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Feasibility of SINR Guarantees for Downlink Transmissions in Relay-Enabled OFDMA Networks

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***Abstract*— We examine the feasibility of the *signal-to-noise-and-interference* (SINR) guarantees for downlink transmissions in relay-enhanced OFDMA networks that feature stationary users. The constraints are as follows: (i) the SINR of every user exceeds a certain threshold and (ii) the transmit power for each transmission is less than a certain threshold. We first derive a set of necessary and sufficient feasibility conditions for the specific case in which a user i served by the relay station shares *at most one* subchannel with a user j served by the base station. These conditions are a function of the target SINR values and the channel gains, and derived using a property of an M-matrix. We then extend these results to the case of networks featuring multiple base stations and multiple relays. Our conditions to check the feasibility can be easily implemented in practice.**

***Index Terms*— OFDMA, downlink transmission, SINR feasibility, M-matrices**

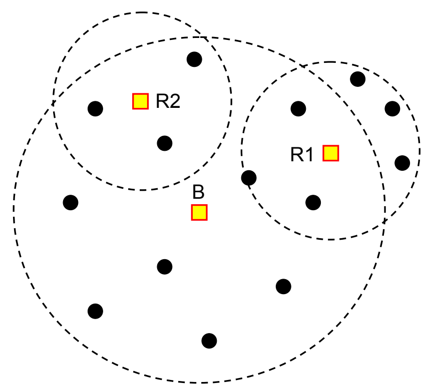

Fig. 1. Two scenarios of a relay-enabled OFDMA network. The squares denote either the base station or a relay whereas the circles denote users. The relay R1 is used to to serve users beyond the coverage range of the base station B while the relay R2 is used to serve the users within the coverage range of the base station who, otherwise, cannot be served due to data rate (or SINR) requirements. The dotted circles denote the coverage range of B, R1, and R2.

I. Introduction

Orthogonal Frequency Division Multiple Access (OFDMA) technology is expected to play a significant part in the next generation wireless networks such as WiMAX, 3GPP-LTE, IEEE 802.22 WRAN (see [1] and [2]). In OFDMA, the base station assigns sets of orthogonal subcarriers, i.e., subchannels, to the users. OFDMA can utilize frequency and multi-user diversities in order to maximize the system capacity. To further improve the system performance, relay stations are often introduced (see [3]): the examples include WiMAX (802.16j) and the 3GPP LTE-Advanced standards (see [4]). The relays are wireless stations that (i) receive the data from a base station and transmit it to a set of users, and (ii) receive the data from a set of users and transmit it to their designated base station. Now, if the SINR requirements of a set of users cannot be satisfied by a base station, could those be satisfied by adding a finite number of relays and, if so, where should the relays be placed? We investigate this problem in the context

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of downlink transmit power allocation in a relay-enabled OFDMA network in which the number of users exceeds the number of subcarriers (see Figure 1).

We build on the framework developed in [5] to answer these questions by using the same properties of the M-matrices that enable it to characterize the maximum number of uplink transmissions that a base station can support in a multi-cell *code division multiple access* (CDMA) network. Let the $\overline{\gamma}_i$ denote the target *signal-to-noise-and-interference* (SINR) values of User i and let L denote the CDMA spreading gain. Then, the base station can support the *uplink* transmissions from M users simultaneously if $\sum_{i=1}^{M} \overline{\gamma}_i/(\overline{\gamma}_i + L) < 1$ (see [5, Lemma 2]). We consider a similar convex optimization problem of minimizing the sum *downlink* transmit power subject to the constraint that the SINR exceeds the desired threshold for all users. We make use of certain properties of M-matrices to establish the required feasibility conditions and extend these results to the general case of networks featuring multiple base stations and muliple relays.

