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Optimal Attitude Estimation Using Set-Valued Observers

S. Brás, P. Rosa, C. Silvestre, and P. Oliveira

***Abstract*— An optimal Set-Valued Observer (SVO) for attitude estimation that merges rate gyro readings with body frame vectorial observations is proposed. The observer provides singularity-free estimates and considers sensor readings corrupted by bounded (but unknown) measurement noise. The suggested solution is an alternative method to the ones available in the literature, with the guarantee that the state is inside the estimated set. Optimal estimates are obtained, provided that there is no uncertainty in the angular velocity measurements, whereas the case with noisy rate gyros measurements is addressed by resorting to a relaxation of the problem. The sensor readings are exploited directly in the observer, without intermediate attitude estimate computation. Moreover, the feasibility of the technique is demonstrated in simulation.**

I. INTRODUCTION

Attitude estimation is an essential element in many modern platforms such as aircrafts, satellites, unmanned air vehicles, and underwater autonomous robots. Typical solutions to this problem require either the exact measurement of the sensed variables or the knowledge of a stochastic description for the exogenous disturbances and measurement noise. However, if this information is not available *a priori*, while norm bounds on the disturbances are known, it may be desirable to compute explicit bounds on the attitude of the vehicle. Such bounds are suitable, for instance, in robust control designs, where *worst-case* guarantees are provided regarding the performance of the closed-loop system – see, for instance, [22], [17].

There is a wide variety of attitude estimation techniques presented in the literature. Some, like the nonlinear observers, have a deterministic nature [8], [18], [19], while others use a stochastic description of the exogenous disturbances and measurement noise to provide optimal estimates of the attitude [21], [11], [7]. However, this stochastic characterization may not be available and only magnitude bounds be known. The work in [15] discusses the state estimation for systems with bounded inputs, while in [3], [9] a similar problem, but using a set-membership description for model uncertainty, is addressed. New advances in the framework of these estimators, known as Set-Valued Observers (SVOs) [1], are presented in [16], [12], [13]. The SVOs consider that the set-valued estimate of the state can be described by polytopes, while the so-called interval analysis, where those sets are over-bounded by hyper-parallelepipeds, thus yielding more conservative results. The work in [14] exploits a different approach and proposes an attitude estimator where uncertainty ellipsoids bound the sensor measurements and the filter states. However, this estimator relies on the linearization of the system to propagate the uncertainty ellipsoids. In the context of deterministic estimators, the work in [20] proposes a nonlinear observer for attitude estimation. This estimator exploits information from exact vector observations and biased angular velocity measurements and prove exponential convergence of the estimation errors to the origin.

The main contribution of the work presented in this paper is the development of an optimal attitude estimator based on SVOs which rely on vector observations and rate gyros measurements, where the sensor measurements are assumed to be corrupted by bounded noise. We propose a solution that considers uncertainties defined by polytopes and that guarantees that the true state of the system is inside the estimated set, as long as the assumptions on the bounds on the measurements are satisfied. No linearization is required and nonconservative estimates are derived for the case where no uncertainty is present in the angular velocity measurements. The case with noisy angular velocity measurements is tackled by relaxing the problem.

The remainder of this article is organized as follows. In Section II, the attitude estimation problem is introduced and the available sensor information is described. The SVOs for attitude estimation, with noisy and noise-free angular velocity measurements are derived in Section III. The interpretation of the estimated set using different $\mathrm{SO}(3)$ parameterizations is discussed in Section IV. In Section V, the yaw, pitch and roll bounds are illustrated in simulation for a typical trajectory, as well as a comparison between the SVOs and the deterministic observer proposed in [20]. Finally, Section VI presents some concluding remarks and a few comments on the future work.

NOMENCLATURE

To enhance the readability of this paper, we introduce the following notation. The set of special orthogonal matrices is denoted by $\mathrm{SO}(3)$ and the associated algebra is denoted by $\mathfrak{so}(3)$. The skew-symmetric operator in $\mathbb{R}^3$ is denoted by $[.]_\times : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$, and satisfies $[\mathbf{v}]_\times \mathbf{w} = \mathbf{v} \times \mathbf{w}$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. The Kronecker product of matrices is denoted by $\mathbf{A} \otimes \mathbf{B}$ (for further details see [22, p. 25]). The 3×3 matrices whose elements are zeros except the element ij and whose all elements are ones, are denoted by $\mathbf{E}_{i,j}$ and $\mathbf{1}_{3\times 3}$, respectively. The operator $\mathrm{vec}(\mathbf{M})$ stacks the columns of the 3×3 matrix $\mathbf{M}$ into a long 9×1 vector and the inverse operation of $\mathrm{vec}(.)$ is denoted as $\mathrm{mat}(.)$. The matrix norm $||.||_{\max}$ is defined as the maximum of the absolute value of all matrix elements, i.e., $\|\mathbf{A}\|_{\max} := \max\{|a_{ij}|\}$. Consider a polytope defined by $\{\mathbf{x} \in \mathbb{R}^{n_x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$. Then, define the Fourier-Motzkin projection [6] as $(\bar{\mathbf{A}}, \bar{\mathbf{b}}) := \mathrm{FM}(\mathbf{A}, \mathbf{b}, n)$, where $n = n_x - \bar{n}_x > 0$, and $\bar{\mathbf{A}}$ and $\bar{\mathbf{b}}$ satisfy, for all $\bar{\mathbf{x}} \in \mathbb{R}^{\bar{n}_x}$, $\bar{\mathbf{A}}\bar{\mathbf{x}} \leq \bar{\mathbf{b}} \Leftrightarrow \exists_{\mathbf{x}\in\mathbb{R}^n} : \mathbf{A}[\bar{\mathbf{x}}^T \ \mathbf{x}^T]^T \leq \mathbf{b}$. Finally define $\mathrm{MatSet}(\mathbf{M}, \mathbf{m}) := \{\mathbf{Y} \in \mathbb{R}^{a\times b} : \mathbf{M}\,\mathrm{vec}(\mathbf{Y}) \leq \mathbf{m}\}$.

