$F_k^L(z) = \underbrace{\frac{\sqrt{1-a^2}}{z-a}}_{\substack{\text{first order filter} \\ \text{with pole } a}} \underbrace{\left[\frac{1-az}{z-a}\right]^{k-1}}_{\substack{\text{all-pass filter} \\ \text{with pole } a}}, \qquad -1 < a < 1, \quad k = 1, \ldots, n\,. \tag{1}$$

These Laguerre filters have a real valued pole a with multiplicity k. This makes it difficult to model an oscillating system with a small number of Laguerre filters [4]. For that reason we will replace the Laguerre basis functions with the Takenaka-Malmquist basis functions (see section III).

The second section of the Wiener model is a multivariate polynomial function $g(x_1, x_2, \ldots, x_n)$ of degree Q, which contains all possible products of Hermite polynomials [1]

$$H_{k_i}[x_i(t)] = \sum_{m=0}^{[k_i/2]} \frac{(-1)^m k_i!}{m!(k_i-2m)!} \left(\frac{\sigma_{x_i}^2}{2}\right)^m x_i^{k_i-2m}(t) \tag{2}$$

of degree k_i, in which $\sigma_{x_i}^2$ is the variance of $x_i(t)$. Since $g(x_1, x_2, \ldots, x_n)$ is of degree Q, the condition $\sum_{i=1}^n k_i \leq Q$ must be met. The coefficients of $g(x_1, x_2, \ldots, x_n)$ are estimated using a Gaussian input signal $u(t)$, since the Hermite polynomials are then orthogonal. Although the model is built using a Gaussian time function, the resulting model is valid for any input [1].

III. TAKENAKA-MALMQUIST BASIS FUNCTIONS

In contrast to the Laguerre basis functions, the Takenaka-Malmquist basis functions $F_k^{TM}(z)$ can contain complex

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valued poles ξ_k. They are given by [2]

$$F_k^{TM}(z) = \underbrace{\frac{\sqrt{1-|\xi_k|^2}}{z-\xi_k}}_{\substack{\text{first order filter}\\ \text{with pole } \xi_k}} \prod_{i=1}^{k-1} \underbrace{\left[\frac{1-\xi_i^* z}{z-\xi_i}\right]}_{\substack{\text{all-pass filter}\\ \text{with pole } \xi_i}}, \tag{3}$$
$$k = 1,\ldots,n, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| < 1\,.$$

Similarly as the k^{th} Laguerre basis function $F_k^L(z)$ in (1), the k^{th} Takenaka-Malmquist basis function $F_k^{TM}(z)$ is constructed as the product of a first order filter $G_k^{TM}(z)$ with pole ξ_k and the product of the all-pass filters $H_i^{TM}(z)$ with the previous poles ξ_i.

A. Real filters

In general, the Takenaka-Malmquist basis functions are complex filters. This means that a real signal filtered by a Takenaka-Malmquist basis function is in general a complex signal. However, if $\xi_{j+1} = \xi_j^*$ it is possible to form linear combinations of $F_j^{TM}(z)$ and $F_{j+1}^{TM}(z)$ to obtain two orthonormal real filters, which span the same space [2]. The so-called real Kautz form is obtained. It is sufficient to replace

$$\begin{array}{rcll}
G_j^{TM}(z) & = & \frac{\sqrt{1-|\xi_j|^2}}{z-\xi_j} & \text{by} \\
G_j(z) & = & \frac{\sqrt{1-c^2}(z-b)}{z^2+b(c-1)z-c} & , \\
G_{j+1}^{TM}(z) & = & \frac{\sqrt{1-|\xi_{j+1}|^2}}{z-\xi_{j+1}} & \text{by} \\
G_{j+1}(z) & = & \frac{\sqrt{1-c^2}\sqrt{1-b^2}}{z^2+b(c-1)z-c} & , \\
H_j^{TM}(z) & = & \frac{1-\xi_j^* z}{z-\xi_j} & \text{by} \\
H_j(z) & = & 1 & \text{and} \\
H_{j+1}^{TM}(z) & = & \frac{1-\xi_{j+1}^* z}{z-\xi_{j+1}} & \text{by} \\
H_{j+1}(z) & = & \frac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c} & ,
\end{array}$$

in which the coefficients b and c are given by

$$b = \frac{\xi_j + \xi_{j+1}}{\xi_j \xi_{j+1} + 1} \tag{4}$$
$$c = -\xi_j \xi_{j+1} \quad . \tag{5}$$

We denote the obtained real versions of the Takenaka-Malmquist basis functions by $F_k(z)$. They are still orthonormal.

