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A state observer approach to filter stochastic nonlinear differential systems

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***Abstract*— This paper investigates the state estimation problem for stochastic nonlinear differential systems with multiplicative noise. Our purpose is to combine the noise filtering properties of the Extended Kalman Filter with the global convergence properties of high-gain observers. We propose an observer-based algorithm and provide conditions under which a bound on the estimation error can be guaranteed. Simulations show that this algorithm reveals to be more efficient than the Extended Kalman Bucy filter for systems with large measurement errors.**

I. INTRODUCTION

This work considers the filtering problem for nonlinear stochastic differential systems described by the Itô equations:

$$\begin{aligned} dx_t &= \phi(x_t)dt + g(x_t)(u_t dt + F dW_t^1), \quad x_0 = \bar{x}, \\ dy_t &= h(x_t)dt + G dW_t^2, \quad y_0 = 0, \ a.s., \end{aligned} \tag{1}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $x_t \in I\!R^n$ is the state vector, $y_t \in I\!R^q$ is the measured output, $u_t \in I\!R^s$ is a known deterministic input, $dW_t^1 \in I\!R^s$ and $dW_t^2 \in I\!R^q$ are independent standard Wiener processes with respect to a family of increasing σ-algebras $\{\mathcal{F}_t, \ t \geq 0\}$ (i.e., the components of vectors dW_t^1 and dW_t^2 are a set of independent standard Wiener processes). $\phi : I\!R^n \mapsto I\!R^n$, $g : I\!R^n \mapsto I\!R^{n \times s}$ and $h : I\!R^n \mapsto I\!R^q$ are smooth nonlinear maps. The initial state $\bar{x}$ is an $\mathcal{F}_0$-measurable random variable, independent of both dW_t^1 and dW_t^2. In order to avoid singular filtering problems, see [6], the standard assumption of nonsingular output-noise covariance is made here, i.e. $\mathrm{rank}(GG^T) = q$.

It is well known that the minimum variance state estimate requires the knowledge of the conditional probability density, whose computation, in the general nonlinear case, is a difficult infinite-dimensional problem (see, e.g., [7], [27]). Only in few cases the optimal filter has a finite dimension, [26]. For this reason a great deal of work has been made to devise suboptimal implementable filtering algorithms (see, e.g., [11], [15], [16], [19], [20], [21]).

Another approach consists in considering the time discretization of the original system and then to apply suboptimal filtering procedures like the well-known Extended Kalman Filter (EKF), the most widely used algorithm in nonlinear filtering problems (see, e.g., [2], [12], [17], [23]), or more recent techniques like particle filters (see [25]), the Unscented Kalman Filter (UKF) (see [24]), Gaussian sum approximations (see [22]), or polynomial filters (see [9], [10], [18]).

In this paper an observer-based algorithm is proposed for the state estimation problem of system (1). High gain observers based upon such idea were proposed in [1], [3], [8]. The common denominator is that the correction gain is computed offline. More recently, an adaptive gain observer has been proposed in [5]. We extend these approaches to the case of multiplicative state noise by means of a fixed high-gain Luenberger-like observer. The main contribution of this paper is a result on the boundedness of the estimation error, that illustrates sufficient conditions required by a high-gain observer to be effective in this context. To our knowledge, this is the first result of this kind for stochastic nonlinear systems with multiplicative state noise. In this note, theoretical results concern the case of scalar outputs, that is $y_t \in I\!R$. The extension to the comprehensive case of multi-output ($q > 1$) is a work in progress by the same authors.

Numerical simulations show the effectiveness of the proposed methodology and the improvements with respect to the standard Kalman-Bucy Filter applied to the linear approximation of the stochastic nonlinear system, in terms of the reduction of the mean square estimation error.

II. PRELIMINARY RESULTS

The proposed observer is inspired by the one developed in [13] and [14]. Therefore, according to the same notation, we define $Q(x)$ the Jacobian of the *observability map* $\Theta(x)$:

$$Q(x) = \frac{d\Theta}{dx}, \qquad \Theta(x) = \begin{bmatrix} h(x) \\ L_\phi h(x) \\ \vdots \\ L_\phi^{n-1} h(x) \end{bmatrix} \tag{2}$$

with $L_\phi^k h(x)$ denoting the Lie derivative of order $k \geq 0$ of the scalar function $h(x)$ along the vector field $\phi(x)$, recursively defined as:

$$L_\phi^0 h(x) = h(x), \qquad L_\phi^k h(x) = \frac{dL_\phi^{k-1} h}{dx} \phi(x). \tag{3}$$

For what follows it is useful to introduce the following definition.

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Definition 1. *System* (1) *is said to have a **strong k relative degree property** if:*

$$\left(\frac{d}{dx}L_\phi^i h(x)\right)g(\xi)=0,\quad \forall x,\xi,\in \mathbb{R}^n,\quad 0\le i\le k-2 \tag{4}$$

and

$$\left(\frac{d}{dx}L_\phi^{k-1} h(x)\right)g(\xi)\neq 0,\quad \text{for some } x,\xi,\in \mathbb{R}^n \tag{5}$$

•

Remark 2. Notice that Definition 1 is a stronger property than the usual k relative degree property, which requires conditions (4-5) hold true for $x=\xi$. Nevertheless, there are many significant cases that satisfy it, such as the case of $g(x)\equiv\bar{g}$, $\forall x$, a constant function.