II. OFDMA System Description

We assume that the OFDMA network differentiates between the uplink and downlink traffics, either in time (e.g., using time-domain duplexing) or frequency (e.g., using frequency-domain duplexing). In addition, we assume that, the downlink transmissions from a base station to the relays (i.e., the so-called *relay zones*) are separated from the transmissions from the base station or relays to the users (i.e., the so-called *access zones*), again in time or frequency. This is commonly used, e.g. in WiMAX relay operations (see [6]), so that users do not require any modifications to support relay operations. Thus, the downlink transmission progresses as the following repeating sequence of two time-slots: (i) in the first time-slot, the base station transmits the data to the relay station, and (ii) in the next time-slot, the base station and the relay transmit the data to their users.

TABLE I
NOTATION

Symbol	Meaning
$\mathcal{M}_1$	The set of users served by the base station
$\mathcal{M}_2$	The set of users served by the relay
$\phi(i)$	The set of users interfering with user i
$s(i)$	Sender (B or R) of the downlink transmission to user i
$h_{is(i)}$	The channel gain for transmission to user i from its sender $s(i)$ (which is either B or R)
$h_{is(\phi(i))}$	The channel gain for transmission to user i from the sender $s(\phi(i))$ of $\phi(i)$
$h_{\phi(i)s(i)}$	The channel gain for transmission to user $\phi(i)$ from the sender $s(i)$ of i
$\overline{\gamma}_i$	The target SINR for user i
p_i	Downlink transmission power for user i
$\text{diag}(x_i)$	Diagonal matrix with x_i as its elements
$\mathcal{N}_M$	Relay-enabled network with 1 base station and M users s.t. a user served by B shares at most one subchannel with a user served by R
$\mathcal{N}_{M\ell}$	Relay-enabled network with 1 base station and M users s.t. a user served by B shares up to ℓ subchannels with a user served by R
$A_{(ij)}$	(i,j)-th element of the matrix A
$\mathcal{Z}$	Set of non-negative integers

The OFDMA wireless network comprises a base station "B", a relay station "R", N subchannels and M users, indexed $1, 2, \ldots, M$, with $2N > M > N$. Without loss of generality, we assume that $M_1 = N$ users receive the data directly from the base station; let us refer to this set of users as $\mathcal{M}_1$. The remaining $M_2 = M - N$ users receive the data from the relay station; let us refer to this set of users as $\mathcal{M}_2$. The base station assigns an orthogonal channel each to the $\mathcal{M}_1$ users while the relay station assigns an orthogonal channel each to the $\mathcal{M}_2$ users. Thus, a $\mathcal{M}_1$ user may experience interference from at most one $\mathcal{M}_2$ user and vice versa. So, given a user i, let $\phi(i)$ denote the user interfering with it. Let $\mathcal{M}$ denote the union of $\mathcal{M}_1$ and $\mathcal{M}_2$. Without loss of generality, we index the users in $\mathcal{M}$ such that the first N users are from $\mathcal{M}_1$. Let us denote the transmission power of the base station for the i-th user as p_{iB} and that of the relay station for the i-th user as p_{iR}. With a slight abuse of notation, we will, at times, refer to p_{iB} and p_{iR} as p_i. All transmit powers p_i obey the constraint $p_i \in [0, p_{max}]$ for some pre-defined p_{max}; we assume p_{max} is sufficiently large. The channel gain from the base station to the i-th user is denoted h_{iB} and the channel gain from the relay to the i-th user is denoted h_{iR}. At times, we shall refer to these channel gains as $h_{is(i)}$, i.e., channel gain for the transmission of the sender $s(i)$ of user i to user i — the sender $s(i)$ can be either the base station B or the relay station R. Both h_{iB} and h_{iR} are constrained to lie between 0 and 1. The received signal at the i-th $\mathcal{M}_1$ user is $x_i = h_{iB}p_i + h_{\phi(i)R}p_{\phi(i)}$. Likewise, the received signal at the i-th $\mathcal{M}_2$ user is $x_i = h_{iR}p_i + h_{\phi(i)B}p_{\phi(i)}$. Thus, the SINR for the i-th $\mathcal{M}_1$ user is given as