B. One extra basis function

We will also introduce one extra basis function, namely $F_{n+1}(z) = 1$, to make it possible to model systems that have a direct feedthrough of the input to the output. Eventually the model structure shown in Fig. 2 is obtained.

IV. BEST LINEAR APPROXIMATION

As stated before, the poles ξ_k in (3) should be close to the system poles so that a minimal set of basis functions is needed [2]. We will estimate the system poles by means of the best linear approximation (BLA) of the system.

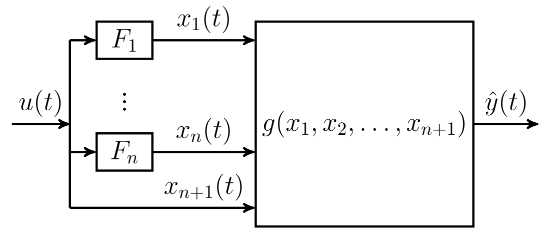

Fig. 2. The proposed model structure

First we will define the BLA of a nonlinear system. Consider a single input single output (SISO) nonlinear system with input signal $u(t)$ and output signal $y(t)$. This system can be linearized around its operating point $(E\{u(t)\}, E\{y(t)\})$. The obtained linear system $g(t)$ that minimizes the mean square error

$$E\{|\tilde{y}(t) - g(t) * \tilde{u}(t)|^2\} \quad \text{with} \begin{cases} \tilde{y}(t) = y(t) - E\{y(t)\} \\ \tilde{u}(t) = u(t) - E\{u(t)\} \end{cases}, \tag{6}$$

is called the best linear approximation g_{BLA} of the nonlinear system. The BLA can be obtained with correlation methods, which in the frequency domain corresponds to the calculation of the frequency response function (FRF) [5]:

$$G_{BLA}(j\omega) = \frac{S_{\tilde{Y}\tilde{U}}(j\omega)}{S_{\tilde{U}\tilde{U}}(j\omega)} \quad , \tag{7}$$

in which $S_{\tilde{Y}\tilde{U}}(j\omega)$ is the cross-power spectrum between $\tilde{y}(t)$ and $\tilde{u}(t)$ and $S_{\tilde{U}\tilde{U}}(j\omega)$ is the auto-power spectrum of $\tilde{u}(t)$.

We state here the following properties of the BLA:

Property 1 (cascade rule): The best linear approximation G_{BLA} of the cascade of two systems equals the product of the best linear approximations of each of the systems if at least one of the systems is linear. [7]

Property 2 (BLA of a static nonlinear system): The best linear approximation of a static nonlinear system $y(t) = f(u(t))$ with $u(t)$ a signal belonging to the class of excitations with a Gaussian distribution is a constant. [6]

We will now show that, under certain conditions, the poles of the BLA are equal to the true system poles. If the system to be identified is the cascade of linear systems G_i and one nonlinear system f (e.g. Wiener, Hammerstein or Wiener-Hammerstein), then Property 1 can be applied successively on the BLA of the system, so that

$$G_{BLA} = G_{BLA,f} \prod_i G_i \quad , \tag{8}$$

in which $G_{BLA,f}$ is the BLA of the nonlinear system f. Moreover, if f is static, then for input signals with a Gaussian distribution, Property 2 can be applied as well:

$$G_{BLA} = c \prod_i G_i \quad , \tag{9}$$

in which c is an unknown constant. We see that the BLA and the system itself have the same poles. Hence the poles of the estimated $\hat{G}_{BLA}$ match the system poles closely, which makes them suitable to be used in the construction of the Takenaka-Malmquist orthonormal basis functions.

V. EXAMPLE

We will now identify a Wiener system using the Takenaka-Malmquist orthonormal basis functions and the BLA.