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Lemma 1. *Assume that system (1) has a full ($k=n$) strong relative degree property, and define $z_t=\Theta(x_t)$ according to the observability map defined in (2). Then, the differential dz_t can be written as:*

$$\begin{aligned}dz_t=&\Big(A_b z_t+B_bL_\phi^n h(x_t)\Big)dt+B_bL_gL_\phi^{n-1}h(x_t)(u_tdt+\\ &FdW_t^1)+\frac{1}{2}B_b\sum_{i=1}^{s}F_i^Tg^T(x_t)H_{\phi h}(x_t)g(x_t)F_idt\end{aligned} \tag{6}$$

where F_i is the i-th column of matrix F, A_b and B_b are Brunowski matrices of size n

$$A_b=\begin{bmatrix}0_{(n-1)\times 1} & I_{n-1}\\ 0 & 0_{1\times(n-1)}\end{bmatrix},\ B_b=\begin{bmatrix}0_{(n-1)\times 1}\\ 1\end{bmatrix} \tag{7}$$

and $H_{\phi h}(x_t)$ is the Hessian of $L_\phi^{n-1}h(x_t)$, that is:

$$[H_{\phi h}(x)]_{(i,j)}=\frac{\partial^2}{\partial x_i\,\partial x_j}L_\phi^{n-1}h(x),\ i,j=1,\dots,n. \tag{8}$$

Proof. By applying the Itô formula, according to the Kronecker formalism (cfr. [19]), it is:

$$\begin{aligned}dz_t=&\Bigg((\nabla_x\otimes\Theta)\Big|_{x_t}\big(\phi(x_t)+g(x_t)u_t\big)\\ &+\frac{1}{2}\big(\nabla_x^{[2]}\otimes\Theta\big)\Big|_{x_t}\tilde{g}_2(x_t)\Bigg)dt+(\nabla_x\otimes\Theta)\Big|_{x_t}g(x_t)FdW_t^1\end{aligned} \tag{9}$$

with:

$$\tilde{g}_2(x_t)=\sum_{i=1}^{s}\big(g(x_t)F_i\big)^{[2]}=g^{[2]}(x_t)F_0,\qquad F_0=\sum_{i=1}^{s}F_i^{[2]}, \tag{10}$$

and the differential operator $\nabla_x^{[i]}\otimes$ applied to a generic function $f:\mathbb{R}^n\mapsto\mathbb{R}^p$ is defined as follows:

$$\nabla_x^{[0]}\otimes f=f,\quad \nabla_x^{[i+1]}\otimes f=\nabla_x\otimes\big(\nabla_x^{[i]}\otimes f\big),\quad i\ge 1, \tag{11}$$

with $\nabla_x=[\partial/\partial x_1\ \cdots\ \partial/\partial x_n]$ and $\nabla_x\otimes f$ the Jacobian of the vector function f (see [19] for more details).

Note that, by suitably exploiting the observability map definition (2), it is:

$$\big(\nabla_x\otimes\Theta\big)\Big|_{x_t}\phi(x_t)=A_bz_t+B_bL_\phi^n h(x_t), \tag{12}$$

and, accordingly, by suitably exploiting the full relative degree hypotheses, it is:

$$\big(\nabla_x\otimes\Theta\big)\Big|_{x_t}g(x_t)=B_bL_gL_\phi^{n-1}h(x_t), \tag{13}$$

where, given a scalar function $\chi:\mathbb{R}^n\mapsto\mathbb{R}$, it is:

$$L_g\chi(x)=\frac{d\chi}{dx}\big[g_1\ \cdots\ g_s\big]=\big[L_{g_1}\chi(x)\ \ \cdots\ \ L_{g_s}\chi(x)\big], \tag{14}$$

with g_i the i-th column of g. It has to be stressed that equation (13) is achieved without using the *strong* full relative degree property, but just the standard one. The strong full relative degree property will be required in the following. Indeed, rewrite the second order derivative term in (9) as follows:

$$\begin{aligned}&\big(\nabla_x^{[2]}\otimes\Theta\big)\Big|_{x_t}\cdot g^{[2]}(x_t)\\ &=\big(\nabla_x\otimes(\nabla_x\otimes\Theta)\big)\Big|_{x_t}\cdot\big(g(x_t)\otimes g(x_t)\big)\\ &=\Big[\Big(\nabla_x\otimes\big(\nabla_x\otimes\Theta(x)\big)\Big)\cdot\big(g(\xi)\otimes g(\xi)\big)\Big]_{\substack{x=x_t\\ \xi=x_t}}\end{aligned} \tag{15}$$

