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Stabilizability of constrained uncertain linear systems via smooth control Lyapunov R-functions

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Abstract— **The stabilization problem of constrained uncertain linear systems is addressed via the class of control Lyapunov R-functions that are obtained reformulating the classic geometric intersection operator in terms of R-functions. The feasibility test of the proposed smooth control Lyapunov functions can be casted into (bi)linear matrix inequalities conditions. Like polyhedral Lyapunov functions, the maximal estimate of the controlled invariant state space set is achieved. The advantage of the proposed approach is that the inner sublevel sets are smooth and can be made everywhere differentiable. This smoothing technique is very general and it can be used to smooth both polyhedral and truncated ellipsoidal control Lyapunov functions to improve the control performances, as shown in some benchmark examples.**

I. INTRODUCTION

The state feedback stabilizability of constrained uncertain linear systems covers both theoretic and practical control problems, characterized by saturation of the control inputs, state constraints and model uncertainties. The solution of the problem is equivalent to the design of a suitable robust Control Lyapunov Function (CLF) for the system, associated to a state feedback control law. Moreover, the choice of the candidate CLF leads to a particular estimation of the controlled invariant set. Ideally, the exact solution of the problem consists in finding the largest admissible controlled invariant region of the state space, according to both state and control constraints. Quadratic CLFs (QCLFs) can only provide a conservative solution to the stabilizability problem, namely the largest controlled invariant ellipsoidal set included inside the admissible state space region, easily computable via standard Linear Matrix Inequality (LMI) techniques. However, usually the shape of the maximal controlled invariant set can not be described by ellipsoidal functions and more complex classes of candidate CLFs are needed. For instance, Polyhedral CLFs (PCLFs) [1] [2] can approximate the maximal controllable invariant set with arbitrary precision [3]. PCLFs can be smoothed with standard high-order norms [4] in order to obtain an everywhere differentiable smoothed PCLF that can be used together with nonlinear gradient-based continuous controllers [5].

Recently, the class of Truncated Ellipsoids (TEs) [6] [7] has been proposed as candidate LFs and CLFs for constrained uncertain linear systems. The advantage of using TEs is that a quite good approximation of the feasible region is provided with a considerably smaller number of parameters [6]. In [7] a linear state feedback control is designed to solve a set of sufficient Bilinear Matrix Inequalities (BMIs), maximizing the volume of the estimated controlled invariant set.

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In this paper, the class of Control Lyapunov R-Functions (CLRFs) is proposed for the state feedback stabilization problem. The main contribution is the extention of the results of [8], [9] for constrained uncertain linear systems. Moreover, in [9] the solution of a nonlinear optimization problem is proposed for the feasibility test of the candidate R-composed CLF for a fixed smoothing parameter, while here the feasibility test is casted into an easily-tractable LMI problem that is valid for all admissible values of the smoothing parameter. CLRFs can smooth both PCLFs and TEs in a non-homothetic way and can be made everywhere differentiable. This is an important novelty because nonlinear gradient-based controllers can not be associated to TEs for of the lack of differentiability. Since the smooth composition of PCLFs and QCLFs is investigated, the robust quadratic stabilizability is assumed for the unconstrained system.

The proposed smoothing technique follows by the interpretation of the intersection of ellipsoidal and polyhedral regions in the framework of R-functions [10], referred in the next section, which are real-valued functions that admit a generalization of the standard pointwise *min* and *max* operators. Section 3 shows the design of a candidate CLRF, together with the proposed feasibility theorems. In Section 4, a standard nonlinear gradient-based control is associated to both the smooth CLRF and classic ones. In the numerical simulations, the control performances of the proposed CLRF are compared to the ones of a smooth PCLF and also of a TE. In the last section the main results are summarized and interesting future lines of research are outlined.

A. Notation

I_n denotes the $n \times n$ identity matrix. The closed k-level set of a continuous function $V : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$, i.e. $\{x \in \mathcal{X} : \ V(x) \leq k\}$, is denoted by $\mathcal{L}[V,k]$. A set $\mathcal{S} \subseteq \mathbb{R}^n$ is called $\mathcal{C}$-set if it is a convex and compact set including the origin in its interior [3]. $\mathbb{I}_r$ denotes $\{n \in \mathbb{Z}^+ : \ n \leq r\}$. $\overline{1}_r$ denotes the vector $[1, 1, \cdots, 1]^\top \in \mathbb{N}^r$.

II. ON THE USE OF R-FUNCTIONS IN SYSTEMS AND CONTROL THEORY

A. R-functions

The use of R-functions for the state feedback stabilization of control systems has been firstly proposed in [8]. Here only a brief description of the framework to compose LFs is provided.