$$\gamma_i = \frac{h_{iB}p_i}{h_{\phi(i)R}p_{\phi(i)} + \sigma_i^2},$$

and the SINR for the i-th $\mathcal{M}_2$ user is given as

$$\gamma_i = \frac{h_{iR}p_i}{h_{\phi(i)B}p_{\phi(i)} + \sigma_i^2},$$

where σ_i^2 is the background noise on the subchannel assigned to the i-th user. We refer to this network as $\mathcal{N}_M$. To better illustrate our notation, let us consider the case of an OFDMA wireless network having 2 subchannels, say f_1 and f_2, and a single base station enhanced by a relay station. Let there be 3 users, indexed $1, 2, 3$. Thus, in this example, $N = 2$ and $M = 3$. Suppose the base station assigns the subchannels f_1 and f_2 to users 1 and 2, respectively, and does not serve the third user, who is served by the relay station instead. Suppose the relay station assigns the subchannel f_2 to user 3. Thus, user 1 experiences no interference whereas the users 2 and 3 interfere with each other. The SINRs for the users are: $\gamma_1 = \frac{h_{1B}p_1}{\sigma_1^2}$, $\gamma_2 = \frac{h_{2B}p_2}{h_{2R}p_3+\sigma_2^2}$, $\gamma_3 = \frac{h_{3R}p_3}{h_{3B}+\sigma_3^2}$. Let us refer to this network as $\mathcal{N}_3$.

III. Problem Formulation

Let M denote the number of users in the OFDMA networks. Then, the optimization problem that we wish to solve is as follows:

$$\min_p \sum_{i=1}^{M} C_i(p_i) \quad \text{s. t.} \quad \gamma_i \geq \overline{\gamma}_i, 0 \leq p_i \leq p_{max} \quad \forall i, \tag{1}$$

where $C_i(\cdot)$ is a convex continuously differentiable function, $p \doteq [p_1 \ p_2 \ \ldots \ p_M]^T$ is the vector of the downlink transmit powers and $\overline{\gamma}_i$ is the target SINR for user i. To better illustrate our approach, we first derive the results for the network $\mathcal{N}_3$ described above and then generalize the results to arbitrarily large M. For this network, $M = 3$. Let

$$\begin{aligned} A &\doteq \begin{bmatrix} h_{1B} & 0 & 0 \\ 0 & h_{2B} & -h_{2R}\overline{\gamma}_2 \\ 0 & -h_{3B}\overline{\gamma}_3 & h_{3R} \end{bmatrix}, \qquad (2) \\ b &\doteq [\overline{\gamma}_1\sigma_1^2 \ \ \overline{\gamma}_2\sigma_2^2 \ \ \overline{\gamma}_3\sigma_3^2]^T, \\ \Omega &\doteq \{p \in \mathbb{R}^N : Ap \geq b, \ p_i \in [0, p_{max}] \ \forall i\}. \end{aligned}$$

Then, the optimization problem given by (1) is recast as

$$\min_p \sum_{i=1}^{M} C_i(p_i) \quad \text{subject to} \quad p \in \Omega. \tag{3}$$

We say that the optimization problem given by (3) is *feasible* if Ω is non-empty. For the general case of M users, the matrix A, described by equation (2) for the 3-user network $\mathcal{N}_3$, is as follows:

$$A = H_D + H_{OD}, \tag{4}$$

$$H_D \doteq \text{diag}(h_{is(i)}), \tag{5}$$

$$(H_{OD})_{ij} \doteq \begin{cases} -h_{jB}\overline{\gamma}_i & \text{if } i \in \mathcal{M}_1 \text{ and } j = \phi(i); \\ -h_{jR}\overline{\gamma}_i & \text{if } i \in \mathcal{M}_2 \text{ and } j = \phi(i); \\ 0 & \text{else.} \end{cases} \tag{6}$$

IV. Upper Bound on Simultaneous Downlink Transmissions in $\mathcal{N}_3$ and $\mathcal{N}_M$

To determine the largest target SINR values that can be supported by a relay-enhanced base station in downlink transmissions in $\mathcal{N}_3$, we now extend the technique used in [5] to prove its Lemma 3.1 (also see [7]). We first note that a matrix C is said to be an *M-matrix* if it can be decomposed as $C = kI - B$ where k is a positive real number, I is the identity matrix of suitable size, and B is a matrix comprising non-negative elements such that the spectral radius of B is strictly less than k (see [8] and [9] for details on M-matrices).