This way, by formally applying the following Kronecker product property $(A\otimes B)\cdot(C\otimes D)=(A\cdot C)\otimes(B\cdot D)$ to eq.(15), it is:

$$\begin{aligned}&\big(\nabla_x^{[2]}\otimes\Theta\big)\Big|_{x_t}\cdot g^{[2]}(x_t)\\ &=\Big[\big(\nabla_x\cdot g(\xi)\big)\otimes\Big(\big(\nabla_x\otimes\Theta(x)\big)\cdot g(\xi)\Big)\Big]_{\substack{x=x_t\\ \xi=x_t}}\end{aligned} \tag{16}$$

with $\big(\nabla_x\cdot g(\xi)\big)\otimes$ the differential operator (shortly denoted by $\nabla_x^g\otimes$ in the following), which applied to a function $\eta(x):\mathbb{R}^n\mapsto\mathbb{R}^{n\times s}$ is such that:

$$\nabla_x^g\otimes\eta=\big[\nabla_x^{g_1}\otimes\eta\ \ \cdots\ \ \nabla_x^{g_s}\otimes\eta\big] \tag{17}$$

with:

$$\nabla_x^{g_i}\otimes\eta=\sum_{j=1}^{n}g_{ji}\frac{\partial\eta}{\partial x_j}. \tag{18}$$

Then, according to the *strong* full relative degree property, it is:

$$\big(\nabla_x\otimes\Theta(x)\big)\cdot g(\xi)=B_b\left(\frac{d}{dx}L_\phi^{n-1}h(x)\right)g(\xi) \tag{19}$$

so that, by taking into account that $\nabla_x^g\otimes$ does not differentiate $g(\xi)$, it is:

$$\begin{aligned}&\Big[\nabla_x^{g_i}\otimes\Big(\big(\nabla_x\otimes\Theta(x)\big)\cdot g(\xi)\Big)\Big]_{\substack{x=x_t\\ \xi=x_t}}\\ &=B_b\sum_{j=1}^{n}g_{ji}(x_t)\left[\frac{\partial}{\partial x_j}\frac{d}{dx}L_\phi^{n-1}h(x)\right]_{x=x_t}g(x_t)\\ &=B_bg_i^T(x_t)H_{\phi h}(x_t)g(x_t)\end{aligned} \tag{20}$$

By exploiting the following property involving the stack of suitably sized matrices

$$\mathrm{st}(A\cdot B\cdot C)=\big(C^T\otimes A\big)\cdot\mathrm{st}(B) \tag{21}$$

it is:

$$
\begin{aligned}
\left(\nabla_x^{[2]} \otimes \Theta\right)\Big|_{x_t} & \tilde{g}_2(x_t) \\
&= B_b\big[g_1^T H_{\phi h} g \quad \cdots \quad g_s^T H_{\phi h} g\big] F_0 \\
&= B_b \left(\mathrm{st} \begin{bmatrix} g_1^T H_{\phi h} g_1 & \cdots & g_s^T H_{\phi h} g_1 \\ \vdots & \ddots & \vdots \\ g_1^T H_{\phi h} g_s & \cdots & g_s^T H_{\phi h} g_1 \end{bmatrix} \right)^T F_0 \\
&= B_b \Big(\mathrm{st}\big(g^T H_{\phi h} g\big)\Big)^T F_0 = B_b F_0^T \mathrm{st}\big(g^T H_{\phi h} g\big) \\
&= B_b \sum_{i=1}^{s} \big(F_i^T \otimes F_i^T\big)\big(g^T \otimes g^T\big)\mathrm{st}(H_{\phi h}) \\
&= B_b \sum_{i=1}^{s} \big(F_i^T g^T(x_t)\big)^{[2]} \mathrm{st}(H_{\phi h}(x_t))
\end{aligned} \tag{22}
$$

Finally, by exploiting again (21), it is:

$$
\begin{aligned}
\left(\nabla_x^{[2]} \otimes \Theta\right)\Big|_{x_t} \tilde{g}_2(x_t) &= B_b \sum_{i=1}^{s} \mathrm{st}\big(F_i^T g^T H_{\phi h} g F_i\big) \\
&= B_b \sum_{i=1}^{s} F_i^T g^T H_{\phi h} g F_i
\end{aligned} \tag{23}
$$

which completes the proof.

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III. The observer-based state estimator

The state estimator, $\hat{x}_t$, proposed for system (1) obeys the following differential equations:

$$
\begin{aligned}
d\hat{x}_t &= \phi(\hat{x}_t)dt + g(\hat{x}_t)u_t dt \\
&+ \frac{1}{2} Q^{-1}\big(\hat{x}_t\big) B_b \sum_{i=1}^{s} F_i^T g^T(\hat{x}_t) H_{\phi h}(\hat{x}_t) g(\hat{x}_t) F_i dt \\
&+ Q^{-1}\big(\hat{x}_t\big) K \big(dy_t - h(\hat{x}_t)dt\big)
\end{aligned} \tag{24}
$$

with Q the Jacobian of the observability map Θ, defined in (2). The main result is the proof that there exists a bound to the state estimate error, in the mean square sense.