The authors are with the Institute for Systems and Robotics (ISR), Instituto Superior Técnico, Lisbon, Portugal. E-mails: {sbras, prosa, cjs, pjcro}@isr.ist.utl.pt
Tel: (+351) 21-8418054, Fax: (+351) 21-8418291.

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II. Problem Formulation

The attitude of a rigid body is often mathematically modeled as a rotation matrix, which is a linear transformation that maps coordinates between reference frames with the same origin. Denote vector $\mathbf{x}$ expressed in reference frame $\{I\}$ as ${}^{I}\mathbf{x} \in \mathbb{R}^3$. We obtain this vector expressed in reference frame $\{B\}$, ${}^{B}\mathbf{x} \in \mathbb{R}^3$, by using the relationship ${}^{B}\mathbf{x} = {}^{B}_{I}\mathbf{R}\,{}^{I}\mathbf{x}$, where ${}^{B}_{I}\mathbf{R} \in \mathrm{SO}(3)$ is the rotation matrix from $\{I\}$ to $\{B\}$. Let $\mathcal{R} := {}^{B}_{I}\mathbf{R}$ and assume the reference frame $\{I\}$ to be inertial. The kinematics of the attitude is given by

$$\dot{\mathcal{R}}(t) = [\boldsymbol{\omega}(t)]_{\times}\mathcal{R}(t), \tag{1}$$

where $\boldsymbol{\omega}$ is the angular velocity of the inertial frame with respect to the rigid body. This continuous-time model is not suitable to be implemented in a digital system. However, for a sufficiently small sampling period, T, we can approximate the angular velocity between sampling times by a constant function, and use the Euler approximation (see [5, p. 126]) of (1), which is given by

$$\mathcal{R}(k+1) = \exp(T\boldsymbol{\omega}(k))\mathcal{R}(k), \tag{2}$$

where $\exp(.) : \mathbb{R}^3 \mapsto \mathrm{SO}(3)$ is the exponential map on the special orthogonal group. An advantage of this approximation is the linearity on $\mathcal{R}$ of the discrete-time model in (2).

Suppose there is a triad of rate gyros fixed in reference frame $\{B\}$ which measures $\boldsymbol{\omega}$, and on-board sensors such as magnetometers, star trackers, among others, which provide vector observations expressed in body frame coordinates, i.e.,

$${}^{B}\mathbf{v}_i = \mathcal{R}\,{}^{I}\mathbf{v}_i, \tag{3}$$

where $i = 1, \ldots, N_v$, and N_v is the number of vector observations and that no three of which are collinear, or, in the matrix form,

$${}^{B}\mathbf{V} = \mathcal{R}\,{}^{I}\mathbf{V}, \tag{4}$$

where ${}^{B}\mathbf{V} = [{}^{B}\mathbf{v}_1 \ldots {}^{B}\mathbf{v}_{N_v}]$ such ${}^{I}\mathbf{V} = [{}^{I}\mathbf{v}_1 \ldots {}^{I}\mathbf{v}_{N_v}]$. If the linear acceleration is neglectable in comparison with the gravity, accelerometers are also suitable to be used as vector observations. We assume that the sensor measurements are corrupted by noise and that a measurement $\mathbf{q} \in \mathbb{R}^n$ belongs to the convex polytope defined as $\mathrm{Set}(\mathbf{M}, \mathbf{m}) := \{\mathbf{q} : \mathbf{M}\mathbf{q} \leq \mathbf{m}\}$, where $\mathbf{M}$ is a $m \times n$ real matrix and $\mathbf{m} \in \mathbb{R}^m$. Hence, at each time k we obtain a *set of observations*, rather than a vector observation.

The objective of the present work is to optimally estimate the attitude of a rigid body using the available sensor suite, i.e., to obtain the set-valued attitude estimate with the smallest possible uncertainty.

III. Attitude Estimation Using SVOs

In this section, we propose a methodology for the attitude estimation problem with bounded sensor noise. We present two solutions based on SVOs, namely one for the case with noisy vector observations and another for the case in which the angular velocity measurements are also corrupted by bounded noise. The first result is stronger since the minimal set that contains the state is obtained, whereas in the latter case the existence of uncertainties in the angular velocity measurements hinders the estimation problem and requires the introduction of conservatism in the solution.

A. SVO with uncertainties in the vector observations

We firstly develop an SVO for attitude estimation when there is uncertainty solely in the vector observations.

Definition 1: We say that $\mathcal{R}$ *is compatible with a set of observations*, S, if there exists ${}^{B}\mathbf{v} \in S$ such that (3) is satisfied.

In the next lemma, we show how the output of the system relates with the state, i.e., the time-varying rotation matrix.

