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A Faà di Bruno Hopf Algebra for a Group of Fliess Operators with Applications to Feedback

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***Abstract*— A Faà di Bruno type Hopf algebra is developed for a group of integral operators known as Fliess operators, where operator composition is the group product. The result is applied to analytic nonlinear feedback systems to produce an explicit formula for the feedback product, that is, the generating series for the Fliess operator representation of the closed-loop system written in terms of the generating series of the Fliess operator component systems. This formula is employed to provide a proof that local convergence is preserved under feedback.**

I. Introduction

Let f and g be two functions with convergent Taylor series about $x = 0$ which leave the origin invariant, say $f(x) = \sum_{n\geq 1} f_n x^n/n!$ and $g(x) = \sum_{n\geq 1} g_n x^n/n!$. The composition $h = f \circ g$ has the same nature as f and g, and the well known Faà di Bruno formula provides its Taylor series coefficients, specifically,

$$h_n = \sum_{k=1}^{n} \frac{f_k}{k!} \sum_{j} \frac{n!k!}{j_1!j_2!\cdots j_n!} \frac{g_1^{j_1} g_2^{j_2} \cdots g_n^{j_n}}{(1!)^{j_1}(2!)^{j_2}\cdots(n!)^{j_n}}, \quad (1)$$

where the second sum is over all $j_1, j_2, \ldots, j_n \geq 0$ such that $j_1 + j_2 + \cdots + j_n = k$. In the event that the series are not convergent, the functions involved can be interpreted as formal functions rather than as analytic functions. In either case, if $f_1 \neq 0$ then f has a compositional inverse, f^{-1}, and therefore, the corresponding set of functions forms a group under composition. In the special case where $f_1 = 1$, the coordinate functions $a_n : f \mapsto f_n$, $n \geq 1$ on the corresponding subgroup form a graded connected Hopf algebra, a so called Faà di Bruno Hopf algebra [2], [3], [6], [20]. The antipode of this Hopf algebra acts on each coordinate function to produce a polynomial expression for the coordinates of the compositional inverse. It turns out that this algebra has great utility in quantum field theory and related areas [6].

In this paper, an analogous Faà di Bruno Hopf algebra is developed for a group of integral operators known as Fliess operators. Such an operator, F_c, is normally written in terms of a generating series c over a noncommutative alphabet $X = \{x_0, x_1, \ldots, x_m\}$ [7], [8]. It was shown in [19] that a noncommutative version of (1) describes the input-output map $F_c : u \mapsto y$ when u is described by a Taylor series (in one variable). In contrast, the focus here is on system interconnections. First, it is shown that the set of operators

$$\mathscr{F}_\delta := \{I + F_c \ : \ c \in \mathbb{R}\langle\langle X\rangle\rangle\},$$

where I denotes the identity operator, and $\mathbb{R}\langle\langle X\rangle\rangle$ is the set of all formal power series over X, forms a group under operator composition when $m = 1$. It is worth noting that the elements of $\mathscr{F}_\delta$ bear some resemblance to the group of diffeomorphisms on $\mathbb{R}$ having the form $f(x) = x + O(x^2)$, as well as to the noncommutative compositional groups that appear in [3], [9]. In the latter case, however, composition refers to the direct composition of power series, a notion which is entirely distinct from the composition product used here to describe Fliess operator composition [4], [5], [12], [21], [22]. Furthermore, an element like $I + F_c$ is *not*, strictly speaking, a Fliess operator since I has no integral representation. Nevertheless, tools already exist for handling this modest generalization of the Fliess operator concept in the context of operator composition since $\mathscr{F}_\delta$ naturally arises in the study of analytic nonlinear feedback systems [12], [21]. Next, a graded Faà di Bruno bialgebra is systematically constructed for the coordinate functions of $\mathscr{F}_\delta$. Since the generating series are completely arbitrary, F_c may only be a formal Fliess operator and not necessarily convergent in any sense [7], [15]. It will be shown subsequently that convergent operators form a subgroup of $\mathscr{F}_\delta$. Next, the existence of an antipode is addressed. It is shown that while the bialgebra under consideration is *not* connected, a well defined antipode *does* exist so as to render a graded Faà di Bruno Hopf algebra. This class of combinatorial Hopf algebras is quite distinct from those normally associated with the Cauchy product and shuffle product [10], [16]–[18], [24], which for the most part involve a finite alphabet. Finally, it is shown that the subgroup of operators having proper generating series, i.e., generating series with a zero constant term, leads to a connected Faà di Bruno Hopf subalgebra. This structure is most similar to the one in the classical case. As an application, it is demonstrated that the antipode formula naturally appears in the context of feedback theory for Fliess operators. Specifically, it was shown via a fixed point argument in [12], [15] that any feedback connection involving two Fliess operators F_c and F_d always produces a closed-loop system with a Fliess operator representation. The fixed point, represented by the generating series $c@d$, defines a formal series product of c and d referred to as the *feedback product*. Such an approach, however, does not provide an explicit formula for computing this product. It will be shown here that a suitable formula can be derived in terms of the Faà di Bruno Hopf algebra antipode associated with $\mathscr{F}_\delta$. Aside from the obvious computational benefits, it will be used to provide a proof of the fact that feedback preserves local convergence. This result was recently proved in [13], [26] for the special case of unity feedback systems, that is, when F_d is replaced with I in the feedback path. Here the general case is addressed. Finally, it should be noted that the journal version of this paper is available as [11]. So a majority of the proofs are suppressed here for brevity.