TABLE I

CORRESPONDENCE BETWEEN LOGIC FUNCTIONS AND R-FUNCTIONS

BOOLEAN	R-COMPOSITION
NOT $\neg$	$-r$
AND $\overset{\alpha}{\wedge}$	$\dfrac{r_1 + r_2 - \sqrt{r_1^2 + r_2^2 - 2\alpha r_1 r_2}}{2 - \sqrt{2 - 2\alpha}}$
OR $\overset{\alpha}{\vee}$	$\dfrac{r_1 + r_2 + \sqrt{r_1^2 + r_2^2 - 2\alpha r_1 r_2}}{2 + \sqrt{2 - 2\alpha}}$

Definition 1: A function $r : \mathbb{F}^n \subseteq \mathbb{R}^n \to \mathbb{R}$ is an R-function if there exists a Boolean function $\mathcal{R} : \mathbb{B}^n \to \mathbb{B}$, where $\mathbb{B} = \{0, 1\}$, such that the following equality is satisfied:

$$h\left(r\left(x_1, x_2, \ldots, x_n\right)\right) = \mathcal{R}\left(h\left(x_1\right), h\left(x_2\right), \ldots, h\left(x_n\right)\right),$$

where $h(\cdot)$ is the standard Heaviside step function.

Informally, a real function r is an R-function if it can change its sign only when some of its arguments change the sign [8]. The parallelism between logic functions and R-functions becomes more evident when classic Boolean operators are recovered as described in Table I [9].

For instance, according to Table I, the interpretation of the AND composition is that the composed function is positive when evaluated in x if and only if both $r_1(x)$ *and* $r_2(x)$ are positive. The result is obtained by exploiting the triangle inequality and the law of cosines, and it holds for all values of $\alpha \in [0, 1] \subset \mathbb{R}$ [11]. The terms at the denominator in Table I are normalizing factors, i.e., at a given x, the composed function $r_{\overset{\alpha}{\wedge}}(x) = 1$ if and only if $r_1(x) = 1$ and $r_2(x) = 1$. Also, $r_{\overset{\alpha}{\vee}}(x) = 1$ if and only if $r_1(x) = 1$ or $r_2(x) = 1$.

Remark 1: When $\alpha = 1$, $r_1 \overset{1}{\wedge} r_2 = \min\{r_1, r_2\}$ and $r_1 \overset{1}{\vee} r_2 = \max\{r_1, r_2\}$.

In the following, we consider only the AND composition rule, since we are concerned with 0-symmetric controlled $\mathcal{C}$-sets. A geometric interpretation of R-functions is now provided. Consider the polyhedral function $V_1(x) = \max\left\{x^\top F_1^\top F_1 x,\ x^\top F_2^\top F_2 x\right\}$ and the quadratic function $V_2(x) = x^\top P x$ where

$$F = \begin{bmatrix} 1.50 & -0.50 \\ -0.50 & 1.50 \end{bmatrix}, \quad P = \begin{bmatrix} 2.07 & 0.66 \\ 0.66 & 2.07 \end{bmatrix}, \qquad (1)$$

being F_i the i^{th} row of matrix F.

To compose the positive definite functions V_1 and V_2 in their 1-level sets, respectively $\mathcal{L}[V_1, 1]$ and $\mathcal{L}[V_2, 1]$, define the R-functions $R_1(x) = 1 - V_1(x)$ and $R_2(x) = 1 - V_2(x)$. Without loss of generality, these functions have been normalized so that their maximum value is 1. Then compute the *R-intersection* (AND rule $\overset{\alpha}{\wedge}$) $R_{\overset{\alpha}{\wedge}} = R_1 \overset{\alpha}{\wedge} R_2$, according to the equation of Table I, for an arbitrary value of $\alpha \in [0, 1]$. The composed function $R_{\overset{\alpha}{\wedge}}$ is the (smoothed) *intersection* between the polyhedral function and the quadratic one in the sense that $R_{\overset{\alpha}{\wedge}}$ is positive inside the geometric intersection region $\mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1]$, it is zero on the boundary, negative outside, and its maximum value is 1 at the origin. The positive definite function associated to $R_{\overset{\alpha}{\wedge}}$ is $V_{\overset{\alpha}{\wedge}} = 1 - R_{\overset{\alpha}{\wedge}}$. The sublevel sets of the function $V_{\overset{\alpha}{\wedge}}$ are shown in Figure 1, for the case of $\alpha = 1$ (truncated ellipsoid [6], [7]) and $\alpha = 0$.

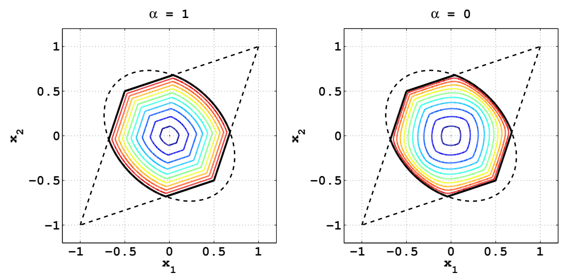

Fig. 1. On the left side the sublevel sets of the composed function for $\alpha = 1$ (truncated ellipsoid), while on the right the sublevel sets of the composed function for $\alpha = 0$.