Lemma 1: Assume that the vector observations ${}^{B}\mathbf{v}_i(k)$, $i = 1, \ldots, N_v$, at each time k satisfy ${}^{B}\mathbf{v}_i(k) \in \mathrm{Set}(\mathbf{M}_{v_i}(k), \mathbf{m}_{v_i}(k))$. Then, there exist a matrix $\mathbf{M}$ and a vector $\mathbf{m}$ such that

$$\mathbf{M}(k)\mathrm{vec}(\mathcal{R}(k)) \leq \mathbf{m}(k),$$

if and only if $\mathcal{R}(k)$ is compatible with the set of observations, $\mathrm{Set}(\mathbf{M}_{\mathbf{v}}(k), \mathbf{m}_{\mathbf{v}}k)$, where

$$\mathbf{M}_{\mathbf{v}}(k) = \begin{bmatrix} \mathbf{M}_{v_1}(k) & & 0 \\ & \ddots & \\ 0 & & \mathbf{M}_{v_{N_v}}(k) \end{bmatrix}, \quad \mathbf{m}_{\mathbf{v}}(k) = \begin{bmatrix} \mathbf{m}_{v_1}(k) \\ \vdots \\ \mathbf{m}_{v_{N_v}}(k) \end{bmatrix}.$$

Proof: The vector observations ${}^{B}\mathbf{v}_i(k)$, $i = 1, \ldots, N_v$, at each time k, satisfy ${}^{B}\mathbf{v}_i(k) \in \mathrm{Set}(\mathbf{M}_{v_i}(k), \mathbf{m}_{v_i}(k))$, i.e., $\mathrm{vec}({}^{B}\mathbf{V}(k))$ satisfy $\mathbf{M}_{\mathbf{v}}(k)\,\mathrm{vec}({}^{B}\mathbf{V}(k)) \leq \mathbf{m}_{\mathbf{v}}(k)$. On the other hand, it follows from (4) that $\mathrm{vec}({}^{B}\mathbf{V}(k)) = \mathbf{Q}(k)\mathrm{vec}(\mathcal{R}(k))$, where $\mathbf{Q}(k)$ is given by

$$\mathbf{Q}(k) = \begin{bmatrix} {}^{I}v_{11}\mathbf{I} & {}^{I}v_{21}\mathbf{I} & {}^{I}v_{31}\mathbf{I} \\ \vdots & \vdots & \vdots \\ {}^{I}v_{1N_v}\mathbf{I} & {}^{I}v_{2N_v}\mathbf{I} & {}^{I}v_{3N_v}\mathbf{I} \end{bmatrix}$$

and ${}^{I}v_{ij}$ is the element of line i and column j of matrix ${}^{I}\mathbf{V}(k)$. Hence, we have that

$$\underbrace{\mathbf{M}_{\mathbf{v}}(k)\mathbf{Q}(k)}_{\mathbf{M}(k)}\mathrm{vec}(\mathcal{R}(k)) \leq \underbrace{\mathbf{m}_{\mathbf{v}}(k)}_{\mathbf{m}(k)}.$$

∎

Remark 1: For the sake of comprehension, the problem of observability is not addressed in this paper. Thus, we consider that three or more non-coplanar vector observations are available, which guarantees observability [2], [7]. The case with vector observations that solely span a line or a plane are still under research.

We are now in conditions of stating the following theorem, which can be used to optimally obtain an SVO for the estimation of $\mathcal{R}(k)$.

Theorem 1: Assume that the angular velocity $\boldsymbol{\omega}(.)$ is constant between sampling times, that it is measured by rate gyros with no uncertainty, and that measurements ${}^{B}\mathbf{v}_i$, $i = 1, \ldots, N_v$ satisfying the conditions of Lemma 1 are available. Also assume that there exist a matrix $\mathbf{M}_s(k)$ and a vector $\mathbf{m}_s(k)$, such that the possible values of the attitude at time k are defined by all $\mathcal{R}(k)$ satisfying

$$\mathcal{R}(k) \in \mathrm{MatSet}(\mathbf{M}_s(k), \mathbf{m}_s(k)) \cap \mathrm{SO}(3).$$

Moreover, define $\mathbf{M}_s(k+1)$ and $\mathbf{m}_s(k+1)$ such that

$$\begin{aligned} &\mathrm{Set}(\mathbf{M}_s(k+1), \mathbf{m}_s(k+1)) = \\ &\quad \mathrm{Set}(\mathbf{M}(k+1), \mathbf{m}(k+1)) \cap \mathrm{Set}(\mathbf{M}_s(k)\boldsymbol{\Omega}^{-1}(k), \mathbf{m}_s(k)), \end{aligned}$$

where $\boldsymbol{\Omega}(k) = \mathbf{I}_{3\times 3} \otimes \exp(T\boldsymbol{\omega}(k))$ and where $\mathbf{M}(k+1)$ and $\mathbf{m}(k+1)$ satisfy $\mathcal{R}(k+1)$ compatible with the observations $\Leftrightarrow \mathrm{vec}(\mathcal{R}(k+1) \in \mathrm{Set}(\mathbf{M}(k+1), mmb(k+1)))$. Then,

$\mathcal{R}(k+1)$ satisfies (2) and is compatible with the current set of observations if and only if

$$\mathcal{R}(k+1) \in \mathrm{MatSet}(\mathbf{M}_s(k+1), \mathbf{m}_s(k+1)) \cap \mathrm{SO}(3)$$

Proof: If the rate gyros measure the angular velocity $\boldsymbol{\omega}(k)$ without uncertainty, then the predict equation (2) can be rewritten in the form

$$\mathbf{x}(k+1) = \boldsymbol{\Omega}(k)\mathbf{x}(k), \tag{5}$$

where $\mathbf{x}(k) = \mathrm{vec}(\mathcal{R}(k))$. By assumption, the set of states at time k is defined as

$$X(k) := \left\{\mathbf{x} \in \mathbb{R}^9 : \mathbf{M}_s(k)\mathbf{x} \leq \mathbf{m}_s(k), \mathrm{mat}(\mathbf{x}) \in \mathrm{SO}(3)\right\}. \tag{6}$$