A. The system to be identified

Consider the Wiener system shown in Fig. 3. The linear time-invariant system $G_1(z)$ is a third order low-pass Chebyshev Type I filter with 10 dB peak-to-peak ripple in the passband and with a normalized passband edge frequency $\omega_p = 0.2846$. Its FRF is shown in Fig. 4. The coefficients α_2 and α_3 of the static nonlinearity $f(x)$ are equal to 0.8 and 0.7 respectively. The input signal $u(t)$ is a random phase multisine [6]

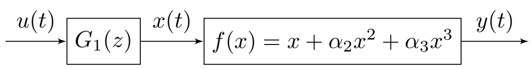

Fig. 3. Wiener system with $G_1(z)$ a third order Chebyshev filter and $f(x)$ a static nonlinearity

$$u(t) = \sum_{k=-N_F}^{N_F} U_k e^{j2\pi f_{max} kt/N_F} \quad , \tag{10}$$

with $U_k = U_{-k}^* = |U_k|e^{j\phi_k}$; f_{max} the maximum frequency of the excitation signal, chosen equal to one sixth of the sampling frequency f_s; $N_F = 1365$ the number of frequency components; and the phases ϕ_k uniformly distributed in the interval $[0, 2\pi[$. The amplitudes $|U_k|$ are chosen equal to each other and such that the rms value of $u(t)$ is equal to 1. The system has a complex conjugate pole pair $p_1 = p_2^* \approx 0.6720 + j\ 0.6805$ and a real valued pole $p_3 \approx 0.9004$.

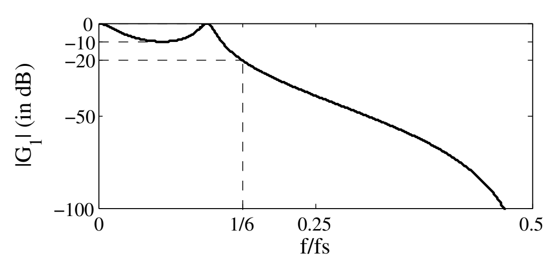

Fig. 4. FRF of the third order Chebyshev filter $G_1(z)$ in Fig. 3

B. Identification procedure

1) Identify $\hat{G}_{BLA}$: First the BLA is estimated using $M = 2, 5, 10, 100$ realizations of the random phase multisine u. Since the input is a periodic signal, (7) boils down to the division of the output spectrum $Y(j\omega)$ by the input spectrum $U(j\omega)$, so that the BLA is estimated as

$$\hat{G}_{BLA}(j\omega_k) = \frac{1}{M} \sum_{m=1}^{M} \frac{Y^{[m]}(k)}{U^{[m]}(k)} \quad , \tag{11}$$

in which $Y^{[m]}(k)$ and $U^{[m]}(k)$ are the discrete Fourier transforms (DFTs) of the output and the input, corresponding to the m^{th} realization of the input signal.

2) Calculate the poles of $\hat{G}_{BLA}$: Next a third order parametric model is fitted on $\hat{G}_{BLA}$ and its poles ($\xi_1 = \xi_2^*$ and ξ_3) are calculated. Table I gives an overview of the pole estimates.

TABLE I

POLE ESTIMATES OF THE WIENER SYSTEM SHOWN IN FIG. 3, WITH TRUE SYSTEM POLES $p_1 = p_2^* \approx 0.6720 + j\ 0.6805$ AND $p_3 \approx 0.9004$, FOR DIFFERENT NUMBERS OF REALIZATIONS M OF THE INPUT SIGNAL

M	$\xi_1 = \xi_2^*$	$\frac{\lvert p_1-\xi_1\rvert}{\lvert p_1\rvert}$ (in %)	ξ_3	$\frac{\lvert p_3-\xi_3\rvert}{\lvert p_3\rvert}$ (in %)
2	0.6737 + j 0.6818	0.229	0.8964	0.443
5	0.6716 + j 0.6810	0.060	0.8978	0.285
10	0.6719 + j 0.6802	0.027	0.8995	0.098
100	0.6718 + j 0.6806	0.023	0.9001	0.033

3) Construct the orthonormal basis functions: These poles are then used to construct the Takenaka-Malmquist basis functions (see equation (3)). It can be observed that the Wiener system shown in Figure 3 can be exactly modeled by the proposed model, shown in Figure 2 (see Appendix I). Since the pole estimates are not exactly equal to the true system poles, additional basis functions are needed, besides the essential ones. These allow us to compensate for the modeling errors in the first step.