The paper is organized as follows. In the next section, a brief overview is given of Fliess operator theory. Similarly,

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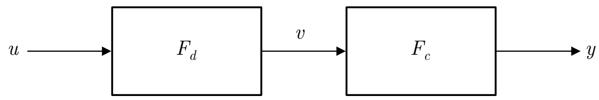


Fig. 1. Cascade connection of two Fliess operators.

the essential elements of Hopf algebra theory employed in the paper are summarized. In Section III, the Faà di Bruno Hopf algebra of interest is constructed. Its application to feedback systems is described in the subsequent section. The main conclusions are summarized in the final section.

II. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is written as X^*. Clearly X^* forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η). Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\, \eta$. A series c is called *proper* when $(c, \emptyset) = 0$. For any language $L \subseteq X^*$, its *characteristic series* is defined as $\mathrm{char}(L) = \sum_{\eta \in L} \eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$, while the set of polynomials over X is designated by $\mathbb{R}\langle X\rangle$. Each set forms an associative $\mathbb{R}$-algebra under the catenation (Cauchy) product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, denoted here by $⧢$ [7].

A. Fliess Operators and Their Interconnections

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Define iteratively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0)$$

[7], [8]. If there exists real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \eta \in X^*,$$

then F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0,\, t_0 + T]$ into $B^\ell_\mathfrak{q}(S)[t_0,\, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [14]. The set of all such *locally convergent* series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$.

When F_c and F_d with $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion as shown in Fig. 1, the composite system $u \mapsto y$ always has a Fliess operator representation, and the composition product can be used to describe its generating series. It is convenient to first define a family of mappings

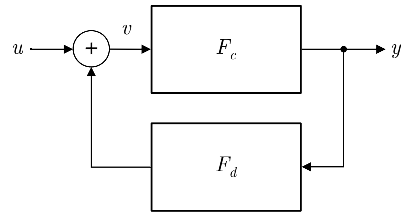


Fig. 2. Feedback connection of two Fliess operators.

$$D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i ⧢ e),$$

where $i = 0, 1, \ldots, m$ and $d_0 := 1$. Let $D_\emptyset$ be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Such maps can be composed in an obvious way so that $D_{x_i x_j} := D_{x_i} D_{x_j}$ provides an $\mathbb{R}$-algebra which is isomorphic to the usual $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ under the catenation product.

Definition 1: [4], [5], [12] The **composition product** of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as

$$(\underbrace{x_{i_k} x_{i_{k-1}} \cdots x_{i_1}}_{\eta}) \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ define

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, D_\eta(1).$$

The composition product is associative, distributes to the left over the shuffle product, and has the key property that $F_c \circ F_d = F_{c \circ d}$ [4], [5]. In addition, the composition product preserves local convergence [12], and the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$ in an ultrametric sense [4], [12].

In the event that two Fliess operators are interconnected to form a feedback system as shown in Fig. 2, the output y must satisfy the feedback equation

$$y = F_c[v] = F_c[u + F_d[y]]$$

for every admissible input u. It was shown in [12], [15] that there always exists a unique generating series e so that $y = F_e[u]$. In which case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d \circ e}[u]] = F_{c \tilde{\circ} (d \circ e)}[u],$$

where $\tilde{\circ}$ denotes the *modified* composition product. That is, the product

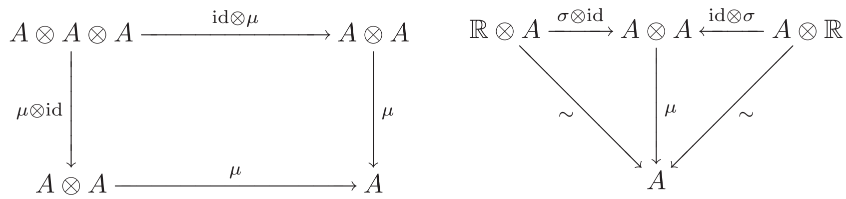
$$c \tilde{\circ} d = \sum_{\eta \in X^*} (c, \eta)\, \tilde{D}_\eta(1),$$

where

$$\tilde{D}_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_i e + x_0(d_i ⧢ e)$$

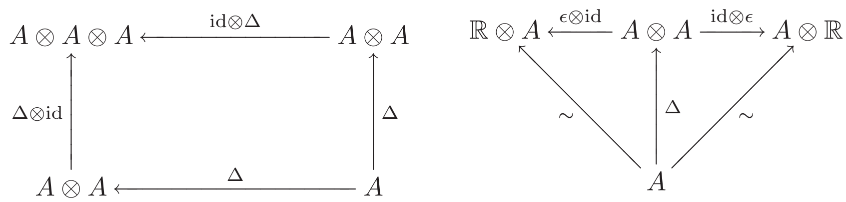
with $d_0 := 0$. It was shown in [12], [21] that the modified composition product preserves local convergence and that the mapping $d \mapsto c \,\tilde{\circ}\, d$ is also an ultrametric contraction. The *feedback product* of c and d, namely $c@d$, is defined as the unique fixed point of the contractive iterated map

$$\tilde{S} : e_i \mapsto e_{i+1} = c \tilde{\circ} (d \circ e_i).$$



(a) Associative property (b) Unitary property

Fig. 3. Defining properties of an $\mathbb{R}$-algebra (A, μ, σ).