From (5) and (6), we derive the following constraint in the state at time $k+1$

$$\begin{aligned}&\mathbf{x}(k+1) \in \\ &\left\{\mathbf{x} : \mathbf{M}_s(k)\boldsymbol{\Omega}^{-1}(k)\mathbf{x} \leq \mathbf{m}_s(k), \mathrm{mat}(\boldsymbol{\Omega}^{-1}(k)\mathbf{x}) \in \mathrm{SO}(3)\right\}\end{aligned} \tag{7}$$

By Lemma 1, and by resorting to the vector observations ${}^B\mathbf{v}_i$, $i = 1, \ldots, N_v$, at each time $k+1$, there exists $\mathbf{M}(k+1)$ and $\mathbf{m}(k+1)$ such that $\mathbf{x}(k+1)$ is compatible with the observations, if and only if,

$$\mathbf{M}(k+1)\mathbf{x}(k+1) \leq \mathbf{m}(k+1) \tag{8}$$

Enclosing the constraints (7) and (8), we can write

$$\underbrace{\begin{bmatrix}\mathbf{M}_s(k)\boldsymbol{\Omega}^{-1}(k)\\ \mathbf{M}(k+1)\end{bmatrix}}_{\mathbf{M}_s(k+1)}\mathbf{x}(k+1) \leq \underbrace{\begin{bmatrix}\mathbf{m}_s(k)\\ \mathbf{m}(k+1)\end{bmatrix}}_{\mathbf{m}_s(k+1)},$$
$$\mathrm{mat}(\boldsymbol{\Omega}^{-1}(k)\mathbf{x}(k)) \in \mathrm{SO}(3).$$

The condition $\mathrm{mat}(\boldsymbol{\Omega}^{-1}(k)\mathbf{x}(k)) \in \mathrm{SO}(3)$ can be replaced by $\mathbf{x}(k+1) \in \mathrm{SO}(3)$. To show this, we start by stating a property of the SO(3) manifold. For any matrix $\mathbf{A} \in \mathrm{SO}(3)$, $\mathbf{AB} \in \mathrm{SO}(3)$ if and only if $\mathbf{B} \in \mathrm{SO}(3)$. Since (5) is equivalent to $\mathrm{mat}(\mathbf{x}(k+1)) = \exp(T\boldsymbol{\omega}(k))\,\mathrm{mat}(\mathbf{x}(k))$, and $\exp(T\boldsymbol{\omega}(k)), \mathrm{mat}(\mathbf{x}(k+1)) \in \mathrm{SO}(3)$ we conclude that $\mathrm{mat}(\mathbf{x}(k)) \in \mathrm{SO}(3)$. Thus, we conclude that at each time $k+1$, $\mathrm{mat}(\mathbf{x}(k+1)) = \mathcal{R}(k+1)$ satisfies

$$\mathcal{R}(k+1) \in \mathrm{MatSet}(\mathbf{M}_s(k+1), \mathbf{m}_s(k+1)) \cap \mathrm{SO}(3),$$

if and only if (2) is satisfied and $\mathcal{R}(k+1)$ is compatible with the current set of observations. ∎

Remark 2: It should be noticed that this observer is optimal in the sense that, at each time k, it provides the smallest set that contains all the solutions to (2).

B. SVO with uncertainties in the vector observations and in the angular velocity measurements

In this section, we assume that the angular velocity measurements, $\boldsymbol{\omega}_r(k)$, are corrupted by bounded noise and are given by

$$\boldsymbol{\omega}_r(k) = \boldsymbol{\omega}(k) + \mathbf{n}(k),$$

where $\boldsymbol{\omega}(k) \in \mathbb{R}^3$ is the true angular rate at time k, and $\mathbf{n}(k) \in \mathbb{R}^3$ is a vector with the noise components such that for all $k \geq 0$, there exists $\bar{n} \in \mathbb{R}^+$, with $\|\mathbf{n}(k)\|_\infty \leq \bar{n}$. We also assume that the angular velocity is bounded, i.e., for all $k \geq 0$, there exists $\bar{\omega} \in \mathbb{R}^+$, such that $\|\boldsymbol{\omega}(k)\|_\infty \leq \bar{\omega}$. The kinematics of the attitude is given by

$$\mathcal{R}(k+1) = \exp\left(T\left(\boldsymbol{\omega}_r(k) - \mathbf{n}(k)\right)\right)\mathcal{R}(k),$$

which can be rewritten as

$$\mathbf{x}(k+1) = \boldsymbol{\Omega}(k)\mathbf{x}(k).$$

where $\mathbf{x}(k) = \mathrm{vec}(\mathcal{R}(k))$ and $\boldsymbol{\Omega}(k) = \mathbf{I}_{3\times 3} \otimes \exp\left(T(\boldsymbol{\omega}_r(k) - \mathbf{n}(k))\right)$.

Theorem 2: Assume that the angular velocity, $\boldsymbol{\omega}(.)$, is constant between sampling times and that tri-axial rate gyros provide angular velocity measurements corrupted by noise bounded by $\bar{n}$, and that measurements ${}^B\mathbf{v}_i$, $i = 1, \ldots, N_v$ under the conditions of Lemma 1 are available. Also assume that there exist a matrix $\mathbf{M}_s(k)$ and a vector $\mathbf{m}_s(k)$, such that

$$\mathcal{R}(k) \in \mathrm{MatSet}(\mathbf{M}_s(k), \mathbf{m}_s(k)) \cap \mathrm{SO}(3).$$

Let $\mathbf{M}_s(k+1)$ and $\mathbf{m}_s(k+1)$ be defined as in the proof of this theorem, then

i) $\mathcal{R}(k+1)$ satisfy (2), and

ii) $\mathcal{R}(k+1) \in \mathrm{MatSet}(\mathbf{M}_s(k+1), \mathbf{m}_s(k+1)) \cap \mathrm{SO}(3)$, if $\mathcal{R}(k+1)$ is compatible with the current observations.