As explained in section III-A, real filters are formed out of the Takenaka-Malmquist basis functions and the filtered input signals $x_k(t)$ are calculated.

4) Estimate the nonlinearities: Finally the coefficients of the multivariate polynomial function $g(x_1, x_2, \ldots, x_{n+1})$ are estimated by solving the linear least squares problem

$$\hat{g} = \arg\min_{g} MSE(g) \quad , \tag{12a}$$

in which the mean square error MSE is equal to

$$\frac{1}{N} \sum_{k=1}^{N} [y(k) - g(x_1(k), x_2(k), \ldots, x_{n+1}(k))]^2 \quad . \tag{12b}$$

Note that the static nonlinearity is described by a multivariate polynomial, instead of a univariate one ($f(x)$ in Fig. 3). This means that more parameters are introduced. However, $g(x_1, x_2, \ldots, x_{n+1})$ can describe the static nonlinearity perfectly.

C. Results

In order to validate the obtained model, a random phase multisine $u_{val}(t)$ is applied to the input of the system given in Fig. 3. Its output $y_{val}(t)$ is estimated with the obtained model. Fig. 5 shows the rms error for different numbers of realizations M and the number of parameters to be estimated as a function of the number of basis functions.

We observe that the rms error on the modeled output can be made arbitrarily small by increasing the number of basis

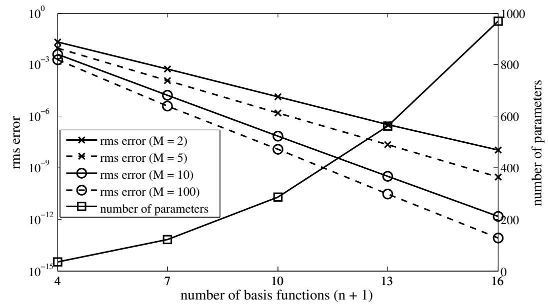

Fig. 5. Rms error on the modeled output of the Wiener system shown in Fig. 3 for different numbers of realizations M and the number of parameters to be estimated as a function of the number of basis functions $n+1$

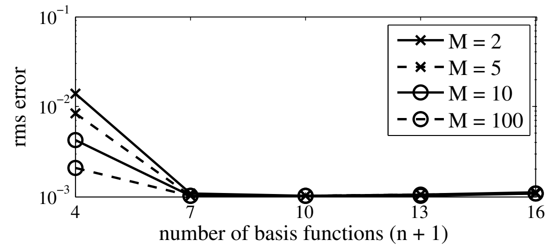

Fig. 7. Rms error on the modeled output of the Wiener system shown in Fig. 6 with $\sigma_{n_y} = 0.001$ as a function of the number of basis functions $n+1$ for different numbers of realizations M

functions. As the number of realizations increases, the pole estimates are better (see Table I) and less basis functions are needed. We also observe that the number of parameters to be estimated increases rapidly as the number of basis functions increases. This shows the importance of good pole estimates.

VI. DISCUSSION

A. Noise sensitivity

In this section the Wiener system shown in Fig. 3 is identified in the presence of disturbing output noise (see Fig. 6). The output noise $n_y(t)$ is zero mean white Gaussian noise with standard deviation $\sigma_{n_y} = 0.001$.

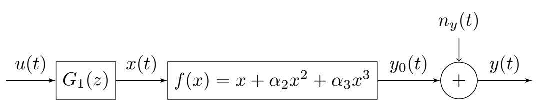

Fig. 6. Wiener system with $G_1(z)$ a third order Chebyshev filter, $f(x)$ a static nonlinearity and $n_y(t)$ output noise

The identification procedure explained in section V-B is followed. From Table II it can be observed that the BLA still gives a consistent estimate of the pole locations in the presence of disturbing output noise.