(a) Coassociative property (b) Counitary property

Fig. 4. Defining properties of an $\mathbb{R}$-coalgebra (A, Δ, ϵ).

Specifically, $c@d = e$, where $e = c\tilde{\circ}(d \circ e)$. In the case of a unity feedback system, this equation reduces to $e = c\tilde{\circ}e$. Given arbitrary c and d, there is no general method for computing $c@d$ explicitly.

B. Hopf Algebra Fundamentals

The basic elements of Hopf algebra theory used in the paper are now summarized. The treatment is based on [1], [2], [6], [25]. The starting point is a systematic statement of what it means for a set A to be an associative $\mathbb{R}$-algebra. Let A be an $\mathbb{R}$-vector space and consider an $\mathbb{R}$-bilinear map and an $\mathbb{R}$-linear map,

$$\mu : A \otimes A \to A, \quad \sigma : \mathbb{R} \to A,$$

respectively, which satisfy the associative property and unitary property as described by the commutative diagrams in Fig. 3. Here id denotes the identity map on A, and the symbol $\sim$ denotes the canonical isomorphism between the vector spaces A and $A \otimes \mathbb{R}$. These diagrams are equivalent to, respectively, the identities

$$(ab)c = a(bc), \quad a, b, c \in A$$
$$1_A\, a = a = a1_A, \quad a \in A,$$

where $\mu(a \otimes b) = ab$ and $\sigma(1) = 1_A$ is the unit of A. Traditionally, μ is called the *multiplication map*, and σ is called the *unit map*. The triple (A, μ, σ) is an associative $\mathbb{R}$-algebra. Next suppose there exist two $\mathbb{R}$-linear maps

$$\Delta : A \to A \otimes A, \quad \epsilon : A \to \mathbb{R},$$

which satisfy the coassociative property and the counitary property as illustrated in Fig. 4. These commutative diagrams are the same as the ones depicted in Fig. 3 except that the directions of the arrows have been reversed. In this case, Δ is called the *comultiplication map*, and ϵ is the *counit map*. The triple (A, Δ, ϵ) is called an $\mathbb{R}$*-coalgebra*. In this setting, consider the following definition.

Definition 2: A **morphism** between two $\mathbb{R}$-algebras (A_1, μ_1, σ_1) and (A_2, μ_2, σ_2) is any $\mathbb{R}$-linear map $\psi : A_1 \to A_2$ such that

$$\psi \circ \mu_1 = \mu_2 \circ (\psi \otimes \psi)$$
$$\psi \circ \sigma_1 = \sigma_2.$$

An analogous definition can be given for a morphism between two $\mathbb{R}$-coalgebras. Using either concept, one can produce the notion of a bialgebra as described next.

Definition 3: The five-tuple $(A, \mu, \sigma, \Delta, \epsilon)$ is called an **$\mathbb{R}$-bialgebra** when Δ and ϵ are both $\mathbb{R}$-algebra morphisms.

Specifically this means that the mapping $\Delta : A \to A \otimes A$ must be an $\mathbb{R}$-algebra morphism between the $\mathbb{R}$-algebras (A, μ, σ) and $(A \otimes A, \mu_{A\otimes A}, \sigma_{A\otimes A})$, where

$$\begin{aligned}\mu_{A\otimes A} &: (A \otimes A) \otimes (A \otimes A) \to A \otimes A\\ &: (a_1 \otimes a_2) \otimes (a_3 \otimes a_4) \mapsto \mu(a_1 \otimes a_3) \otimes \mu(a_2 \otimes a_4)\\ \sigma_{A\otimes A} &: \mathbb{R} \to A \otimes A\\ &: k \mapsto \sigma(k) \otimes 1_A.\end{aligned}$$

In which case, it follows directly that

1. $\Delta\circ\mu = \mu_{A\otimes A}\circ(\Delta\otimes\Delta) = (\mu\otimes\mu)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\Delta)$
2. $\Delta \circ \sigma = \sigma_{A\otimes A} = \sigma \otimes \sigma$,

where $\tau : A \otimes A \to A \otimes A : a \otimes a' \mapsto a' \otimes a$. Similarly, $\epsilon : A \to \mathbb{R}$ must be an $\mathbb{R}$-algebra morphism between the $\mathbb{R}$-algebras (A, μ, σ) and $(\mathbb{R}, \mu_{\mathbb{R}}, \sigma_{\mathbb{R}})$. Therefore,

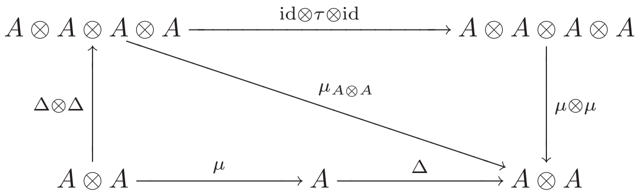
3. $\epsilon \circ \mu = \mu_{\mathbb{R}} \circ (\epsilon \otimes \epsilon) = \epsilon^2$
4. $\epsilon \circ \sigma = \sigma_{\mathbb{R}} = 1$.