Theorem. *Assume the following hypotheses are satisfied:*

H1 *the observability map $z_t = \Theta(x_t)$ defined in (2) is a global diffeomorphism, with the inverse map $\Theta^{-1}(\cdot)$ globally Lipschitz, with Lipschitz coefficient γ_θ;*

H2 *system (1) has a strong full (i.e. equal to n) relative degree;*

H3 *functions $L_{g_i} L_\phi^{n-1} h(x)$ are bounded in the mean square sense, that is: there exists a positive constant γ_1 such that*

$$
\max_{i=1,\dots,s} \sup_x \mathbb{E}\Big[\|L_{g_i} L_\phi^{n-1} h(x)\|^2\Big] \le \gamma_1 \tag{25}
$$

H4 *functions $L_\phi^n h(x)$, $L_g L_\phi^{n-1} h(x)$ and $g^T(x) H_{\phi h}(x) g(x)$ are globally Lipschitz, with Lipschitz coefficients γ_2, γ_3 and γ_4, respectively;*

H5 *u_t is uniformly bounded, that is: there exists a positive constant U_M such that*

$$
\sup_{t \ge 0} \|u_t\|^2 \le U_M. \tag{26}
$$

Then, there exists a gain vector $K \in \mathbb{R}^{n\times 1}$ such that the observer defined by (24) has a bounded error (in the mean square sense), that is: there exists a positive constant L such that

$$
\mathbb{E}\big[\|x_t - \hat{x}_t\|^2\big] \le L. \tag{27}
$$

Proof. Define the following observer of z_t as:

$$
\begin{aligned}
d\hat{z}_t = \Big(& A_b \hat{z}_t + B_b L_\phi^n h\left(\Theta^{-1}(\hat{z}_t)\right) \\
&+ B_b L_g L_\phi^{n-1} h\left(\Theta^{-1}(\hat{z}_t)\right) u_t\Big) dt \\
&+ \frac{1}{2} B_b \sum_{i=1}^{s} F_i^T g^T\left(\Theta^{-1}(\hat{z}_t)\right) \\
&\quad \cdot H_{\phi h}\left(\Theta^{-1}(\hat{z}_t)\right) g\left(\Theta^{-1}(\hat{z}_t)\right) F_i dt \\
&+ K(dy_t - C_b \hat{z}_t dt),
\end{aligned} \tag{28}
$$

where $C_b = [1\ 0\ \cdots\ 0] \in \mathbb{R}^{1\times n}$. According to hypothesis **H2**, Lemma 1 holds true and allows to write the differential equation for the error $\epsilon_t = z_t - \hat{z}_t$ as:

$$
\begin{aligned}
d\epsilon_t = &\big(A_b - KC_b\big)\epsilon_t dt + B_b \Delta_\epsilon(z_t, \hat{z}_t, u_t)dt \\
&+ B_b L_g L_\phi^{n-1} h\left(\Theta^{-1}(z_t)\right) F dW_t^1 - KG dW_t^2,
\end{aligned} \tag{29}
$$

where the following notation has been adopted for the sake of simplicity:

$$
\begin{aligned}
\Delta_\epsilon(z_t, \hat{z}_t, u_t) &= L_\phi^n h\left(\Theta^{-1}(z_t)\right) + L_g L_\phi^{n-1} h\left(\Theta^{-1}(z_t)\right) u_t \\
&+ \frac{1}{2} \sum_{i=1}^{s} F_i^T g^T\big(\Theta^{-1}(z_t)\big) H_{\phi h}\left(\Theta^{-1}(z_t)\right) g\left(\Theta^{-1}(z_t)\right) F_i \\
&- \Big(L_\phi^n h\left(\Theta^{-1}(\hat{z}_t)\right) + L_g L_\phi^{n-1} h\left(\Theta^{-1}(\hat{z}_t)\right) u_t \\
&+ \frac{1}{2} \sum_{i=1}^{s} F_i^T g^T\big(\Theta^{-1}(\hat{z}_t)\big) H_{\phi h}\left(\Theta^{-1}(\hat{z}_t)\right) g\left(\Theta^{-1}(\hat{z}_t)\right) F_i\Big).
\end{aligned} \tag{30}
$$

According to the Brunowski matrices definition, A_b and B_b constitute an observable pair, so that matrix K can be set in order to choose negative real and distinct eigenvalues for $A_b - KC_b$. Denote such a spectrum as $\{\lambda_i,\ i = 1,\dots,n\}$, and define $\tilde{\epsilon}_t = V(\lambda)\epsilon_t$, where $V(\lambda)$ is the Vandermonde matrix

$$
V(\lambda) = \begin{bmatrix} \lambda_1^{n-1} & \cdots & \lambda_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_n^{n-1} & \cdots & \lambda_n & 1 \end{bmatrix} \tag{31}
$$

By definition, $V(\lambda)$ is a coordinate transformation matrix, which diagonalize $A_b - KC_b$, that is $\Lambda = \mathrm{diag}\{\lambda_1,\dots,\lambda_n\} = V(\lambda)(A_b - KC_b)V^{-1}(\lambda)$ so that, according to (29) the $\tilde{\epsilon}_t$ dynamics is given by:

$$
\begin{aligned}
d\tilde{\epsilon}_t = &\Lambda \tilde{\epsilon}_t dt + \underline{1}\Delta_\epsilon(z_t, \hat{z}_t, u_t)dt \\
&+ \underline{1} L_g L_\phi^{n-1} h\left(\Theta^{-1}(z_t)\right) F dW_t^1 - V(\lambda) K G dW_t^2,
\end{aligned} \tag{32}
$$

where $\underline{1} = [1 \ 1 \ \cdots \ 1]^T$ is the n-dimensional column vector of ones. Then, the integral equation associated to (32) is:

$$\begin{aligned}\tilde{\epsilon}_t = e^{\Lambda t}\tilde{\epsilon}_0 &+ \int_0^t e^{\Lambda(t-\tau)}\underline{1}\Delta_\epsilon(z_\tau,\hat{z}_\tau,u_\tau)d\tau \\ &+ \sum_{i=1}^{s}\int_0^t e^{\Lambda(t-\tau)}\underline{1}L_{g_i}L_\phi^{n-1}h\left(\Theta^{-1}(z_\tau)\right)FdW^1_{\tau,i} \\ &- \int_0^t e^{\Lambda(t-\tau)}V(\lambda)KGdW^2_\tau\end{aligned} \tag{33}$$

where $W^1_{t,i}$, $i = 1,\ldots,s$, is the i-th component of vector W^1_t. Since dW^1_t and dW^2_t are uncorrelated, the expected value of the square norm of $\tilde{\epsilon}_t$ can be written as

$$\begin{aligned}I\!\!E\left[\|\tilde{\epsilon}_t\|^2\right] &= I\!\!E\left[\tilde{\epsilon}_t^T\tilde{\epsilon}_t\right] \le I\!\!E\left[\left\|e^{\Lambda t}\tilde{\epsilon}_0\right\|^2\right] \\ &+2I\!\!E\left[\left\|e^{\Lambda t}\tilde{\epsilon}_0\right\|\cdot\int_0^t\left\|e^{\Lambda(t-\tau)}\underline{1}\Delta_\epsilon(z_\tau,\hat{z}_\tau,u_\tau)\right\|d\tau\right] \\ &+I\!\!E\Bigg[\int_0^t\int_0^t\left\|e^{\Lambda(t-\tau)}\underline{1}\Delta_\epsilon(z_\tau,\hat{z}_\tau,u_\tau)\right\| \\ &\qquad\cdot\left\|e^{\Lambda(t-\theta)}\underline{1}\Delta_\epsilon(z_\theta,\hat{z}_\theta,u_\theta)\right\|d\tau d\theta\Bigg] \\ &+I\!\!E\left[\sum_{i=1}^{s}\int_0^t\left\|e^{\Lambda(t-\tau)}\underline{1}L_{g_i}L_\phi^{n-1}h\left(\Theta^{-1}(z_\tau)\right)F\right\|^2d\tau\right] \\ &+\int_0^t\left\|e^{\Lambda(t-\tau)}\ V(\lambda)KG\right\|^2d\tau.\end{aligned} \tag{34}$$

Let us take into account the terms in the right hand side of (34). As for the first term, it is:

$$I\!\!E\left[\left\|e^{\Lambda t}\tilde{\epsilon}_0\right\|^2\right] \le e^{2\lambda_M t}I\!\!E\left[\|\tilde{\epsilon}_0\|^2\right], \tag{35}$$

where λ_M denotes the largest eigenvalue of Λ.

As for the second term in the right hand side of eq.(34), note that hypotheses **H1** and **H4** imply that $L_\phi^n h\left(\Theta^{-1}(z)\right)$, $L_g L_\phi^{n-1}h\left(\Theta^{-1}(z)\right)$ and $g^T\left(\Theta^{-1}(z)\right)H_{\phi h}\left(\Theta^{-1}(z)\right)g\left(\Theta^{-1}(z)\right)$ are Lipschitz with Lipschitz constants $\gamma_2\gamma_\theta$, $\gamma_3\gamma_\theta$ and $\gamma_4\gamma_\theta$, respectively. Indeed:

$$\left|L_\phi^n\big(\Theta^{-1}(z_1)\big) - L_\phi^n\big(\Theta^{-1}(z_2)\big)\right| \le \gamma_2\gamma_\theta\|z_1 - z_2\| \tag{36}$$

and analogously for the other two functions. Then, according also to hypothesis **H5**, it is

$$\|\Delta_\epsilon(z_t,\hat{z}_t,u_t)\| \le \gamma_\Delta\|\epsilon_t\|^2 \tag{37}$$

with

$$\gamma_\Delta = \gamma_\theta\left(\gamma_2 + \gamma_3 U_M + \frac{\gamma_4}{2}\sum_{i=1}^{s}\|F_i\|^2\right). \tag{38}$$

Then:

$$\begin{aligned}&2I\!\!E\left[\left\|e^{\Lambda t}\tilde{\epsilon}_0\right\|\cdot\int_0^t\left\|e^{\Lambda(t-\tau)}\underline{1}\Delta_\epsilon(z_\tau,\hat{z}_\tau,u_\tau)\right\|d\tau\right] \\ &\le 2I\!\!E\left[e^{\lambda_M t}\|\tilde{\epsilon}_0\|\int_0^t e^{\lambda_M(t-\tau)}\sqrt{n}\gamma_\Delta\|V^{-1}(\lambda)\|\cdot\|\tilde{\epsilon}_\tau\|d\tau\right] \\ &\le \sqrt{n}\gamma_\Delta\|V^{-1}(\lambda)\|e^{\lambda_M t}\int_0^t e^{\lambda_M(t-\tau)} \\ &\qquad\cdot\Big(I\!\!E\left[\|\tilde{\epsilon}_0\|^2\right] + I\!\!E\left[\|\tilde{\epsilon}_\tau\|^2\right]\Big)d\tau \\ &\le \sqrt{n}\gamma_\Delta\|V^{-1}(\lambda)\|e^{\lambda_M t}\Bigg(\tfrac{1-e^{\lambda_M t}}{|\lambda_M|}I\!\!E\left[\|\tilde{\epsilon}_0\|^2\right] \\ &\qquad+\int_0^t e^{\lambda_M(t-\tau)}I\!\!E\left[\|\tilde{\epsilon}_\tau\|^2\right]d\tau\Bigg) \\ &\le \sqrt{n}\gamma_\Delta\|V^{-1}(\lambda)\|e^{\lambda_M t}\Bigg(\tfrac{I\!\!E\left[\|\tilde{\epsilon}_0\|^2\right]}{|\lambda_M|} \\ &\qquad+\int_0^t e^{\lambda_M(t-\tau)}I\!\!E\left[\|\tilde{\epsilon}_\tau\|^2\right]d\tau\Bigg),\end{aligned} \tag{39}$$