TABLE II
POLE ESTIMATES OF THE WIENER SYSTEM SHOWN IN FIG. 6, WITH TRUE SYSTEM POLES $p_1 = p_2^* \approx 0.6720 + j\,0.6805$ AND $p_3 \approx 0.9004$, FOR DIFFERENT NUMBERS OF REALIZATIONS M OF THE INPUT SIGNAL

M	$\xi_1 = \xi_2^*$	$\frac{\lvert p_1-\xi_1\rvert}{\lvert p_1\rvert}$ (in %)	ξ_3	$\frac{\lvert p_3-\xi_3\rvert}{\lvert p_3\rvert}$ (in %)
2	0.6736 + j 0.6810	0.185	0.9004	0.002
5	0.6721 + j 0.6803	0.024	0.9025	0.242
10	0.6721 + j 0.6803	0.023	0.8994	0.104
100	0.6718 + j 0.6805	0.016	0.8999	0.048

Because of the output noise, the rms error on the modeled output is at least equal to the noise level σ_{n_y}, as shown in Fig. 7. The noise level can be reduced by applying $P \geq 2$ consecutive periods of the input signal $u(t)$ for every one of the M realizations. The BLA is then estimated as

$$\hat{G}_{BLA}(j\omega_k) = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{1}{P}\sum_{p=1}^{P}\frac{Y^{[m,p]}(k)}{U^{[m,p]}(k)}\right) \quad , \tag{13}$$

in which $Y^{[m,p]}(k)$ and $U^{[m,p]}(k)$ are the DFTs of the output and the input, corresponding to the m^{th} realization and the p^{th} period of the input signal. The coefficients of the multivariate polynomial function $g(x_1, x_2, \ldots, x_{n+1})$ are estimated by solving the linear least squares problem in (12), in which $y(t)$ is replaced by

$$\frac{1}{P}\sum_{p=1}^{P} y^{[p]}(t) \quad , \tag{14}$$

in which $y^{[p]}(t)$ is the output signal corresponding to the p^{th} period of the input signal. Consider for example the identification of the Wiener system shown in Fig. 6 with $n_y(t)$ zero mean white Gaussian noise with standard deviation $\sigma_{n_y} = 0.1$. Fig. 8 shows the rms error on the modeled output for $M = 2$ realizations and $P = 10, 100, 1000$ periods of the input signal. The noise level drops with a factor $1/\sqrt{P}$ and can be made arbitrarily small at a cost of increased measuring time. It can be concluded that the method is robust to output noise.

From Figs. 7 and 8, it can be seen that as the number of basis functions increases, the rms error on the modeled output first decreases and then slighly increases. In the beginning the extra basis functions reduce the modeling error, but once the noise level is reached, the model becomes more sensitive to the noise, due to the use of too many parameters.

B. Sensitivity to the distribution of the input

If the intermediate signal $x(t)$ of the Wiener system shown in Fig. 3 doesn't have a Gaussian distribution, then Property 2 no longer holds. Consequently the BLA and the system itself will not have the same poles anymore. We will now see the effect on the identification method.

Consider the input signal $u(t)$ to be uniformly distributed noise with zero mean and standard deviation equal to 1. Then the intermediate signal $x(t)$ is filtered uniformly distributed noise. But since the filter $G_1(z)$ is a third order filter, the distribution of $x(t)$ is almost Gaussian (see Fig. 9). For that

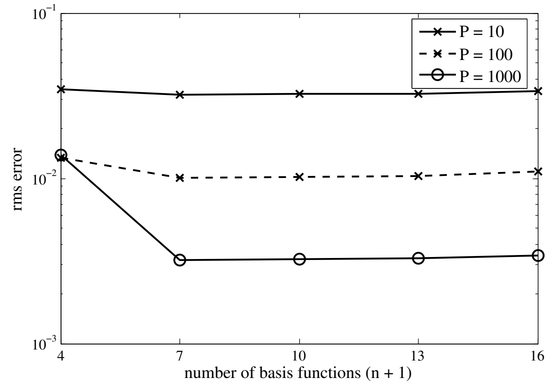

Fig. 8. Rms error on the modeled output of the Wiener system shown in Fig. 6 with $\sigma_{n_y} = 0.1$ as a function of the number of basis functions $n+1$ for $M = 2$ realizations and for different numbers of periods P