Proof: By assumption, the set of states at time k is defined as

$$X(k) := \left\{\mathbf{x} \in \mathbb{R}^9 : \mathbf{M}_s(k)\mathbf{x} \leq \mathbf{m}_s(k), \mathrm{mat}(\mathbf{x}) \in \mathrm{SO}(3)\right\}.$$

Also, define $X(k+1)$ as the set of all possible states of the system at time $k+1$. By the definition of exponential map we have

$$\exp(T(\boldsymbol{\omega}_r - \mathbf{n})) = \mathbf{I} + [T\boldsymbol{\omega}_r]_\times - [T\mathbf{n}]_\times + \sum_{k=2}^{\infty}\frac{[T\boldsymbol{\omega}]_\times^k}{k!}.$$

The dynamics of the system can be written in the form

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \sum_{i=1}^{9}\mathbf{A}_i(k)\Delta_i(k)\mathbf{x}(k), \tag{9}$$

for some $\Delta_i(k)$, where $|\Delta_i(k)| \leq 1$, $i = 1, \ldots, 9$, and

$$\mathbf{A}(k) = \mathbf{I}_{3\times 3} \otimes (\mathbf{I} + T[\boldsymbol{\omega}_r]_\times)$$
$$\mathbf{A}_i(k) = \begin{cases}\mathbf{I}_{3\times 3} \otimes \gamma_1\mathbf{E}_{i,j} & \text{if } m = n;\\ \mathbf{I}_{3\times 3} \otimes \gamma_2\mathbf{E}_{i,j} & \text{otherwise,}\end{cases}$$

where $m, n = 1, 2, 3$, $i = 3(n-1) + m$, and $\gamma_1 = \frac{1}{2}\left(\exp(2T\bar{\omega}) - 1\right) - T\bar{\omega}$, $\gamma_2 = \gamma_1 + \bar{n}$, and the following relation was used

$$\|[\mathbf{a}]_\times^k\|_{\max} \leq \frac{(2\bar{a})^k}{2}, \tag{10}$$

for $\mathbf{a} \in \mathbb{R}^3$, $\bar{a} = \|\mathbf{a}\|_\infty$, and $k \in \mathbb{N}$.

Due to the noise present in the angular velocity measurements, the set of feasible states is not convex and hence cannot be described by a polytope. Despite of this, we will see next that, considering specific realizations of (9) and using SVOs to obtain the polytope that contains the state for each particular realization, we can derive a set that contains the true state $\mathbf{x}(k+1)$.

Consider a realization of (9) where $\Delta_i(k) = \Delta_i^*$, $|\Delta_i^*| \leq 1$, $i = 1, \ldots, 9$ and denote by $\mathbf{A}_{\Delta^*}$ the corresponding uncertainty map, i.e., $\mathbf{A}_{\Delta^*} = \mathbf{A}_1^*\Delta_1^* + \cdots + \mathbf{A}_9^*\Delta_9^*$. For each $\mathbf{A}_{\Delta^*}$, the technique in [16] can be used to design an SVO which computes a set-valued estimate of the state of the system. Indeed, if the matrix $\mathbf{A}(k) + \mathbf{A}_{\Delta^*}$ is non-singular, we can write the following inequality as a constraint for the state $\mathbf{x}(k+1)$

$$\underbrace{\begin{bmatrix}\mathbf{M}_s(k)(\mathbf{A}(k)+\mathbf{A}_{\Delta^*})^{-1}\\ \mathbf{M}(k+1)\end{bmatrix}}_{\mathbf{M}_*(k+1)}\mathbf{x}(k+1) \leq \underbrace{\begin{bmatrix}\mathbf{m}_s(k)\\ \mathbf{m}(k+1)\end{bmatrix}}_{\mathbf{m}_*(k+1)}. \tag{11}$$

If, however, $\mathbf{A}(k)+\mathbf{A}_{\Delta^*}$ is singular or ill-conditioned one can write the inequality

$$\underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{A}(k)-\mathbf{A}_{\Delta^*}(k) \\ -\mathbf{I} & \mathbf{A}(k)+\mathbf{A}_{\Delta^*}(k) \\ \mathbf{M}(k+1) & 0 \\ 0 & \mathbf{M}_s(k) \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{x}(k) \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbf{m}(k+1) \\ \mathbf{m}_s(k) \end{bmatrix}}_{\mathbf{q}}, \quad (12)$$

and then use the Fourier-Motzkin projection [6] to compute $\mathbf{M}_*(k+1)$ and $\mathbf{m}_*(k+1)$ such that $\mathbf{M}_*(k+1)\mathbf{x}(k+1) \leq \mathbf{m}_*(k+1)$, i.e.,

$$(\mathbf{M}_*(k+1), \mathbf{m}_*(k+1)) = \mathrm{FM}(\mathbf{P}, \mathbf{q}, 3).$$

Let v_i, $i = 1, \ldots, 2^9$ denote a vertex of the hypercube $H := \{\delta \in \mathbb{R}^9 : |\delta| \leq 1\}$, where $v_i = v_j \Leftrightarrow i = j$. Then, we denote by $\hat{X}_{v_i}(k+1)$ the set of points $\mathbf{x}(k+1)$ that satisfy (11) (or (12)) where $\mathbf{A}_{\Delta^*} = \mathbf{A}_{v_i}$ and with $\mathbf{x}(k) \in \hat{X}(k)$. Further define $\hat{X}(k+1) := \mathrm{co}\left\{\hat{X}_{v_1}(k+1), \ldots, \hat{X}_{v_{2^9}}(k+1)\right\}$, where $\mathrm{co}\{p_1, \ldots, p_m\}$ is the smallest convex set containing the points $p_1, \ldots, p_m$, also known as convex hull of $p_1, \ldots, p_m$. Since $X(k+1)$ is, in general, non-convex even if $X(k)$ is convex, we are going to use $\hat{X}(k+1)$ to overbound the set $X(k+1)$.