Lemma 1: [SINR Guarantees in $\mathcal{N}_3$]
The optimization problem given by (2) and (3) is feasible if and only if

$$\frac{h_{2R}}{h_{2B}}\overline{\gamma}_2 < 1 \quad \text{and} \quad \frac{h_{3B}}{h_{3R}}\overline{\gamma}_3 < 1. \tag{7}$$

Furthermore, if (7) holds, then every p satisfying $Ap \geq b$ satisfies $p > 0$. □

Proof: To prove the result, we use the approach used in [5] to prove [5, Lemma 3.1]. Let us rewrite $Ap \geq b$ as $\widetilde{A}x \geq b$ where $x \doteq [x_1\ x_2\ x_3]^T$, $x_i \doteq h_{is(i)}p_i\ (i = 1, 2, 3)$, and

$$\widetilde{A} \doteq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{h_{2R}}{h_{2B}}\overline{\gamma}_2 \\ 0 & -\frac{h_{3B}}{h_{3R}}\overline{\gamma}_3 & 1 \end{bmatrix}. \tag{8}$$

Now, if the condition (7) holds, then $\widetilde{A}$ is a nonsingular M-matrix. Note that b is component-wise positive. Hence, it follows from the result [8, Chapter 6, N39] that $\widetilde{A}x \geq 0$ implies $x \geq 0$, i.e., $p \geq 0$ since the channel gains $h_{is(i)} \in (0, 1]$. This proves that if the condition (7) holds, then every p satisfying $Ap \geq b$ also satifies $p \geq 0$ whence Ω is non-empty, i.e., the optimization problem given by (2) and (3) is feasible.

To prove that if the condition (7) does not hold then Ω is empty, let us use a contradiction. Suppose, on the contrary, the condition (7) does not hold but Ω is non-empty. Thus, by assumption, $\widetilde{A}$ is not an M-matrix and there exists $x \geq 0$ such that $\widetilde{A}x \geq b$. Since each element of b is strictly positive, it follows that x and Ax are componentwise strictly positive. Hence, by [8, Chapter 6, I39], $\widetilde{A}$ is an M-matrix, which is a contradiction. Hence the proof. ■

It may be verified that the feasibility of SINR guarantees in $\mathcal{N}_M$ is characterized as follows.

Theorem 1: [SINR Guarantees in $\mathcal{N}_M$]
Consider the network $\mathcal{N}_M$. The optimization problem given by (3)–(6) is feasible if and only if

$$\frac{h_{is(\phi(i))}}{h_{is(i)}}\overline{\gamma}_i < 1 \tag{9}$$

for every user i that shares a subchannel with some other user $\phi(i)$. Furthermore, if (9) holds, then every p satisfying $Ap \geq b$ satisfies $p > 0$. □

Next, the consider the general case of multi-cell networks with L base stations. Let $h^{[\ell]}_{is(j)}$ be the transfer function for the communication channel for transmission from $s(j)$ (which is either a base station or a relay) to user i on the ℓ-th subchannel — $h^{[\ell]}_{ij}$ is taken to be zero if $s(j)$ does not cause interference to user i on the ℓ-th subchannel.

Theorem 2: [SINR Guarantees in General Case]
Consider an OFDMA network featuring multiple base stations, multiple relays, and possibly multiple shared subchannels for downlink transmissions to users. The generalized optimization problem is feasible if and only if

$$\sum_{j \in \phi(i)} \sum_{\ell \in \Xi(i)} \frac{h^{[\ell]}_{is(j)}}{h^{[\ell]}_{is(i)}}\overline{\gamma}_i < 1 \tag{10}$$

for every user i. Furthermore, if (10) holds, then every p satisfying $Ap \geq b$ satisfies $p > 0$. □