Note that properties 1 and 2 can be expressed in terms of the commutative diagrams shown in Fig. 5, and, likewise, properties 3 and 4 are shown in Fig. 6. If instead one introduces the notion of a $\mathbb{R}$-coalgebra morphism as suggested above, then an equivalent characterization of a bialgebra is one where μ and σ are both $\mathbb{R}$-coalgebra morphisms, yielding properties 1 and 3, and properties 2 and 4, respectively.

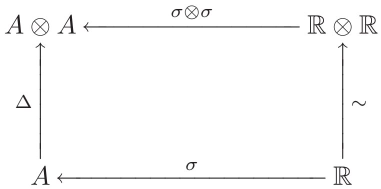
To complete the development of the Hopf algebra definition, consider the set of all $\mathbb{R}$-endomorphisms on A, denoted by $\mathrm{End}(A)$. Given two arbitrary $f, g \in \mathrm{End}(A)$, the *Hopf convolution product*,

$$f * g := \mu \circ (f \otimes g) \circ \Delta,$$

defines another element of $\mathrm{End}(A)$. The following theorem is central to the theory.

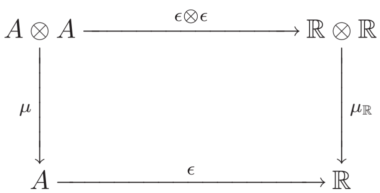


(a) Property 1

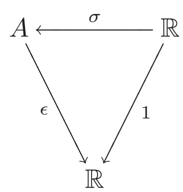


(b) Property 2

Fig. 5. Commutative diagrams describing Δ as an $\mathbb{R}$-algebra morphism.



(c) Property 3



(d) Property 4

Fig. 6. Commutative diagrams describing ϵ as an $\mathbb{R}$-algebra morphism.

Theorem 1: The triple $(\mathrm{End}(A), *, \vartheta)$ forms an associative $\mathbb{R}$-algebra with unit $\vartheta = \sigma \circ \epsilon$.

Finally, an element $\alpha \in \mathrm{End}(A)$ is called an *antipode* of the bialgebra if

$$\mathrm{id} * \alpha = \alpha * \mathrm{id} = \vartheta.$$

Clearly, this implies that an antipode is a convolution inverse of the identity map id. When an antipode exists, it is unique and described by the series

$$\alpha = \mathrm{id}^{*-1} = (\vartheta - (\vartheta - \mathrm{id}))^{*-1} = \sum_{k=0}^{\infty} (\vartheta - \mathrm{id})^{*k}.$$

For any $a, a' \in A$ it follows that $\alpha(aa') = \alpha(a')\alpha(a)$. This final bit of structure culminates in the definition below.

Definition 4: The six-tuple $(A, \mu, \sigma, \Delta, \epsilon, \alpha)$ is called an **$\mathbb{R}$-Hopf algebra**.

The following definitions concerning bialgebras will be important.

Definition 5: An $\mathbb{R}$-bialgebra $(A, \mu, \sigma, \Delta, \epsilon)$ is **filtered** if there exists a nested sequence of $\mathbb{R}$-vector subspaces of A, say $A_0 \subsetneq A_1 \subsetneq \cdots$, such that $A = \cup_{n \geq 0} A_n$ and

$$\Delta A_n \subseteq \sum_{i=0}^{n} A_i \otimes A_{n-i}.$$

The collection $\{A_n\}_{n \geq 0}$ is called a **filtration** of A.

Definition 6: An $\mathbb{R}$-bialgebra that is filtered such that $A_0 = \sigma(\mathbb{R})$ is said to be **connected**.

Definition 7: An $\mathbb{R}$-bialgebra is **graded** if there exists a set of $\mathbb{R}$-vector subspaces of A, say $\{A_{(n)}\}_{n \geq 0}$, such that $A = \oplus_{n \geq 0} A_{(n)}$ with

$$A_{(i)} A_{(j)} \subseteq A_{(i+j)}, \quad \Delta A_{(n)} \subseteq \bigoplus_{i=0}^{n} A_{(i)} \otimes A_{(n-i)},$$

and $\epsilon(A_{(n)}) = 0$, $n > 0$.

Definition 8: Let A be an $\mathbb{R}$-bialgebra. An element $g \in A$ is **group-like** if $\epsilon(g) = 1$ and $\Delta g = g \otimes g$. If A has only one group-like element, then any other element $a \in A$ is **primitive** if $\Delta a = a \otimes g + g \otimes a$.

A number of useful results follow from these definitions. For example, if A has a grading $\{A_{(n)}\}_{n \geq 0}$, then a natural filtration of A is $\{A_n\}_{n \geq 0}$, where

$$A_n = \bigoplus_{i=0}^{n} A_{(i)}.$$

Furthermore, if $A_{(0)} = \sigma(\mathbb{R})$ then A has only one group-like element. Perhaps the most important aspect concerning a connected bialgebra is a key property of its coalgebra. If $A^+ := \ker \epsilon$ and $A_n^+ := A^+ \cap A_n$ then for any $a \in A_n^+$ it follows that

$$\Delta a = a \otimes 1 + 1 \otimes a + \Delta' a, \tag{2}$$

where $\Delta' a \in A_{n-1}^+ \otimes A_{n-1}^+$. From this property, it can be shown that $A = A_0 \oplus A^+$ and that the following theorem holds.