Remark 2: In [7] the term truncated ellipsoid has been introduced to define a candidate LF that corresponds to the intersection of a polyhedral region with an ellipsoidal one. Within the framework of R-functions, the TE is recovered as a special case ($\alpha = 1$) of the R-intersection between a polyhedral function and a quadratic one, see Figure 1.

Parameter α affects the smoothness of the inner sublevel sets of the composed function, while it does not affect the shape of the overall region $\mathcal{L}[V_{\overset{\alpha}{\wedge}}, 1]$. For $\alpha \in [0, 1)$ such smoothing technique yields non homothetic sublevel sets and a differentiable function in the intersection set $\mathcal{L}[V_{\overset{\alpha}{\wedge}}, 1]$.

R-functions can be used to compose general functions and not only polyhedral or quadratic ones. Some examples of different compositions can be found in [11] and [12].

B. Lyapunov R-functions for stability analysis of nonlinear systems

In this subsection, the intersection function $V_{\overset{\alpha}{\wedge}}$ is used as candidate Lyapunov function for the stability analysis of a general dynamical system

$$\dot{x}(t) = f\left(x(t)\right) : \ x(t) \in \mathcal{X} \subseteq \mathbb{R}^n, \ f : \mathcal{X} \to \mathcal{X} \text{ continuous.} \qquad (2)$$

In the following theorem we consider the R-composition of two LFs V_1, V_2 for the system (2) in a given closed subset of the state space. To avoid the lack of differentiability in the set $\{x \in \mathcal{X} : \ V_{\overset{\alpha}{\wedge}}(x) = 1\} = \{x \in \mathcal{X} : \ \max_i\{V_i(x)\} = 1\}$, we consider the $\mathcal{C}$-set $\mathcal{L}[V_{\overset{\alpha}{\wedge}}, 1 - \epsilon]$, for any $\epsilon \in (0, 1) \subset \mathbb{R}^+$.

Theorem 1: Assume that functions $V_i : \ \mathcal{L}[V_i, 1] \subseteq \mathcal{X} \to \mathbb{R}$, $i = 1, 2$, are two strict Lyapunov functions with time derivatives $\dot{V}_i(x(t)) \leq -\eta V_i\left(x(t)\right)$, $i = 1, 2$, along the system trajectory (2), in the 0-symmetric $\mathcal{C}$-set $\mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1]$. Then the R-composed function $V_{\overset{\alpha}{\wedge}}$ is a strict LF with decreasing rate $\eta \in \mathbb{R}^+$ for (2) in the intersection set $\mathcal{L}[V_{\overset{\alpha}{\wedge}}, 1 - \epsilon] = \mathcal{L}[V_1, 1 - \epsilon] \cap \mathcal{L}[V_2, 1 - \epsilon]$, $\forall \alpha \in [0, 1]$, for any $\epsilon \in (0, 1) \subset \mathbb{R}^+$.

Proof: Define the R-functions $R_i(x) = 1 - V_i(x),\ i = 1,2$, and the R-composition $R_{\overset{\alpha}{\wedge}}$, according to the AND rule of Table I:

$$R_{\overset{\alpha}{\wedge}} = \frac{R_1 + R_2 - \sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}}{2 - \sqrt{2 - 2\alpha}}. \tag{3}$$

The candidate LF $V_{\overset{\alpha}{\wedge}}$ is positive definite in the set $\mathcal{L}[V_{\overset{\alpha}{\wedge}}, 1] = \mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1]$ because $R_{\overset{\alpha}{\wedge}}(x) = 1 \Leftrightarrow R_1(x) = R_2(x) = 1 \Leftrightarrow x = 0$. Moreover, $V_{\overset{\alpha}{\wedge}}$ is everywhere differentiable in the set $\mathcal{L}[V_{\overset{\alpha}{\wedge}}, 1 - \epsilon]$, $\forall \alpha \in [0, 1)$, for any $\epsilon \in (0, 1) \subset \mathbb{R}^+$.