TABLE II

SIMULATION PARAMETERS

	Base Station	Relay 1	Relay 2	User 1	User 2	User 3	User 4
Coordinates	$(300, 300)$	$(593, 300)$	$(7, 300)$	$(365, 300)$	$(690, 350)$	$(670, 100)$	$(10, 150)$
Case 1	Used	Used	Unused	$\overline{\gamma}_1 = 10$	$\overline{\gamma}_2 = 10$	$\overline{\gamma}_3 = 10$	$\overline{\gamma}_4 = 10$
Case 2	Used	Used	Used	$\overline{\gamma}_1 = 10$	$\overline{\gamma}_2 = 10$	$\overline{\gamma}_3 = 10$	$\overline{\gamma}_4 = 10$
Case 3	Used	Used	Used	$\overline{\gamma}_1 = 18$	$\overline{\gamma}_2 = 10$	$\overline{\gamma}_3 = 18$	$\overline{\gamma}_4 = 10$
Case 4	Used	Used	Used	$\overline{\gamma}_1 = 18$	$\overline{\gamma}_2 = 18$	$\overline{\gamma}_3 = 18$	$\overline{\gamma}_4 = 18$

TABLE III

VALUES OF TERMS IN THE FEASIBILITY CONDITIONS - CASE 1

	$\frac{h^{[1]}_{1R1}}{h^{[1]}_{1B}}\overline{\gamma}_1$	$\frac{h^{[1]}_{2B}}{h^{[1]}_{2R1}}\overline{\gamma}_2$	$\frac{h^{[2]}_{3B}}{h^{[2]}_{3R1}}\overline{\gamma}_3$	$\frac{h^{[2]}_{3R1}}{h^{[2]}_{3B}}\overline{\gamma}_4$
Case 1	0.813	0.770	2.60	2.94

V. SIMULATION RESULTS

We now present simulation results for a single-cell network comprising a base station, 2 relays and 4 users. The base station has 2 subchannels available to it. We assume that these subchannels have a path-loss exponent of 2. Thus, the gain of the communication channel between a node i and a node j is given as $h_{ji} = 5000/d_{ji}^2$, where d_{ji} is the distance between nodes i and j. The background noise on each subchannel at each node is taken to be $\sigma^2 = 0.1$.

In Case 1 (see Table II), the base station communicates directly with users 1 and 4 on channels 1 and 2, respectively. Relay 1 is used to communicate with users 2 and 3, on channels 1 and 2, respectively, whereas Relay 2 is not used at all. For Case 1, as per our Theorem 2, the target SINR values are feasible for all users if and only

$$\frac{h^{[1]}_{1R1}}{h^{[1]}_{1B}}\overline{\gamma}_1 < 1, \quad \frac{h^{[1]}_{2B}}{h^{[1]}_{2R1}}\overline{\gamma}_2 < 1 \quad \frac{h^{[2]}_{3B}}{h^{[2]}_{3R1}}\overline{\gamma}_3 < 1, \quad \frac{h^{[2]}_{4R1}}{h^{[2]}_{4B}}\overline{\gamma}_4 < 1.$$

Table III shows that these feasiblility conditions are not met. So, we expect that even if an appropriate stabilizing power controller is used, the target SINR values will not be achieved for some of the users. To test this hypothesis, we updated the downlink transmit power at the base station using the provably stable power control algorithm described in [10], viz., [10, Algorithm 2.1]. Specifically, the transmit power p_i of the base station (or the relay station) for the downlink transmission to user i is updated at each time instant k as follows:

$$p_i[k+1] = \frac{\overline{\gamma}_i}{\gamma_i[k]} p_i[k], \quad k \in \mathcal{Z} \tag{11}$$

where $\gamma_i[k]$ is the SINR at user i at time instant k and $\mathcal{Z}$ is the set of integers. The initial power level is taken to be 0.1 for the base station as well as for the relay. The instantaneous SINR values are plotted in Figure 2. Figure 2 validates our hypothesis since the target SINR values are clearly not achievable for users 3 and 4.