where the property $2\|a\|\cdot\|b\| \le \|a\|^2 + \|b\|^2$ has been used.

As for the third term of eq.(34), it is:

$$\begin{aligned}&I\!\!E\Bigg[\int_0^t\int_0^t\left\|e^{\Lambda(t-\tau)}\underline{1}\Delta_\epsilon(z_\tau,\hat{z}_\tau,u_\tau)\right\| \\ &\qquad\cdot\left\|e^{\Lambda(t-\theta)}\underline{1}\Delta_\epsilon(z_\theta,\hat{z}_\theta,u_\theta)\right\|d\tau d\theta\Bigg] \\ &\le \int_0^t\int_0^t e^{\lambda_M(t-\tau)}e^{\lambda_M(t-\theta)}n\gamma_\Delta^2\|V^{-1}(\lambda)\|^2 \\ &\qquad\cdot I\!\!E\Big[\|\tilde{\epsilon}_\tau\|\cdot\|\tilde{\epsilon}_\theta\|\Big]d\theta d\tau \\ &\le \tfrac{n\gamma_\Delta^2\|V^{-1}(\lambda)\|^2}{2}\int_0^t\int_0^t e^{\lambda_M(t-\tau)}e^{\lambda_M(t-\theta)} \\ &\qquad\cdot\Big(I\!\!E\left[\|\tilde{\epsilon}_\tau\|^2\right] + I\!\!E\left[\|\tilde{\epsilon}_\theta\|^2\right]\Big)d\theta d\tau \\ &\le n\gamma_\Delta^2\|V^{-1}(\lambda)\|^2\tfrac{1-e^{\lambda_M t}}{|\lambda_M|}\int_0^t e^{\lambda_M(t-\tau)}I\!\!E\left[\|\tilde{\epsilon}_\tau\|^2\right]d\tau \\ &\le \tfrac{n\gamma_\Delta^2\|V^{-1}(\lambda)\|^2}{|\lambda_M|}\int_0^t e^{\lambda_M(t-\tau)}I\!\!E\left[\|\tilde{\epsilon}_\tau\|^2\right]d\tau.\end{aligned} \tag{40}$$

As for the fourth term of eq.(34), according to hypothesis **H3**, it is:

$$\begin{aligned}&I\!\!E\left[\sum_{i=1}^{s}\int_0^t\left\|e^{\Lambda(t-\tau)}\underline{1}L_{g_i}L_\phi^{n-1}h\left(\Theta^{-1}(z_\tau)\right)F\right\|^2d\tau\right] \\ &\le\int_0^t ne^{2\lambda_M(t-\tau)}\|F\|^2\sum_{i=1}^{s}I\!\!E\left[\left\|L_{g_i}L_\phi^{n-1}h\big(\Theta^{-1}(z_\tau)\big)\right\|^2\right]d\tau \\ &\le s\gamma_1 n\|F\|^2\int_0^t e^{2\lambda_M(t-\tau)}d\tau \quad \le \quad \frac{s\gamma_1 n\|F\|^2}{2|\lambda_M|},\end{aligned} \tag{41}$$

and, finally,for the fifth term of eq.(34), it is:

$$\int_0^t\left\|e^{\Lambda(t-\tau)}\ V(\lambda)KG\right\|^2d\tau \le \frac{\|V(\lambda)KG\|^2}{2|\lambda_M|}. \tag{42}$$

By using inequalities (35,39-42) and re-arranging the terms, we have:

$$\begin{aligned}\mathbb{E}\big[\|\tilde{\epsilon}_t\|^2\big] &\le e^{\lambda_M t}\Big(e^{\lambda_M t} + \frac{\sqrt{n}\gamma_\Delta\|V^{-1}(\lambda)\|}{|\lambda_M|}\Big)\mathbb{E}\big[\|\tilde{\epsilon}_0\|^2\big] \\ &\quad + \Big(\sqrt{n}\gamma_\Delta e^{\lambda_M t} + \frac{n\gamma_\Delta^2\|V^{-1}(\lambda)\|^2}{|\lambda_M|}\Big) \\ &\cdot \int_0^t e^{\lambda_M(t-\tau)}\mathbb{E}\big[\|\tilde{\epsilon}_\tau\|^2\big]d\tau + \frac{s\gamma_1 n\|F\|^2 + \|V(\lambda)KG\|^2}{2|\lambda_M|}.\end{aligned} \tag{43}$$