The assumption is equivalent to $\dot{R}_i\left(x(t)\right) \geq \eta\left(1 - R_i\left(x(t)\right)\right),\ i = 1, 2$, therefore, considering the time derivative

$$\dot{R}_{\overset{\alpha}{\wedge}}(x(t)) = \frac{1}{2 - \sqrt{2 - 2\alpha}} \left[\dot{R}_1 \left(1 + \frac{-R_1 + \alpha R_2}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right) + \right. \\ \left. + \dot{R}_2 \left(1 + \frac{-R_2 + \alpha R_1}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right) \right], \tag{4}$$

the following inequality for the R-intersection holds.

$$\begin{aligned}
\dot{R}_{\overset{\alpha}{\wedge}}(x(t)) \geq{}& \frac{1}{2 - \sqrt{2 - 2\alpha}} \left[\eta(1 - R_1) \left(1 + \frac{-R_1 + \alpha R_2}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right) + \right. \\
&\left. + \eta(1 - R_2) \left(1 + \frac{-R_2 + \alpha R_1}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right) \right] = \\
={}& \frac{\eta}{2 - \sqrt{2 - 2\alpha}} \left[\left(\frac{(\alpha - 1)(R_1 + R_2)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} + 2 \right) + \right. \\
&+ \frac{-R_1 \left(-R_1 + \alpha R_2 + \sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2} \right)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} + \\
&\left. + \frac{-R_2 \left(-R_2 + \alpha R_1 + \sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2} \right)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right] = \\
={}& \eta \left[\frac{1}{2 - \sqrt{2 - 2\alpha}} \left(\frac{(\alpha - 1)(R_1 + R_2)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} + 2 \right) - R_{\overset{\alpha}{\wedge}} \right].
\end{aligned} \tag{5}$$

Finally, $\forall \alpha \in [0, 1]$, $\forall R_1, R_2 \in [0, 1]$, it can be proved that

$$\frac{1}{2 - \sqrt{2 - 2\alpha}} \left(\frac{(\alpha - 1)(R_1 + R_2)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} + 2 \right) \geq 1. \tag{6}$$

Equivalently

$$\frac{(1 - \alpha)(R_1 + R_2)}{\sqrt{R_1^2 + R_2 - 2\alpha R_1 R_2}} \leq \sqrt{2 - 2\alpha} \tag{7}$$

and by taking the square:

$$(1 - \alpha)^2 \left(R_1^2 + R_2^2 + 2R_1 R_2 \right) \leq 2(1 - \alpha) \left(R_1^2 + R_2^2 - 2\alpha R_1 R_2 \right). \tag{8}$$

For $\alpha = 1$ the previous inequality (8) is verified as equality. Considering the case of $\alpha \in [0, 1)$, we can divide both sides of (8) by $(1 - \alpha)$. Then, by simple algebra, we obtain

$$-\alpha R_1^2 - \alpha R_2^2 + 2R_1 R_2 \leq R_1^2 + R_2^2 - 2\alpha R_1 R_2 \Leftrightarrow \\ (1 + \alpha)\left(R_1 - R_2\right)^2 \geq 0. \tag{9}$$

Therefore $\dot{R}_{\overset{\alpha}{\wedge}}(x(t)) \geq \eta(1 - R_{\overset{\alpha}{\wedge}}(x(t)))$ concludes the proof. ■

Formally, in the limit case of $\alpha = 1$, the requirement of differentiability for a valid candidate LF is violated, therefore, in the case of differentiable composing functions, the above results yield differentiable LFs for $\alpha \in [0, 1)$. In fact, according to Remark 1, for $\alpha = 1$ the non smooth $\max$ and $\min$ operators are recovered. For the same reason, we avoid the lack of differentiability of the external level set in which $V_{\overset{\alpha}{\wedge}}(x) = 1$ by considering the set $\mathcal{L}[V_{\overset{\alpha}{\wedge}}, 1 - \epsilon]$.

III. CONTROL LYAPUNOV R-FUNCTIONS FOR CONSTRAINED STABILIZATION

A. Problem statement

Let us consider the constrained stabilization problem for an uncertain linear system:

$$\begin{aligned}
&\dot{x}(t) = A(\mu(t))x(t) + B(\mu(t))u(t) \quad \text{sub. to} \\
&x(t) \in \mathcal{X} \subset \mathbb{R}^n, \quad u(t) \in \mathcal{U} \subset \mathbb{R}^m \\
&A(\mu(t)) \in \mathcal{A} = \{\textstyle\sum_{i=1}^{p} \mu_i A_i : \ \mu_i \geq 0, \sum_{i=1}^{p} \mu_i = 1\} \\
&B(\mu(t)) \in \mathcal{B} = \{\textstyle\sum_{i=1}^{p} \mu_i B_i : \ \mu_i \geq 0, \sum_{i=1}^{p} \mu_i = 1\}
\end{aligned} \tag{10}$$

where $\mu(t) \in \mathbb{R}^p$, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$ $\forall i \in \mathbb{I}_p$. The control objective is to design a state feedback control law $u\left(x(t)\right)$ such that $x(t)$ asymptotically converges to the origin, in accordance to the state and control input constraints.

Let us assume the state and input constraints to be linear, convex and 0-symmetric, of the form

$$\mathcal{X} = \left\{x \in \mathbb{R}^n : \|Lx\| \leq 1\right\}, \ \mathcal{U} = \left\{u \in \mathbb{R}^m : \|u\|_\infty \leq 1\right\}, \tag{11}$$

where $L \in \mathbb{R}^{r \times n}$ is a full-rank matrix.