Since $\hat{X}(k+1)$ is the convex hull of a finite number of polytopes, it can be written in the form $\mathrm{Set}(\mathbf{M}_s(k+1), \mathbf{m}_s(k+1))$. Then, an overbound on the space of possible solutions of $\mathcal{R}$ at time $k+1$ is given by the intersection of the two sets $\mathcal{R}(k+1) \in \mathrm{MatSet}(\mathbf{M}_s(k+1), \mathbf{m}_s(k+1)) \cap \mathcal{R}(k+1) \in \mathrm{SO}(3)$. ■

Either using the inversion of matrices in (11) or using the Fourier-Motzkin algorithm in (12), the sizes of $\mathbf{M}_s(k)$ and $\mathbf{m}_s(k)$ may be increasing very fast with time, which can be problematic. To overcome this issue, one should eliminate the linearly dependent constraints. This can be done by solving several small linear programming problems at each sampling time, which, however, increases the complexity of the practical implementation of this type of observers.

IV. Implementation Issues

The representation of the uncertainty in $\mathcal{R}(k)$ by means of a set may not be suitable in practice. Therefore, this section is devoted to the problem of describing the uncertainty in attitude of the rigid body through: $i)$ a rotation vector; and $ii)$ Euler angles.

A. Representation in terms of a rotation vector

We start by showing how to describe the uncertainty in the attitude of the rigid body by means of a rotation vector, where a vector $\boldsymbol{\lambda} \in \mathbb{R}^3$ that parameterizes a rotation matrix $\mathcal{R}$ satisfies $\mathcal{R} = \exp([\boldsymbol{\lambda}]_\times)$, $||\boldsymbol{\lambda}|| \leq \pi$. The attitude SVOs, introduced in Section III, provide a polytope description in $\mathbb{R}^9$ that contains the true state, i.e., $\mathbf{M}_s(k)$ and $\mathbf{m}_s(k)$ such that $\mathbf{M}_s(k)\,\mathrm{vec}(\mathcal{R}(k)) \leq \mathbf{m}_s(k)$. We can use this result to derive an inequality for the rotation vector parameterization of the attitude such that $\mathbf{M}_\lambda(k)\boldsymbol{\lambda}(k) \leq \mathbf{m}_\lambda(k)$. Since $\mathcal{R} = \exp([\boldsymbol{\lambda}]_\times)$, we can write $\mathbf{M}_s\,\mathrm{vec}(\exp([\boldsymbol{\lambda}]_\times)) \leq \mathbf{m}_s$, where k was omitted for simplicity of the notation. By the definition of exponential map we obtain

$$\mathbf{M}_s\,\mathrm{vec}\left(\mathbf{I} + [\boldsymbol{\lambda}]_\times + \sum_{k=2}^{\infty} \frac{[\boldsymbol{\lambda}]_\times^k}{k!}\right) \leq \mathbf{m}_s.$$

Moving the nonlinear terms to the right-hand side of the inequality, we obtain $\mathbf{M}_s\,\mathrm{vec}([\boldsymbol{\lambda}]_\times) \leq \mathbf{m}_s - \mathbf{M}_s\,\mathrm{vec}(\mathbf{I}) - \mathbf{M}_s\,\mathrm{vec}\left(\sum_{k=2}^{\infty} \frac{[\boldsymbol{\lambda}]_\times^k}{k}\right)$. Noting that $\exists_{\mathbf{V}} : \mathrm{vec}([\boldsymbol{\lambda}]_\times) = \mathbf{V}\boldsymbol{\lambda}$, we conclude

$$\mathbf{M}_\lambda \boldsymbol{\lambda} \leq \mathbf{m}_\lambda,$$

where $\mathbf{M}_\lambda = \mathbf{M}_s\mathbf{V}$, and $\mathbf{m}_\lambda = \mathbf{m}_s - \mathbf{M}_s\,\mathrm{vec}(\mathbf{I}) + \mathbf{M}_s\,\mathrm{vec}(\mu \mathbf{1}_{3\times3})$ where, resorting to (10), we can set $\mu = \frac{1}{2}(\exp(2||\boldsymbol{\lambda}||_\infty) - 1 - 2||\boldsymbol{\lambda}||_\infty)$.

B. Representation in terms of Euler angles

A different and well-known alternative to the one presented in the last section is the Euler angles representation, which is defined by the angles that each axis of the rigid body reference frame has to rotate to match the axis of the inertial reference frame. These angles are named yaw (ψ), pitch (θ), and roll (ϕ), and correspond to the rotation around the z-axis, the y-axis and the x-axis, respectively. For the ZYX-Euler angles, the rotation matrix $\mathcal{R} \in \mathrm{SO}(3)$ can be computed using

$$\mathcal{R} = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & s_\psi s_\phi + c_\psi s_\theta c_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix},$$

where $c_\psi := \cos(\psi)$, $s_\psi := \sin(\psi)$, $c_\theta := \cos(\theta)$, $s_\theta := \sin(\theta)$, $c_\phi := \cos(\phi)$ and $s_\phi := \sin(\phi)$.