To satisfy the target SINR requirement of all users, we shall add a relay station, viz., relay 2, to serve user 4. In all

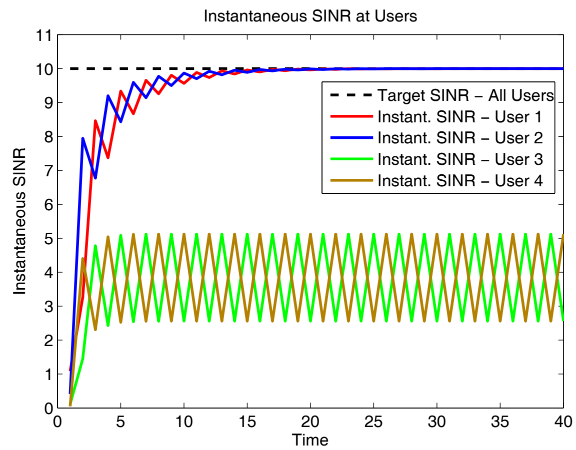

Fig. 2. Case 1: Instantaneous SINR values of all users. Since the conditions in Theorem 2 are *not* satisfied, the target SINR values are infeasible. Using of the transmit power algorithm given by (11), although the target SINR of users 1 and 2 could be met, the instantaneous SINR of users 3 and 4 oscillate around a baseline value of 4.

TABLE IV
VALUES OF TERMS IN THE FEASIBILITY CONDITIONS: CASES 2,3, AND 4

	$\frac{h_{1R1}^{[1]}}{h_{1B}^{[1]}}\overline{\gamma}_1$	$\frac{h_{2B}^{[1]}}{h_{2R1}^{[1]}}\overline{\gamma}_2$	$\frac{h_{3R2}^{[2]}}{h_{3R1}^{[2]}}\overline{\gamma}_3$	$\frac{h_{3R1}^{[2]}}{h_{3R2}^{[2]}}\overline{\gamma}_4$
Case 2	0.813	0.770	0.958	0.621
Case 3	1.46	0.770	1.72	0.621
Case 4	1.46	1.39	1.72	1.12

of the subsequent scenarios, the base station communicates directly with user 1 on channel 1, uses relay 1 to communicate with users 2 and 3 on channels 1 and 2, respectively, and uses relay 2 to communicate with user 4 on channel 2. Now, as per our Theorem 2, the target SINR values are feasible for all users if and only if all of the following conditions are satisfied:

$$\frac{h_{1R1}^{[1]}}{h_{1B}^{[1]}}\overline{\gamma}_1 < 1,\ \frac{h_{2B}^{[1]}}{h_{2R1}^{[1]}}\overline{\gamma}_2 < 1,\ \frac{h_{3R2}^{[2]}}{h_{3R1}^{[2]}}\overline{\gamma}_3 < 1,\ \frac{h_{4R1}^{[2]}}{h_{4R2}^{[2]}}\overline{\gamma}_4 < 1.$$

Now consider 3 different scenarios, viz., Case 2, Case 3, and Case 4, of target SINR specifications as tabulated in Table II. Table IV shows the new feasibility condition values with the addition of relay station 2. We see in Case 2, even though the target SINR values remain the same as Case 1, the feasibility conditions are now met, since all the values are less than 1. Hence, we expect that the target SINR values to be achievable. In Cases 3 and 4 (see Table II), some of the feasibility conditions stated by Theorem 2 are not met so that we expect some of the SINRs to be not achievable. Our simulation results support these conclusions.

VI. CONCLUSION

We have investigated the feasibility of SINR guarantees in relay-enhanced OFDMA networks featuring stationary users. Our approach is based on checking the feasibility of a convex optimization problem on minimizing the total downlink transmit power subject to the constraints that (i) the SINR of every user exceeds a certain threshold and (ii) the transmit power is not to exceed a certain threshold. We first check the feasibility of solution for the specific case (viz., the $\mathcal{N}_M$ network) in which a user i served by the relay station shares at most one subchannel with a user j served by the base station. We obtain analytical conditions (viz., Lemma 1, Theorem 1, and Theorem 2) for the existence of a solution.

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