A classic control strategy for the state feedback constrained stabilization of the uncertain system (10) is the design of a CLF associated to a (linear) state feedback control law $u\left(x(t)\right)$. In the literature, several classes of functions have been proposed as candidate CLFs.

The universal class of PCLFs

$$V_\infty(x) = \|Fx\|_\infty = \max_{i \in \mathbb{I}_s} \left\{|F_i x|\right\}, \tag{12}$$

can estimate the maximal controlled invariant region of the state space with arbitrary precision [13]. Alternatively, a PCLF *of the second order* [14] can be defined as

$$V_\infty(x) = \max_{i \in \mathbb{I}_s} \left\{x^\top F_i^\top F_i x\right\}. \tag{13}$$

A typical control strategy associated to PCLFs is the piecewise-linear control, with one different control vector for each vertex of the controlled invariant polyhedron [2], [14]. However, a good approximation of the largest controlled invariant set can consists of thousands of vertices and planes, making the use of piecewise-linear controllers unsuited for the practical implementation.

Since for PCLF the classic differentiability condition is relaxed, smoothed PCLFs has been proposed in [15], where the smoothing is performed with a standard $2p$-norm

$$V_{2p}(x) = \|Fx\|_{2p} = \sqrt[2p]{\sum_{i=1}^{s} (F_i x)^{2p}}, \tag{14}$$

i.e. another universal class of CLFs for p sufficiently large [4].

The benefit of using a smooth (differentiable) CLF is that nonlinear optimal gradient-based controllers can be used, since the gradient of the CLF is everywhere defined. For instance, the *minimum effort control* [5] is used in [4].

Recently, candidate CLFs corresponding to the intersection between the polyhedron associated to the state constraints and a quadratic function have been proposed in order to provide a relaxed estimation of the maximal invariant set with a reduced number of design parameters. For instance, truncated ellipsoids

$$V_{te}(x) = \max_{i \in \mathbb{I}_s} \left\{ x^\top F_i^\top F_i x,\ x^\top P x \right\}, \tag{15}$$

where $P \in \mathbb{R}^{n \times n}$, $P \succ 0$, have been proposed in [6], [7].

The use of the *max* operator makes V_{te} non differentiable, therefore a smoothing techinique has been proposed in [9] together with a nonlinear gradient-based control. However, the smoothed truncated ellipsoid has not been proved to necessarily be a suitable CLF. Next Section presents some sufficient LMI conditions for both a smoothed polyhedral CLRF and a smoothed truncated ellipsoidal CLRF to be a suitable CLF with guaranteed decreasing rate.

B. Control Lyapunov R-Functions

In this subsection it is shown how to design a suitable CLRF $V_{\overset{\alpha}{\wedge}}$, corresponding to the smoothed intersection of a PCLF $\max_{i \in \mathbb{I}_s} \left\{ x^\top F_i^\top F_i x \right\}$ and a QCLF (for the unconstrained system) $x^\top P x$. Let us define

$$\begin{array}{l} R_1(x) = 1 - \max_{i \in \mathbb{I}_s} \{ x^\top F_i^\top F_i x \} \\ R_2(x) = 1 - x^\top P x \\ R_{\overset{\alpha}{\wedge}} \ = \ R_1 \overset{\alpha}{\wedge} R_2. \end{array} \tag{16}$$

As previously remarked, $R_{\overset{\alpha}{\wedge}}(x) \geq 0 \ \forall x \in \mathcal{L}[V_{\overset{\alpha}{\wedge}}, 1]$ and $\max_x \{R_{\overset{\alpha}{\wedge}}(x)\} = R_{\overset{\alpha}{\wedge}}(0) = 1$. Therefore the candidate (positive definite) CLF is

$$V_{\overset{\alpha}{\wedge}}(x) = 1 - R_{\overset{\alpha}{\wedge}}(x). \tag{17}$$

The following Theorem is referred to the smoothing of the PCLF $V_\infty(x) = \|Fx\|_\infty$ via the R-intersection with the QCLF (for the unconstrained system) associated to the shape of the PCLF, that is $V_2(x) = \frac{1}{s} x^\top \left(\sum_{i=1}^{s} F_i^\top F_i \right) x$. Factor $\frac{1}{s}$ guarantees that $\mathcal{L}[V_\infty, 1] \subseteq \mathcal{L}[V_2, 1]$.