Theorem 2: Let $(A, \mu, \sigma, \Delta, \epsilon)$ be a connected $\mathbb{R}$-bialgebra. Then $(A, \mu, \sigma, \Delta, \epsilon, \alpha)$ is an $\mathbb{R}$-Hopf algebra, where the antipode is given on A^+ by

$$\alpha = -\mathrm{id} + \sum_{k=1}^{\infty} (-1)^{k+1} \mu^k \Delta'^{\,k} \tag{3}$$

with

$$\mu^k : A \otimes A \otimes \cdots \otimes A \to A : a_1 \otimes a_2 \otimes \cdots \otimes a_{k+1} \mapsto a_1 a_2 \cdots a_{k+1}$$
$$\Delta'^{\,n+1} = (\mathrm{id} \otimes \Delta')\Delta'^{\,n} = (\Delta' \otimes \mathrm{id})\Delta'^{\,n}, \ \ n \geq 1.$$

Furthermore, for $k \geq n \geq 1$

$$(\vartheta - \mathrm{id})^{*k+1} a = (-1)^{k+1} \mu^k \Delta'^{\,k} a = 0, \ \ a \in A_n^+,$$

and thus, (3) evaluated at a has at most n nonzero terms. Otherwise, on A_0, $\alpha = \mathrm{id}$.
It is easy to show that the *reduced* coproduct, Δ', inherits its coassociativity property from that of Δ.

III. A Faà di Bruno Hopf Algebra for a Group of Fliess Operators

A. Group of Fliess Operators

For brevity the presentation henceforth is restricted to the single-input, single-output case, i.e., $m = \ell = 1$. Let $X = \{x_0, x_1\}$ and define the set of operators

$$\mathscr{F}_\delta = \{I + F_c \ : \ c \in \mathbb{R}\langle\langle X \rangle\rangle\}.$$

It is convenient to introduce the Dirac symbol δ and the definition $F_\delta = I$ such that $I + F_c = F_{\delta + c} = F_{c_\delta}$ with $c_\delta := \delta + c$. In which case, $c \,\tilde{\circ}\, d = c \circ (\delta + d)$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}\langle\langle X_\delta \rangle\rangle$. The transformation $\omega : c \mapsto \delta + c$ can be viewed as a type of Magnus transformation. That is, ω maps the free semigroup

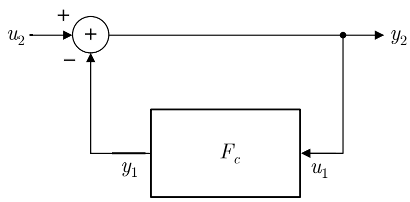


Fig. 7. Compositional inverse of $I + F_c$.

$(\mathbb{R}\langle\langle X\rangle\rangle, \circ, \delta)$ to a free group with generators $\delta + x_i$, $i = 0, 1$ [23, Theorem 5.6]. This suggests that $\mathscr{F}_\delta$ will also form a group under composition. Consider the composition of two elements in $\mathscr{F}_\delta$:

$$\begin{aligned} F_{c_\delta} \circ F_{d_\delta} &= (I + F_c) \circ (I + F_d) \\ &= I + F_d + F_c(I + F_d) \\ &= F_{\delta + d + c\,\tilde{\circ}\,d} \\ &= F_{c_\delta \circ d_\delta}, \end{aligned}$$

where $c_\delta \circ d_\delta := \delta + d + c\,\tilde{\circ}\,d$. It was shown in [21] that the modified composition product on $\mathbb{R}\langle\langle X\rangle\rangle$ is *not* associative. The following lemma, however, holds.

Lemma 1: The composition product on $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ is associative.

In light of the uniqueness of generating series, the semigroups $(\mathscr{F}_\delta, \circ, I)$ and $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ are clearly isomorphic. The next theorem establishes that $(\mathscr{F}_\delta, \circ, I)$ is a group.

Theorem 3: The triple $(\mathscr{F}_\delta, \circ, I)$, or equivalently $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, forms a group.

Example 1: A *linear series* $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is one whose support is a subset of the language $L := \{x_0^{n_1} x_1 x_0^{n_0} : n_i \geq 0\}$. The composition product $c \circ d$ is both left and right $\mathbb{R}$-linear when c is a linear series. It follows directly in this case that $(\delta + c)^{-1} = \delta - c + c^{\circ 2} - c^{\circ 3} + \cdots$. □

Example 2: Suppose $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ has finite Lie rank n. Then $I + F_c$ has an n dimensional input-affine state space realization of the form

$$\begin{aligned} \dot{z} &= g_0(z) + g_1(z)u_1, \quad z(0) = z_0 \\ \tilde{y}_1 &= h(z) + u_1, \end{aligned}$$

where each g_i and h is an analytic vector field and function, respectively, on some neighborhood W of z_0 [8]. It is easily verified that

$$(c, \eta) = L_{g_\eta} h(z_0), \tag{4}$$

where

$$L_{g_\eta} h := L_{g_{i_1}} \cdots L_{g_{i_k}} h, \quad \eta = x_{i_k} \cdots x_{i_1},$$

the *Lie derivative* of h with respect to g_i, is defined as

$$L_{g_i} h : W \to \mathbb{R} : z \mapsto \frac{\partial h}{\partial z}(z)\, g_i(z),$$