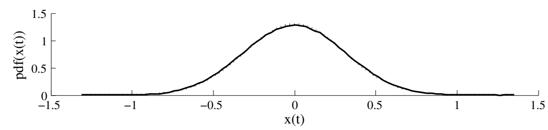

Fig. 9. Probability density function (pdf) of the intermediate signal $x(t)$ of the Wiener system shown in Fig. 3 in case $G_1(z)$ is a third order Chebyshev filter (full line), and pdf of a normal distribution with the same mean and standard deviation (dotted line)

reason we will replace the filter $G_1(z)$ by a first order filter, given by

$$G_1(z) = \frac{1}{z - 0.4} \quad . \tag{15}$$

Fig. 10 shows the distribution of $x(t)$, which is clearly not Gaussian. Consequently the BLA of the nonlinearity f is not constant anymore, but from Fig. 11 it can be seen that the errors made are small. Remark that since the input signal is no longer periodic, the BLA is estimated as

$$\begin{aligned} \hat{G}_{BLA}(j\omega_k) &= \frac{\hat{S}_{YU}(j\omega_k)}{\hat{S}_{UU}(j\omega_k)} \\ &= \frac{\frac{1}{M}\sum_{m=1}^{M} Y^{[m]}(k)\overline{U^{[m]}}(k)}{\frac{1}{M}\sum_{m=1}^{M} \left|U^{[m]}(k)\right|^2} \quad . \end{aligned} \tag{16}$$

The error on the estimated pole locations is much larger than is the case for Gaussian input signals (see Table III). However, this can be compensated with extra basis functions, as can be seen by comparing Figs. 12 and 13. So, we can conclude that the method can compensate for pole variations that are due to a change of the distribution of the signal. So, we can conclude that the method works for an arbitrary distribution of the excitation signal [8].

VII. CONCLUSIONS

The identification of nonlinear systems using a variant of the Wiener G-Functionals has been presented. The use of Takenaka-Malmquist basis functions allowed us to tune the basis functions to the system to be modeled. The approach was illustrated on the identification of a Wiener system,

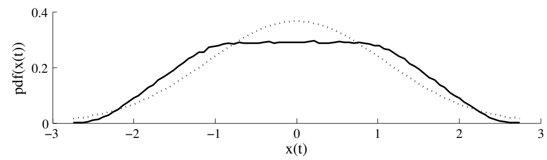

Fig. 10. Probability density function (pdf) of the intermediate signal $x(t)$ of the Wiener system shown in Fig. 3 in case $G_1(z)$ is a first order filter (full line), and pdf of a normal distribution with the same mean and standard deviation (dotted line)

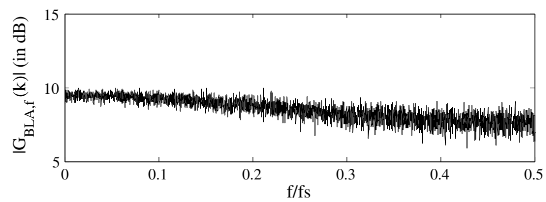

Fig. 11. Estimated BLA of the nonlinearity f in case of a uniformly distributed input signal $u(t)$ with zero mean and standard deviation equal to 1

TABLE III

POLE ESTIMATES OF THE WIENER SYSTEM SHOWN IN FIG. 3, WITH $G_1(z)$ REPLACED BY THE FIRST ORDER FILTER GIVEN BY (15), WITH TRUE SYSTEM POLE $p_1 = 0.4$, FOR DIFFERENT NUMBERS OF REALIZATIONS M OF THE INPUT SIGNAL

	Gaussian input		uniformly distributed input	
M	ξ_1	$\frac{\lvert p_1-\xi_1\rvert}{\lvert p_1\rvert}$ (in %)	ξ_1	$\frac{\lvert p_1-\xi_1\rvert}{\lvert p_1\rvert}$ (in %)
2	0.3966	0.851	0.5410	35.238
5	0.4076	1.891	0.4471	11.763
10	0.3984	0.413	0.4396	9.910
100	0.4003	0.072	0.4387	9.674