Theorem 2: Consider the constrained control problem (10) with $\mathcal{X} = \{ x \in \mathbb{R}^n : \ \|Fx\|_\infty \leq 1 \}$ and $F \in \mathbb{R}^{s \times n}$ full column rank matrix. Assume that there exist $K \in \mathbb{R}^{m \times n}$, $\eta \in \mathbb{R}^+$ and $\gamma_{ijk} = \gamma_{ikj} \in \mathbb{R}_0^+$, for $i = 1, ..., p$, $j, k = 1, ..., s$, such that

$$\begin{aligned} &(A_i + B_i K)^\top F_k^\top F_k + F_k^\top F_k (A_i + B_i K) \preceq \\ &- 2\eta F_k^\top F_k + \sum_{j=1}^{s} \gamma_{ijk} \left(F_j^\top F_j - F_k^\top F_k \right) \quad \forall i \in \mathbb{I}_p, \ \forall k \in \mathbb{I}_s \end{aligned} \tag{18}$$

$$-\bar{1}_m \leq K v^{(l)} \leq \bar{1}_m \quad \forall l, \tag{19}$$

where $v^{(l)}$ are the vertices of the polyhedron $\mathcal{L}[V_\infty, 1]$, and F_k is the k^{th} row of F. Then the smoothed polyhedral CLRFs $V_{\overset{\alpha}{\wedge}}$ (17), being $P = \frac{1}{s} \sum_{i=1}^{s} F_i^\top F_i$, is a strict CLF, with decreasing rate η, for system (10) in the set $\mathcal{L}[V_\infty, 1]$, $\forall \alpha \in [0, 1]$.

Proof: Inequality (18) is a sufficient condition for function $V_\infty(x) = \max_{i \in \mathbb{I}_s} \{ x^\top F_i^\top F_i x \}$ to be a PLF with decreasing rate η for the state constrained closed-loop LDI $\dot{x}(t) = (A_i + B_i K)\, x(t)$, $i \in \mathbb{I}_p$, [7], [14].

Then, thanks to the assumption that $\gamma_{ijk} = \gamma_{ikj}$, by summing over k all left-hand and right-hand sides of inequalities (18), we obtain

$$\sum_{k=1}^{s} \left[(A_i + B_i K)^\top F_k^\top F_k + F_k^\top F_k (A_i + B_i K) \right] \preceq -2\eta \sum_{k=1}^{s} \left(F_k^\top F_k \right) \tag{20}$$

i.e. $V_2(x) = \frac{1}{s} x^\top \sum_{k=1}^{s} \left(F_k^\top F_k \right) x$ is a QLF with decreasing rate η for the unconstrained closed-loop LDI.

Therefore, in view of Theorem 1, also $V_{\overset{\alpha}{\wedge}}$ is a strict LF such that $\dot{V}_{\overset{\alpha}{\wedge}}(x(t)) \leq -\eta V_{\overset{\alpha}{\wedge}}(x(t)) \quad \forall x(t) \in \mathcal{L}\left[V_\infty, 1\right] \cap \mathcal{L}\left[V_2, 1\right] = \mathcal{L}\left[V_\infty, 1\right]$, $\forall \alpha \in [0, 1]$. This means that there exists at least one state feedback controller, $u\left(x(t)\right) = Kx(t)$, satisfying the control input constraints, such that $V_{\overset{\alpha}{\wedge}}$ is a strict CLRF, with decreasing rate η, for the constrained system (10). ∎

Remark 3: The assumption of existence of a linear control $u(t) = Kx(t)$ has been also adopted in the earlier works on stabilization of constrained linear systems by means of polyhedral functions [16], [13], where the Linear Constrained Regulator Problem (LCRP) has been firstly addressed. More recently, the same assumption is required for the feasibility of the BMI problems proposed in [6] for semi-ellipsoidal sets, in [7] for truncated ellipsoids, in [17] for the uniting of two CLFs.

Remark 4: Inequality (18) is non-conservative only if s is allowed to be any integer [14]. Here $s \geq n$ is fixed and although some conservatism is introduced for the assumption that $\gamma_{ijk} = \gamma_{ikj}$, the proposed LMI has been successfully tested in the examples proposed in [16], [13], [18], [15], [4], [6], [7].

More generally, considering the intersection of a PCLF and an arbitrary QCLF (removing the assumption that $\gamma_{ijk} = \gamma_{ikj}$), the following theorem is conclusive for the stabilizability via a smoothed truncated ellipsoidal CLRF.

Theorem 3: Consider the constrained control problem (10) with $\mathcal{X} = \{x \in \mathbb{R}^n : \|Fx\|_\infty \le 1\}$ and $F \in \mathbb{R}^{s\times n}$ full column rank matrix. Assume that there exist $K \in \mathbb{R}^{m\times n}$, $P \in \mathbb{R}^{n\times n}$, $P \succ 0$, $\eta \in \mathbb{R}^+$ and $\gamma_{ijk} \in \mathbb{R}_0^+$, for $i = 1, ..., p$, $j, k = 1, ..., s$, such that

$$(A_i + B_iK)^\top F_k^\top F_k + F_k^\top F_k (A_i + B_iK) \preceq -2\eta F_k^\top F_k + \sum_{j=1}^{s} \gamma_{ijk} \left(F_j^\top F_j - F_k^\top F_k\right)$$
$$(A_i + B_iK)^\top P + P(A_i + B_iK) \preceq -2\eta P \;\; \forall i \in \mathbb{I}_p, \;\; \forall k \in \mathbb{I}_s \tag{21}$$