We are interested in computing the maximum and minimum values of each Euler angle along time. This can be cast into a nonlinear optimization problem and solved using the techniques presented in [4], [10]. For instance, the minimum value of the roll angle can be obtained by solving the following problem

$$\min \phi(k), \qquad \text{s.t. } \mathbf{M}_s(k)\,\mathrm{vec}(\mathcal{R}(k)) - \mathbf{m}_s(k) \leq 0.$$

As an applicability example, this method will be used in the following section.

V. Simulation Results

In this section, we present simulation results that illustrate the feasibility of the proposed solution. The simulated trajectory is characterized by an angular velocity with the following oscillatory profile

$$\begin{aligned} \omega_x(k) &= 0.07\sin(2\pi 0.05kT) \\ \omega_y(k) &= -0.05\sin(2\pi 0.04kT) \\ \omega_z(k) &= 0.06\sin(2\pi 0.02kT) \\ \boldsymbol{\omega}(k) &= \alpha[\omega_x(k)\ \omega_y(k)\ \omega_z(k)]^T, \end{aligned}$$

where α is a scaling factor. The directions for the sensed vectors in the inertial reference frame are given by ${}^I\mathbf{v}_1 = [1\ 0\ 0]^T$, ${}^I\mathbf{v}_2 = [0\ 1\ 0]^T$, ${}^I\mathbf{v}_3 = [0\ 0\ 1]^T$. These vectors observations are measured in the body reference frame and each component is corrupted by uniform noise with absolute value bounded by 0.1.

A. Noise-free angular rate measurements

In the first set of simulations a high-end triad of rate gyros measure the exact angular velocity. In order to assess the performance of the proposed solution, we carry out simulations under different operating conditions, namely, different sampling periods, T, and magnitude of the angular velocity, which is controlled by the scaling factor α.

In Table I, we present the RMS (root-mean-square) error of the yaw, pitch, and roll angles of a typical Monte-Carlo run. The upper and lower bounds on these angles are computed according to the method in Section IV-B. In the error computation, we considered the mean of the upper

TABLE I

RMS ERROR OF THE PROPOSED SOLUTION – SIMULATION WITH NOISE-FREE ANGULAR VELOCITY MEASUREMENTS.

T (s)	α	yaw (deg)	pitch (deg)	roll (deg)
0.1	1	0.0375	0.0228	0.0329
0.01	1	0.0118	0.0126	0.0125
0.1	2	0.1112	0.0657	0.1225
0.1	0.1	0.0044	0.0052	0.0043

TABLE II

RMS ERROR OF THE OBSERVER IN [20] – SIMULATION WITH NOISE-FREE ANGULAR VELOCITY MEASUREMENTS.

T (s)	α	yaw (deg)	pitch (deg)	roll (deg)
0.1	1	0.2380	0.1992	0.2714
0.01	1	0.0297	0.0302	0.0499
0.1	2	0.4398	0.3611	0.5387
0.1	0.1	0.0289	0.0319	0.0300

TABLE III

RMS ERROR OF THE WAHBA'S SOLUTION [21].

T (s)	α	yaw (deg)	pitch (deg)	roll (deg)
0.1	1	3.9804	3.1825	3.0327
0.01	1	3.9966	3.1726	3.0268
0.1	2	4.5379	4.1532	3.6878
0.1	0.1	2.2964	2.3912	2.3671

and lower bounds to be the nominal estimate. The results were compared with two solutions presented in the literature, namely, the deterministic nonlinear observer derived in [20] (Table II), and the Wahba's solution [21] (Table III), which is a classical algebraic approach that relies solely on the vector observations.

The nonlinear observer in [20] has a better performance than the Wahba's solution. This fact is not surprising since the former makes use of more information than the latter. Nevertheless, it is worth noting that the accuracy of the estimates are improved by a factor of ten. The proposed solution outperforms the other two approaches presented in the literature. Moreover, it provides upper and lower bounds on the attitude, whereas the other approaches do not.

Figure 1 depicts the Euler angles representation of the true state and the upper and lower bounds on the attitude estimates for the simulation with parameters $T = 0.1$, $\alpha = 1$. We see that the proposed solution provides tight bounds on the attitude of the rigid body. In Figs. 2(a) and 2(b), the estimation error is illustrated, showing that, after the initial transient, the estimation error of each angle is always below 0.1 deg.

B. Angular rate measurements with noise

In the second set of simulations, the rigid body is equipped with low-end rate gyros which measure the angular velocity with additive noise with uniform distribution between -0.573 deg s^{-1} and 0.573 deg s^{-1}. The simulations were performed under the same operating conditions that were used for the simulations with noise-free angular rate measurements.

The RMS errors of the Euler angles of a typical Monte-Carlo run, computed according to the method in Section IV-B, for the proposed SVO and for the nonlinear observer presented in [20], are shown in Table IV and Table V, respectively. In contrast to the first set of simulations, the nonlinear observer in [20] performs better than the proposed solution, when there is noise in the angular velocity. This

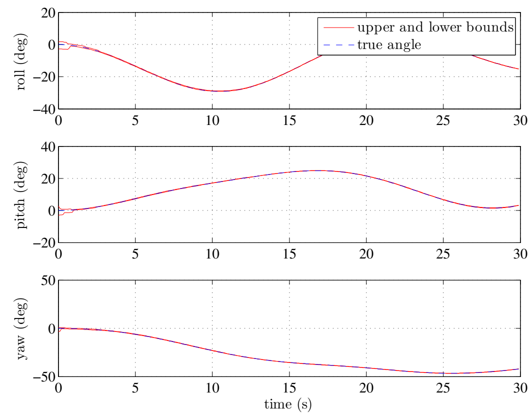
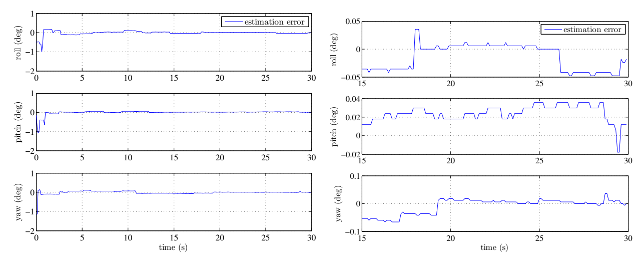


Fig. 1. Euler angles representation of the true state and the upper and lower bounds on the attitude estimates - simulation with noise-free angular velocity measurements.