Note that, by denoting $\beta(t) = e^{-\lambda_M t}\mathbb{E}\big[\|\tilde{\epsilon}_t\|^2\big]$, it is:

$$\beta(t) \le \alpha_1 + \alpha_2 e^{-\lambda_M t} + \alpha_3\int_0^t \beta(\tau)d\tau \tag{44}$$

with

$$\begin{aligned}\alpha_1 &= \Big(1 + \tfrac{\sqrt{n}\gamma_\Delta\|V^{-1}(\lambda)\|}{|\lambda_M|}\Big)\mathbb{E}\big[\|\tilde{\epsilon}_0\|^2\big] \\ \alpha_2 &= \frac{s\gamma_1 n\|F\| + \|V(\lambda)KG\|^2}{2|\lambda_M|} \\ \alpha_3 &= \sqrt{n}\gamma_\Delta e^{\lambda_M t} + \frac{n\gamma_\Delta^2\|V^{-1}(\lambda)\|^2}{|\lambda_M|}\end{aligned} \tag{45}$$

By appropriately using one of the available integral inequalities (see for example Theorem 1.3 in [4], pag. 3) we have the following implication:

$$\begin{aligned}\beta(t) &\le \alpha_1 + \alpha_2 e^{-\lambda_M t} + \int_0^t \big(\alpha_1 + \alpha_2 e^{-\lambda_M s}\big)e^{\int_s^t \alpha_3 d\tau}ds \\ &= \alpha_1 + \alpha_2 e^{-\lambda_M t} + \tfrac{\alpha_1\left(e^{\alpha_3 t}-1\right)}{\alpha_3} \\ &\qquad + \tfrac{\alpha_2}{\alpha_3+\lambda_M}\Big(e^{(\alpha_3+\lambda_M)t} - 1\Big)e^{-\lambda_M t},\end{aligned} \tag{46}$$

from which

$$\begin{aligned}\mathbb{E}\big[\|\tilde{\epsilon}_t\|^2\big] &\le \alpha_1 e^{\lambda_M t} + \alpha_2 + \tfrac{\alpha_1}{\alpha_3}\Big(e^{\alpha_3 t} - 1\Big)e^{\lambda_M t} \\ &\quad + \tfrac{\alpha_2}{\alpha_3+\lambda_M}\Big(e^{(\alpha_3+\lambda_M)t} - 1\Big).\end{aligned} \tag{47}$$

Now, if $\alpha_3 + \lambda_M < 0$, we finally have

$$\mathbb{E}\big[\|\tilde{\epsilon}_t\|^2\big] \le \alpha_1 + \alpha_2 + \frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{|\alpha_3 + \lambda_M|} \tag{48}$$

Notice that the requirement $\alpha_3 + \lambda_M < 0$ is equivalent to

$$\sqrt{n}\gamma_\Delta\left(1 + \frac{\sqrt{n}\gamma_\Delta\|V^{-1}(\lambda)\|^2}{|\lambda_M|}\right) + \lambda_M < 0, \tag{49}$$

and since it is possible (see [13]) to fix λ_M and choose the remaining eigenvalues λ_i in order to have $\|V^{-1}(\lambda)\|$ arbitrarily close to 1, the left-hand side of (49) can assume any prescribed negative value. This way it is shown that there exists a bound for $\mathbb{E}\big[\|\tilde{\epsilon}_t\|^2\big]$, eq.(48), and, therefore, the z_t-observer defined in (28) has a bounded error in the mean square sense, with bound given by:

$$\mathbb{E}\big[\|\epsilon_t\|^2\big] \le \|V^{-1}(\lambda)\|^2\left(\alpha_1 + \alpha_2 + \frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{|\alpha_3 + \lambda_M|}\right) \tag{50}$$

The x_t-observer it is defined by coupling eq.(28) with $\hat{x}_t = \Theta^{-1}(\hat{z}_t)$. Then, since $\Theta^{-1}(\cdot)$ is Lipschitz, it is

$$\mathbb{E}\big[\|x_t - \hat{x}_t\|^2\big] \le \gamma_\theta^2\mathbb{E}\big[\|z_t - \hat{z}_t\|^2\big] \le L \tag{51}$$

with

$$L = \gamma_\theta^2\|V^{-1}(\lambda)\|^2\left(\alpha_1 + \alpha_2 + \frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{|\alpha_3 + \lambda_M|}\right). \tag{52}$$

To complete the proof we show that the x_t-observer equations are given by (24). Indeed:

$$d\hat{x}_t = \frac{d\Theta^{-1}}{dz}\bigg|_{z=\Theta(\hat{x}_t)} \cdot d\hat{z}_t. \tag{53}$$

Since:

$$\frac{d\Theta^{-1}}{dz}\bigg|_{z=\Theta(\hat{x}_t)} = \left[\frac{d\Theta}{dx}\right]^{-1}\bigg|_{\hat{x}_t} = Q^{-1}(\hat{x}_t), \tag{54}$$

it is:

$$\begin{aligned}d\hat{x}_t &= Q^{-1}(\hat{x}_t)\big(A_b\Theta(\hat{x}_t) + B_b L_\phi^n h(\hat{x}_t) \\ &\quad + B_b L_g L_\phi^{n-1}h(\hat{x}_t)u_t\big)dt \\ &\quad + \frac{1}{2}Q^{-1}(\hat{x}_t)B_b\sum_{i=1}^s F_i^T g^T(\hat{x}_t)H_{\phi h}(\hat{x}_t)g(\hat{x}_t)F_i dt \\ &\quad + Q^{-1}(\hat{x}_t)K\big(dy_t - C_b\Theta(\hat{x}_t)dt\big).\end{aligned} \tag{55}$$