and $L_{g_\emptyset} h = h$. It is not difficult to see that the compositional inverse $(I + F_c)^{-1} = I + F_{c^{-1}} : u_2 \mapsto y_2$ is described by the feedback system in Fig. 7. A straightforward calculation gives a realization for $F_{c^{-1}}$, namely, $(g_0 - g_1 h, g_1, -h, z_0)$. Using this realization and (4), one can compute as many coefficients of c^{-1} as desired. The first few are:

$$\begin{aligned} (c^{-1}, \emptyset) &= -(c, \emptyset) \\ (c^{-1}, x_0) &= -(c, x_0) + (c, \emptyset)(c, x_1) \\ (c^{-1}, x_1) &= -(c, x_1) \\ (c^{-1}, x_0^2) &= -(c, x_0^2) + (c, \emptyset)(c, x_0 x_1) + \\ &\quad (c, x_0)(c, x_1) + (c, \emptyset)(c, x_1 x_0) - \\ &\quad (c, \emptyset)(c, x_1)^2 - (c, \emptyset)^2 (c, x_1^2) \\ (c^{-1}, x_0 x_1) &= -(c, x_0 x_1) + (c, x_1)^2 + (c, \emptyset)(c, x_1^2) \\ (c^{-1}, x_1 x_0) &= -(c, x_1 x_0) + (c, \emptyset)(c, x_1^2) \\ (c^{-1}, x_1^2) &= -(c, x_1^2). \end{aligned}$$

□

Example 3: For a single-input, single-output linear time-invariant system with transfer function $H(s)$ and state space realization (A, B, C), the corresponding generating series is $c = \sum_{i \geq 0} (c, x_0^i x_1)\, x_0^i x_1$, where $(c, x_0^i x_1) = CA^i B$, $i \geq 0$. In light of the previous example, it follows that

$$(c^{-1}, x_0^i x_1) = -C(A - BC)^i B, \quad i \geq 0.$$

Simply expanding these matrix powers gives

$$\begin{aligned} (c^{-1}, x_1) &= -(c, x_1) \\ (c^{-1}, x_0 x_1) &= -(c, x_0 x_1) + (c, x_1)^2 \\ (c^{-1}, x_0^2 x_1) &= -(c, x_0^2 x_1) + 2(c, x_1)(c, x_0 x_1) - (c, x_1)^3 \\ &\ \ \vdots \end{aligned}$$

□

B. Construction of the Faà di Bruno Hopf Algebra

The goal of this section is to describe a Faà di Bruno Hopf algebra associated with the group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, where the antipode, α, satisfies the identity

$$c_\delta^{-1} = \delta + c^{-1} = \delta + \sum_{\eta \in X^*} (\alpha\, a_\eta)(c)\, \eta$$

with

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c, \eta)$$

denoting the coordinate function for $\eta \in X^*$. Formally extend such mappings to $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ by letting $a_\delta(c_\delta) = 1$. Next define a commutative $\mathbb{R}$-algebra of polynomials denoted by

$$A = \mathbb{R}[a_\eta : \ \eta \in X^* \cup \delta],$$

where the product is defined by

$$a_\eta a_\xi(c_\delta) = a_\eta(c_\delta) a_\xi(c_\delta)$$

for all $\eta, \xi \in X^* \cup \delta$ and any given $c_\delta \in \mathbb{R}\langle\langle X_\delta\rangle\rangle$. The first objective is to produce a bialgebra having commutative product and noncocommutative coproduct

$$\mu : A \otimes A \to A : a_\eta \otimes a_\xi \mapsto a_\eta a_\xi \tag{5}$$

$$\Delta : A \to A \otimes A : a_\nu \mapsto \Delta a_\nu, \tag{6}$$

respectively, such that

$$\begin{aligned} \mu(\Delta a_\nu(d_\delta \otimes c_\delta)) &= a_\nu(c_\delta \circ d_\delta) = (c_\delta \circ d_\delta, \nu) \\ &= (\delta, \nu) + (d, \nu) + \sum_{\eta \in X^*} (\tilde{D}_\eta(1), \nu)(c, \eta). \end{aligned}$$

It is clear that the associativity of the composition product on $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ supplies the required coassociativity property for Δ. Observe that since $\tilde{D}_{x_i} : e \mapsto x_i e + x_0(d_i \sqcup\!\sqcup\, e)$, the

$\mathrm{ord}(\tilde{D}_{x_i}\tilde{D}_\eta(1)) \geq \mathrm{ord}(\tilde{D}_\eta(1)) + 1$ for any letter $x_i \in X$. Therefore, $\mathrm{ord}(\tilde{D}_\eta(1)) \geq |\eta|$ for any $\eta \in X^*$, and one can write instead the finite sum

$$\mu(\Delta a_\nu(d_\delta \otimes c_\delta)) = (\delta,\nu)+(d,\nu)+\sum_{k=0}^{|\nu|}\sum_{\eta\in X^k}(\tilde{D}_\eta(1),\nu)(c,\eta).$$