(a) (b)

Fig. 2. Estimation error - simulation with noise-free angular velocity measurements.

indicates that the mean of the lower and upper bounds is not the most likely attitude inside the set. Regardless of this fact, the proposed solution has the advantage of providing explicit bounds on the attitude of the rigid body.

Figure 3 depicts the Euler angles representation of the true state and the upper and lower bounds on the attitude estimates for the simulation with parameters $T = 0.1$, $\alpha = 1$. We see that the proposed solution provides tight bounds on the attitude of the rigid body. In Fig. 4 the estimation error is illustrated, showing that it is always below 1 deg.

The simulations with different operating conditions evidence that, as could be anticipated, for the nonlinear ob-

TABLE IV

RMS ERROR OF THE PROPOSED SOLUTION – SIMULATION WITH NOISY ANGULAR VELOCITY MEASUREMENTS.

T (s)	α	yaw (deg)	pitch (deg)	roll (deg)
0.1	1	0.3551	0.3457	0.2892
0.01	1	0.1057	0.1041	0.0877
0.1	2	0.3929	0.3775	0.3524
0.1	0.1	0.2561	0.2683	0.2667

TABLE V

RMS ERROR OF THE OBSERVER IN [20] - SIMULATION WITH NOISY ANGULAR VELOCITY MEASUREMENTS.

T (s)	α	yaw (deg)	pitch (deg)	roll (deg)
0.1	1	0.2169	0.2542	0.3418
0.01	1	0.0760	0.0611	0.0811
0.1	2	0.3657	0.4184	0.5955
0.1	0.1	0.1518	0.1476	0.1466

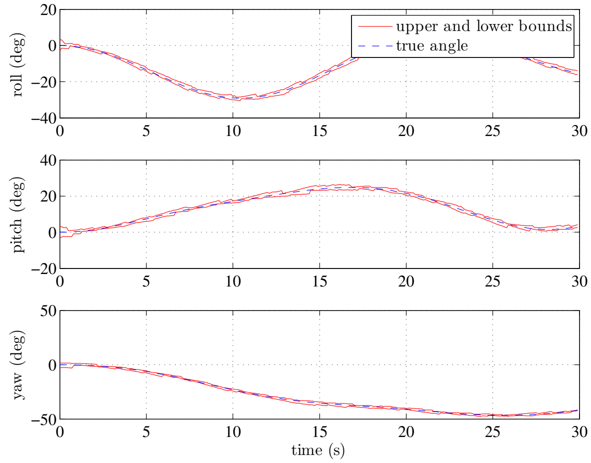


Fig. 3. Euler angles representation of the true state and the upper and lower bounds on the attitude estimates - simulation with noisy angular velocity measurements.

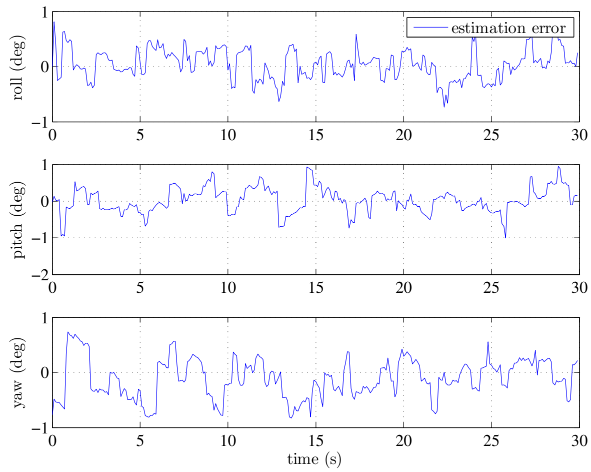


Fig. 4. Estimation error - simulation with noisy angular velocity measurements.

server and the SVOs smaller sampling periods yield better estimates, whereas the Whaba's solution do not improve its accuracy by reducing the sampling period. We also conclude that, greater angular velocity leads to larger estimation errors, and that in the presence of noisy angular velocity measurements, the estimate obtained using the mean of the upper and lower bounds produces larger errors, and that the difference between these bounds is also larger, when compared to the results obtained for the case of noise-free gyroscopes. Hence, as expected, the uncertainty in the angular velocity measurements results in uncertainty in the estimates of the attitude.

VI. CONCLUSIONS

This work addresses the problem of attitude estimation of a rigid body assuming that the sensor measurements have uncertainties characterized by a polytope. The proposed solution is based on set-valued observers and relies on rate gyros and vector observations. The observer has no singularities since the attitude is given by the rotation matrix and is global in the sense that is valid for any initial conditions. If there is measurement noise only on the vector observations, the uncertainty on the estimate is guaranteed to be the smallest possible. If uncertainty is also present in the rate gyros, the nonlinearities of the plant are tackled by adding conservatism. Nevertheless, the simulation results indicate that the solution is implementable, while yielding levels of performance of the order of magnitude of the alternatives in the literature and providing set-valued estimates for the state of the system.

Further efforts are being deployed to reduce the estimation errors when there is bias in the measurements provided by the rate gyros.

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