By suitably exploiting the full relative degree property, it is:

$$Q(x)\phi(x) = A_b\Theta(x) + B_b L_\phi^n h(x) \tag{56}$$

$$Q(x)g(x) = B_b L_g L_\phi^{n-1}h(x) \tag{57}$$

so that the x_t-observer equations (24) are readily obtained. •

IV. Example

Let us consider the following nonlinear stochastic system, which is a version of the Michaelis-Menten model in pharmakokinetics

$$\begin{aligned}dx_{1,t} &= \left(-k_1 x_{1,t} + \frac{v_m x_{2,t}}{k_m + x_{2,t}}\right)dt \\ dx_{2,t} &= -\left(\frac{v_m x_{2,t}}{k_m + x_{2,t}}\right)dt + u_t dt + F dW_t^1 \\ dy_t &= \sqrt{k_2 + x_{1,t}^2}dt + G dW_t^2,\end{aligned} \tag{58}$$

where $k_1 = 0.1$, $k_2 = 1.0$, $k_m = 5.0$, and $v_m = 10.0$ are positive parameters. The input function is $u_t = 5(1 + \sin t)$. From the system equations it is readily seen that the Jacobian of the observability map is nonsingular for $x_{1,t} \ne 0$ and that system (58) has a strong full relative degree property, since $g(x) = [0\ 1]^T$ and $L_g h(x) = (dh/dx)\ g(x) = 0$. Moreover,

$$Fg^T(x_t)H_{\phi h}(x_t)g(x_t)F = \frac{\partial h(x_t)}{\partial x_{1,t}}\left(\frac{-2F^2 v_m k_m}{(k_m + x_{2,t})^3}\right) \tag{59}$$

We have run 100 simulations in the time interval $[0,\ 100]$ for several choices of noise parameters using the Euler-Maruyama method, with the integration step $dt = 5 \cdot 10^{-3}$.

F	G	λ_M	$\mu_o \pm s_o$	$\mu_{KB} \pm s_{KB}$	$10^2 \cdot (\mu_o - \mu_{KB})/\mu_{KB}$
0.25	3	$-4 \cdot 10^{-2}$	0.366 $\pm$ 0.113	0.385 $\pm$ 0.141	-4.73 %
0.25	5	$-2 \cdot 10^{-2}$	0.658 $\pm$ 0.232	0.866 $\pm$ 0.334	-25.79 %
0.25	7	$-1 \cdot 10^{-2}$	0.909 $\pm$ 0.338	1.123 $\pm$ 0.415	-19.58 %
0.5	3	$-5 \cdot 10^{-2}$	0.887 $\pm$ 0.260	0.911 $\pm$ 0.284	-2.70 %
0.5	5	$-4 \cdot 10^{-2}$	1.251 $\pm$ 0.382	1.353 $\pm$ 0.494	-7.55 %
0.5	7	$-2 \cdot 10^{-2}$	1.423 $\pm$ 0.500	1.686 $\pm$ 0.655	-15.57 %

TABLE I

MEAN SQUARE ESTIMATION ERROR WITH STANDARD DEVIATION FOR THE PROPOSED OBSERVER-BASED ESTIMATE (μ_o) AND THE EKBF (μ_{KB})

The performance of the observer is compared with an Extended Kalman-Bucy filter (EKBF) comparing the mean of the squared estimation error (MSE)

$$\mu_x = \frac{1}{N+1} \sum_{k=0}^{N} \| x_{t_k} - \hat{x}_{t_k} \|^2 \tag{60}$$

with $N = t_{max}/dt$, and results are shown in Table I.

In these simulations the observer and the EKBF have a fixed initial state $\hat{x}_0 = [10,\ 10]^T$, whereas the initial condition of the system is chosen around the point $[5,\ 3]$ by adding a random variables with mean 0 and variance 1,$x_0 = [5 + n(0,1),\ 3 + n(0,1)]^T$. The third column in Table I reports the maximum eigenvalue used to compute the observer gain. The impact of the initial transient is not included, since the MSE is computed only for time instants $t > 10$. Results of Table I focus on situations with a high measurement noise and a medium state noise, since in these cases the high-gain observer behaves better than the EKBF.

V. CONCLUSIONS AND FUTURE WORKS

We have analyzed the conditions under which a bound on the estimation error can be provided by a high-gain observer. From the simulations reported in the previous section we may draw several conclusions. A high-gain nonlinear observer displays good performance when the ratio between measurement and state noise is high. Even if in several cases the performance improvement over a EKBF is moderate or questionable, the observer consistently produces a smaller variance of the MSE in all the scenarios. The estimate error is thus less sensitive to the initial conditions and to the specific realization of the stochastic system with respect to an EKBF. The high-gain observer behaves much better in the transient phase and it can thus be useful for systems for which transient situations are frequent.

A promising approach that we are currently investigating is to use a fixed high-gain observer to provide a starting estimate to a filter in charge of evaluating the displacement of the stochastic system from the observer estimate.

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