With the aid of this expression and using the equivalence $a_\delta \sim 1$, the first eight coproducts are found to be:

$$\begin{aligned}
\Delta 1 &= 1\otimes 1\\
\Delta a_\emptyset &= a_\emptyset \otimes 1 + 1\otimes a_\emptyset\\
\Delta a_{x_0} &= a_{x_0}\otimes 1 + 1\otimes a_{x_0} + a_\emptyset\otimes a_{x_1}\\
\Delta a_{x_1} &= a_{x_1}\otimes 1 + 1\otimes a_{x_1}\\
\Delta a_{x_0^2} &= a_{x_0^2}\otimes 1 + 1\otimes a_{x_0^2} + a_\emptyset\otimes a_{x_0x_1} + a_{x_0}\otimes a_{x_1} +\\
&\quad a_\emptyset\otimes a_{x_1x_0} + a_\emptyset^2\otimes a_{x_1^2}\\
\Delta a_{x_0x_1} &= a_{x_0x_1}\otimes 1 + 1\otimes a_{x_0x_1} + a_{x_1}\otimes a_{x_1} + a_\emptyset\otimes a_{x_1^2}\\
\Delta a_{x_1x_0} &= a_{x_1x_0}\otimes 1 + 1\otimes a_{x_1x_0} + a_\emptyset\otimes a_{x_1^2}\\
\Delta a_{x_1^2} &= a_{x_1^2}\otimes 1 + 1\otimes a_{x_1^2}.
\end{aligned}$$

Continuing the construction, define the unit and counit, respectively, as

$$\sigma : \mathbb{R}\to A : \lambda \mapsto \lambda\, a_\delta \sim \lambda\, 1 \qquad (7)$$

$$\epsilon : A \to \mathbb{R} : a_{\eta_1}a_{\eta_2}\cdots a_{\eta_\ell} \mapsto a_{\eta_1}(\delta)a_{\eta_2}(\delta)\cdots a_{\eta_\ell}(\delta). \qquad (8)$$

As required by the definition of a bialgebra, $\sigma(1) = 1$, which is the unit of A, and $\epsilon\circ\sigma = 1$. Furthermore, since $\epsilon(a_\delta) = 1$, it follows that a_δ is group-like. A central result of the paper now follows.

Theorem 4: The five-tuple $(A, \mu, \sigma, \Delta, \epsilon)$ described by (5)-(6) and (7)-(8) is a graded $\mathbb{R}$-bialgebra.

It is important to observe that this bialgebra is not connected, that is, using the natural filtration associated with the given grading, $A_0 \neq \sigma(\mathbb{R}) = \mathrm{span}_{\mathbb{R}}\{a_\delta\}$. For example, $a_\emptyset \in A_0$ but $a_\emptyset \notin \sigma(\mathbb{R})$. It is also clear that the coproduct terms computed above do not satisfy (2). Despite this fact, the following theorem still holds.

Theorem 5: The six-tuple $(A, \mu, \sigma, \Delta, \epsilon, \alpha)$ described by (5)-(6), (7)-(8) and

$$\alpha\, a_\nu = -a_\nu + \sum_{k=1}^{n}(-1)^{k+1}\mu^k\Delta'^{\,k}a_\nu,\ \ \nu\in X^n,\ \ \nu\neq\delta \qquad (9)$$

with $\alpha\, 1 = 1$ is a graded $\mathbb{R}$-Hopf algebra with the grading given by

$$A_{(n)} = \mathrm{span}_{\mathbb{R}}\left\{a_{\eta_1}a_{\eta_2}\cdots a_{\eta_l}\in A\ :\ \sum_{i=1}^{l}|\eta_i| = n\right\},$$

for $n \geq 0$ and $|\delta| := 0$.

The corresponding antipode terms are then found from (9) to be:

$$\begin{aligned}
\alpha\, 1 &= 1 && (10a)\\
\alpha\, a_\emptyset &= -a_\emptyset && (10b)\\
\alpha\, a_{x_0} &= -a_{x_0} + a_\emptyset a_{x_1} && (10c)\\
\alpha\, a_{x_1} &= -a_{x_1} && (10d)\\
\alpha\, a_{x_0^2} &= -a_{x_0^2} + a_\emptyset a_{x_0x_1} + a_{x_0}a_{x_1} + &&\\
&\quad a_\emptyset a_{x_1x_0} - a_\emptyset a_{x_1}^2 - a_\emptyset^2 a_{x_1^2} && (10e)\\
\alpha\, a_{x_0x_1} &= -a_{x_0x_1} + a_{x_1}^2 + a_\emptyset a_{x_1^2} && (10f)\\
\alpha\, a_{x_1x_0} &= -a_{x_1x_0} + a_\emptyset a_{x_1^2} && (10g)\\
\alpha\, a_{x_1^2} &= -a_{x_1^2}. && (10h)
\end{aligned}$$

These terms agree exactly with those for c^{-1} computed from Lie derivatives in Example 2, where it was assumed that c had finite Lie rank. In the present context, however, no such assumption is required.

The following corollary establishes a direct analogy to the classical Faà di Bruno Hopf algebra.

Corollary 1: The set of proper series forms a subgroup of $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, and the corresponding Faà di Bruno Hopf subalgebra is connected and graded.