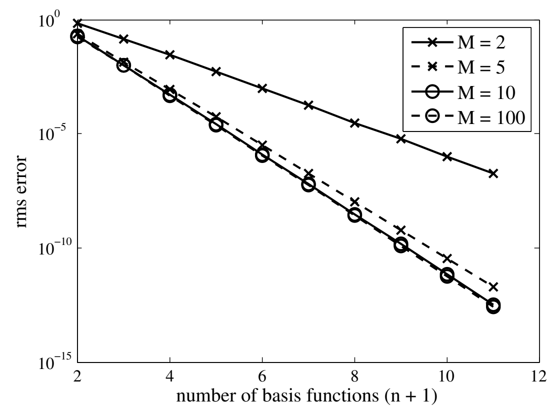

Fig. 12. Rms error on the modeled output of the Wiener system shown in Fig. 3, with $G_1(z)$ replaced by the first order filter, given by (15), as a function of the number of basis functions $n+1$ for different numbers of realizations M (uniformly distributed input)

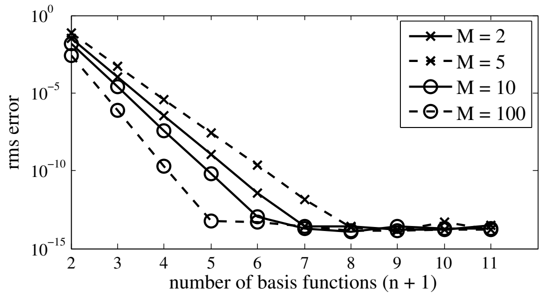

Fig. 13. Rms error on the modeled output of the Wiener system shown in Fig. 3, with $G_1(z)$ replaced by the first order filter, given by (15), as a function of the number of basis functions $n+1$ for different numbers of realizations M (random phase multisine input)

where it was shown that the rms error could be made arbitrarily small. The method was also shown to be robust to output noise and to the distribution of the input.

APPENDIX I
EXACT REPRESENTATION OF THE WIENER SYSTEM BY THE MODEL

Consider the basis functions F_k^B given by

$$F_k^B(z) = \frac{1}{z - p_k} \quad k = 1, 2, 3 \quad . \tag{17}$$

Applying the Gram-Schmidt procedure to these basis functions results in the Takenaka-Malmquist basis functions [2] F_k^{TM}, which are linear combinations of the F_k^B's. The orthonormal basis functions F_k ($k = 1, 2, 3$) are in their turn linear combinations of the F_k^{TM}'s, and therefore linear combinations of the F_k^B's. When the partial fraction expansion of the third order Chebyshev filter $G_1(z)$ is made, one obtains

$$G_1(z) = \sum_{i=1}^{3} \frac{r_i}{z - p_i} + k_{direct} \tag{18}$$

$$= \left(\sum_{i=1}^{3} r_i F_i^B(z) \right) + k_{direct} \quad . \tag{19}$$

Since the F_k's ($k = 1, 2, 3$) are linear combinations of the F_k^B's and $F_4(z) = 1$,

$$G_1(z) = \sum_{i=1}^{4} a_i F_i(z) \quad , \tag{20}$$

in which the coeficients a_i are unknown constants. The intermediate signal $x(t)$ of the Wiener system is equal to

$$x(t) = G_1(z)u(t) \tag{21}$$

$$= \sum_{i=1}^{4} a_i x_i(t) \quad . \tag{22}$$

The output signal $y(t)$ of the Wiener system is equal to

$$y(t) = x(t) + \alpha_2 x^2(t) + \alpha_3 x^3(t) \tag{23}$$

$$= \sum_{i=1}^{4} a_i x_i(t) + \alpha_2 \left(\sum_{i=1}^{4} a_i x_i(t) \right)^2 + \alpha_3 \left(\sum_{i=1}^{4} a_i x_i(t) \right)^3 \tag{24}$$

$$= \sum_{i=1}^{4} \beta_i x_i(t) + \sum_{i=1}^{4} \sum_{j=i}^{4} \beta_{ij} x_i(t) x_j(t) + \sum_{i=1}^{4} \sum_{j=i}^{4} \sum_{k=j}^{4} \beta_{ijk} x_i(t) x_j(t) x_k(t) \quad , \tag{25}$$

which, like the output signal of the model $\hat{y}(t)$, is a multivariate polynomial function of the filtered input signals $x_i(t)$. When the coefficients of both polynomial functions are chosen equal to each other, then the model is an exact representation of the Wiener system.

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