$$-\bar{1}_m \le Kv^{(l)} \le \bar{1}_m \;\; \forall l, \tag{22}$$

where $v^{(l)}$ are the vertices of the polyhedron $\mathcal{L}[V_\infty, 1]$, and F_k is the k^{th} row of F. Then the smoothed polyhedral CLRFs $V_{\overset{\alpha}{\wedge}}$ (17) is a strict CLF, with decreasing rate η, for system (10) in the set $\mathcal{L}[V_\infty, 1] \cap \mathcal{L}[V_q, 1]$, $\forall \alpha \in [0, 1]$.

Proof: Analogously to Theorem 18, $V_\infty(x)$ is a PLF for the constrained closed-loop LDI and $V_2(x) = x^\top Px$ is a QLF for the unconstrained closed-loop LDI, both with decreasing rate η. Therefore, according to Theorem 1, $V_{\overset{\alpha}{\wedge}}$ is a valid CLRF for the constrained system (10) in the set $\mathcal{L}[V_\infty, 1] \cap \mathcal{L}[V_2, 1]$, $\forall \alpha \in [0, 1]$. ■

Remark 5: Inequality (21) is a BMI in the variables K, P, η, γ_{ijk}. Also in [7] a BMI problem has to be solved for the synthesis of an *unsmooth* TE CLF together with a linear state feedback controller (replacing A with $A + BK$ in equations (18), (19) of [7]). The benefit of the proposed approach with respect to [7] is that if the BMI is feasible, then a *smooth* truncated ellipsoidal CLF is obtained and nonlinear gradient-based controllers can be used improving control performances, as shown in Example 2 of Section IV. If matrix P is fixed, then (21) becomes an LMI.

Corollary 1: Under the same assumptions of Theorem 3, the R-intersection of the smooth PCLF (of the second order) $V_{2p}^2 = \|Fx\|_{2p}^2$ (14) and an arbitrary QCLF $V_2(x) = x^\top Px$ for the unconstrained system, both with decreasing rate η, yields an everywhere differentiable CLRF $V_{\overset{\alpha}{\wedge}}$ with decreasing rate η in $\mathcal{L}[V_{2p}, 1] \cap \mathcal{L}[V_q, 1]$, $\forall \alpha \in [0, 1]$.

The results of Theorem 3 and Corollary 1 are important novelties, since the standard $2p$-norm can not be used to smooth the composition of polyhedral and quadratic functions. Moreover, the everywhere differentiability property outlined in Corollary 1 makes minimum effort control problems (associated to the proposed CLRFs) well-defined [4] and nonlinear gradient-based controllers can be used to improve the control performances of linear ones.

C. Lyapunov-based control

For the explicit derivation of the control law, we consider only the case of certain matrix B, due to the lack of space. The case of uncertain input matrix B is addressed in [4].

The control law $u(t)$ that approximately minimizes $\dot{V}_{\overset{\alpha}{\wedge}}(x(t), u(t))$ at each time instant, over the set $\mathcal{U}$, is the gradient-based control

$$u(x(t)) = -sat\left(\kappa B^\top \nabla V_{\overset{\alpha}{\wedge}}(x(t))^\top\right), \tag{23}$$

where sat is the component-wise vector saturation function and $\kappa \in \mathbb{R}^+$ is sufficiently large.

It is particularly convenient to associate a gradient-based control to an everywhere differentiable CLF, because the corresponding control law is continuous over time [5].

	ISE	ISTE	IADU	T
Example 1				
2P-NORM [18]	1	1	1	1
CLRF	0.96	1.10	**0.54**	1.14
Example 2				
TE [7]	1	1	1	1
CLRF	**0.85**	**0.83**	**0.01**	0.98

TABLE II

THE AVERAGE CONTROL PERFORMANCES ARE RESPECTIVELY NORMALIZED WITH RESPECT TO THE RESULTS OF [18] (*Example 1*) AND [7] (*Example 2*). RESULTS HAVE BEEN OBTAINED AVERAGING OVER 100 SIMULATIONS STARTING FROM RANDOM INITIAL STATES INSIDE THE ADMISSIBLE STATE REGION.

IV. SIMULATIONS

Example 1. Consider the constrained control of the following uncertain linear system [18], with $|\mu(t)| \le 0.5$.