Proof: The first claim follows directly from the identities $(c_\delta \circ d_\delta, \emptyset) = (c, \emptyset) + (d, \emptyset)$ and (10b). The second claim is evident from the fact that under the properness assumption, $A_0 \sim \mathbb{R}$. ■

Example 4: In the state space setting employed in Example 2, c is proper if and only if $z_0 = 0$. This is precisely the case for the linear system described in Example 3. □

A simple calculation shows that $c^{-1} = (-c)@\delta$ (see Fig. 7). Thus, the following theorem from [26] establishes that local convergence is preserved by the compositional inverse operation.

Theorem 6: For any $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ it follows that

$$|(c@\delta,\eta)| \leq K\,(\mathcal{A}(K_c)M_c)^{|\eta|}\,|\eta|!,\ \ \eta\in X^*,$$

for some $K > 0$ and

$$\mathcal{A}(K_c) = \frac{1}{1-K_c\ln(1+1/K_c)}.$$

IV. An Explicit Formula for the Feedback Product

Given two Fliess operators F_c and F_d which are linear time-invariant systems with transfer functions G and H, respectively, the closed-loop system in Fig. 2 has the transfer function

$$G(I-HG)^{-1} = G\sum_{k=0}^{\infty}(HG)^k.$$

The next theorem gives a nonlinear generalization of this type of closed-loop system representation.

Theorem 7: For any $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that

$$c@d = c\,\tilde{\circ}\,(-d\circ c)^{-1} = c\circ(\delta - d\circ c)^{-1}. \qquad (11)$$

Proof: Clearly the function v in Fig. 2 must satisfy the identity

$$v = u + F_{d\circ c}[v].$$

Therefore,

$$(I + F_{-d\circ c})\,[v] = u.$$

Applying the compositional inverse $\left(I + F_{(-d\circ c)^{-1}}\right)$ on the left gives

$$v = \left(I + F_{(-d\circ c)^{-1}}\right)[u],$$

and thus, $F_{c@d}[u] = F_c[v] = F_{c\,\tilde{\circ}\,(-d\circ c)^{-1}}[u]$ as desired. ■

Note that (11) also applies when either $c = \delta$ or $d = \delta$, namely, $c@\delta = c\circ(\delta - c)^{-1} = (-c)^{-1}$ and $\delta@d = (\delta - d)^{-1} = \delta - d^{-1}$, respectively. Next it is shown that feedback

TABLE I

COEFFICIENTS OF THE SEQUENCES IN EXAMPLE 5.

η	$\emptyset$	x_0	x_1	x_0^2	x_0x_1	x_1x_0	x_1^2
(c,η)	1	1	1	2	2	2	2
$((-c)^{-1},\eta)$	1	2	1	10	5	4	2
$(c@\delta,\eta)$	1	2	1	10	5	4	2
$L_{g_\eta}h(1)$	1	2	1	10	5	4	2
$(\mathcal{A}(1))^{\|\eta\|}\,\|\eta\|!$	1	3.3	3.3	21.2	21.2	21.2	21.2

preserves local convergence. But the following preliminary result is needed first.

Theorem 8: The triple $(\mathbb{R}_{LC}\langle\langle X_\delta\rangle\rangle,\circ,\delta)$ is a subgroup of $(\mathbb{R}\langle\langle X_\delta\rangle\rangle,\circ,\delta)$.

Theorem 9: If $c,d\in\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ then $c@d\in\mathbb{R}_{LC}\langle\langle X\rangle\rangle$.

Proof: Since the composition product, the modified composition product, and the compositional inverse all preserve local convergence, the claim follows directly from Theorem 7. ■

Example 5: Consider the operator F_c in the feedback configuration shown in Fig. 2, where $c=\sum_{\eta\in X^*}|\eta|!\,\eta$ and $F_d=F_\delta=I$. The first few terms of $(-c)^{-1}$, as shown in Table I, were computed using the antipode formulas (10). After which, $c@\delta$ was computed using (11). As expected, $c@\delta=(-c)^{-1}$. It can be shown that the Lie rank of c is one. To construct a one dimensional state space realization, first observe that

$$c=\sum_{k=0}^{\infty}k!\,\mathrm{char}(X^k)=\sum_{k=0}^{\infty}\mathrm{char}(X)^{\sqcup\!\sqcup\, k}.$$

Therefore,

$$F_c=\sum_{k=0}^{\infty}E_{\mathrm{char}(X)^{\sqcup\!\sqcup\, k}}=\sum_{k=0}^{\infty}E^k_{\mathrm{char}(X)}=\frac{1}{1-E_{\mathrm{char}(X)}}.$$

In which case, defining $z=F_c$, it follows that

$$\dot z=z^2(1+u),\quad z(0)=1$$
$$y=z$$

realizes $y=F_c[u]$, and

$$\dot z=z^2+z^3+z^2v,\quad z(0)=1$$
$$y=z$$

realizes $y=F_{c@\delta}[v]$. The generating series for the closed-loop system can be computed directly using (4) with $(g_0,g_1,h,z_0)=(z^2+z^3,z^2,z,1)$. As expected, it is identical to $c@\delta$ as shown in Table I. Finally, the upper bound on the coefficients of $c@\delta$ as provided by Theorem 6 with $K_c=M_c=1$ is given in the bottom row of the Table I. □

V. CONCLUSIONS

A Faà di Bruno Hopf algebra was constructed for a group of Fliess operators. Its antipode was used to produce an explicit formula for the feedback product of two formal power series. This expression, in turn, facilitated a proof that local convergence is preserved under feedback.

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