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{bmatrix} 0 & -1.5 + \mu(t) \\ -2 & -1 \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u(t) \tag{24}$$

In [19] a smooth PCLF $\|Fx\|_{2p}$ is designed, where

$$F = \begin{bmatrix} 0 & 4.97 & 4.97 \\ 1 & -0.497 & -0.2485 \end{bmatrix}^\top, \quad p = 6. \tag{25}$$

The framework of R-functions is used to smooth the inner sublevel sets of the PCLF $\|Fx\|_{2p}$. The candidate CLRF $V_{\overset{\alpha}{\wedge}}$ (17) is computed with $P = \frac{1}{3}\sum_{i=1}^{3} F_i^\top F_i$ and $\alpha = 0$. The candidate $V_{\overset{\alpha}{\wedge}}$ (for any $\alpha \in [0, 1]$) is a suitable CLF since the LMI problem (18) of Theorem 2 is feasible.

The smooth PCLF $\|Fx\|_{2p}$ and the CLRF are associated to the gradient-based control (23) with $\kappa = 1$. The Runge-Kutta method with step size 0.001 s is used for the numerical simulations. Table II shows the numerical results averaged over 100 simulations starting from random initial states $x_0 \in \mathcal{L}[\|Fx\|_{2p}, 1]$. The control performances are the Integral of the Squared Error (ISE), the Integral Square Time Error (ISTE), the Integral of the Absolute value of the time Derivative of the control signal u (IADU) and the convergence time (T) inside a given threshold.

Although the error dynamics are actually comparable, the control signal to stabilize the system along the CLRF is smoother than the classic one ($\approx -46\%$). This is due to fact that the smoother non-homothetic level curves of the CLRF, obtained R-composing the PCLF $\|Fx\|_{2p}$ and the QCLF (for the unconstrained system) $\frac{1}{s}x^\top\left(\sum_{i=1}^{s} F_i^\top F_i\right)x$, yield a smoother control signal with respect to a high-order $2p$-norm. Some state controlled trajectories converging to the origin are shown in Figure 2.

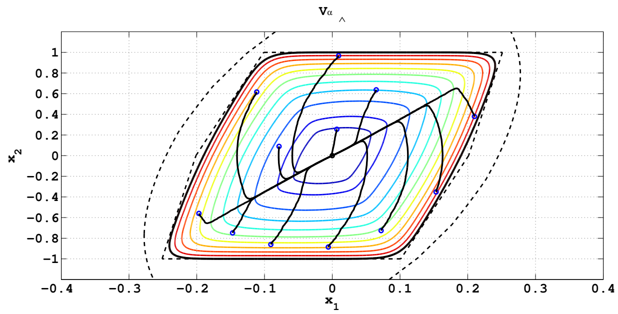

Fig. 2. Some controlled state trajectories converging to the origin in accordance to the sublevel sets of the CLRF.

Example 2. The second example compares the use of a CLRF with respect to a standard TE [7]. The system matrices are

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; F = I_3.$$

In [7] the stability analysis of the closed-loop system $\dot{x}(t) = (A + BK)\, x(t)$ is investigated via a truncated ellipsoidal LF. In particular, the use of a TE improves the volume of the estimated controlled invariant state space region with respect to the semi-ellipsoid proposed in [6]. However, the provided estimation still remains a strict subset of the admissible set $\mathcal{X}$.

On the contrary, the proposed LMI problem (18) is feasible for

$$K = [0,\ 0,\ -1]\,;\ \eta^* = 1.6374 \cdot 10^{-12};$$
$$\gamma_{112} = \gamma_{121} = \gamma_{123} = \gamma_{132} = 1,\ \gamma_{113} = \gamma_{131} = 0. \quad (26)$$

Therefore a smooth CLRF is obtained $\forall \alpha \in [0, 1]$ with *maximal* controlled invariant state space set.

In the numerical simulations, both the standard TE and the smoothed truncated ellipsoidal CLRF are associated to the nonlinear gradient-based controller (23) with $\kappa = 10$. The comparison between the control performances, see Table II, shows that a standard TE can not be used with gradient-based controllers because the control law is actually discontinuous [5].

V. CONCLUSIONS AND FUTURE WORK

In this paper, the class of control Lyapunov R-functions is proposed for the stabilizability of constrained uncertain linear systems. Within the proposed control Lyapunov functions, the maximal estimate of the controlled invariant state space set is obtained. Moreover, the inner sublevel sets can be smoothed and made everywhere differentiable, therefore nonlinear gradient-based controllers can be used as they become well defined. The proposed smoothing technique can be applied to both polyhedral functions and truncated ellipsoids, while a standard high-order norm can not be used to smooth truncated ellipsoidal control Lyapunov functions.

The feasibility problem is addressed via sufficient (bi)linear matrix inequality arguments.

Simulation results show the advantage of having a smoother control Lyapunov function when a nonlinear gradient-based controller is used.

In the control synthesis problem, the quadratic function to be composed with the polyhedral one is either associated to the shape of the polyhedron or left as additional degree of freedom. Future work will investigate the design of a control Lyapunov R-function with inner sublevel sets close to the quadratic optimal ones associated to the nominal unconstrained linear system.

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