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Full Order Observers for Linear DAEs

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Abstract— **Observer design for descriptor systems, or systems of differential algebraic equations (DAEs) as they are also known, is well studied in the linear time invariant (LTI) case. However, those studies do not readily extend to general linear time varying (LTV) or nonlinear descriptor systems. This paper presents an alternative approach for observer design that not only works for the LTI case but also shows great potential for the design of observers for general LTV descriptor systems.**

I. Introduction

Many physical systems are most naturally modeled as systems of differential algebraic equations (DAEs) [4]. Considerable effort has been expended on designing simulation and analysis tools for DAEs. Observers play an important role in control theory, so it is natural to consider observers for DAEs. Work to date has focused on the linear time invariant (LTI) case [2], [9], [10], [11] and does not easily extend to the linear time varying (LTV) or nonlinear cases. Recently there has been progress on generating stabilized completions of LTV and nonlinear DAEs [7], [13]. This work was originally developed with an eye toward numerical simulation of DAEs. In this paper, we begin to consider how these recent results can be used to develop theory and algorithms for observers for linear and nonlinear DAEs by focusing on the LTI case.

Section II presents the general approach and background information. Section III develops connections between completions and observable subspaces. Sections IV and V each discuss a method for generating a stabilized completion. Section VI gives a computational example. Section VII discusses some technical details that must be addressed before extending this approach to LTV DAEs. This approach for observer design also shows great potential for nonlinear DAEs, but that discussion is beyond the scope of this paper. Conclusions are in Section VIII. Due to space limitations we assume that the reader is familiar with basic DAE theory [4].

II. The General Approach

We start with a solvable LTI DAE

$$E\dot{x}(t) + Fx(t) = Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where E, F, B, C, D are appropriately sized matrices. Here (1a) is the process and (1b) is the output equation. Additional known inputs can be included in (1a) without difficulty.

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Our goal is to define an ordinary differential equation (ODE) observer with state estimator $\hat{x}(t)$ such that $\|x(t) - \hat{x}(t)\| = O(\exp(-\delta t)\|x(0) - \hat{x}(0)\|)$ for any $\hat{x}(0)$ and for any consistent initial condition $x(0)$ for (1a). Here $\delta > 0$ gives the convergence rate that we want for our observer. A similar appearing problem was considered in [1], but there are several key differences between [1] and this paper. In [1] $\hat{x}(t)$ was consistent with (1a), but in this paper $\hat{x}(t)$ satisfies (1a) asymptotically. Also, the results on stabilized completions were not yet available so the algorithms in [1] are much more complex. Finally, unlike this paper, the approach of [1] does not have the potential for being extended to nonlinear problems.

A completion of a DAE is a system of ODEs whose solutions include those of the DAE. Ways of converting a DAE into an ODE have been used since at least the 1970s. However, if the problem was not LTI, these approaches required the problem to have explicit constraints and structure. The first general algorithm was based on least squares solutions of the derivative array [5] and formed the least squares completion (LSC). However, a completion has solutions in addition to those of the original DAE, and it was shown that these additional dynamics could be unstable [6]. Recently an investigation into these additional dynamics has begun [7], [13], including showing how to modify the process of computing the completion so that the additional dynamics could have some desired stability properties [7].

The approach of this paper may be summarized as follows. We choose a convergence rate for the observer and construct a stabilized completion whose additional dynamics converge faster than this desired rate. We then design an observer for the stabilized completion with a convergence rate that is at least the desired rate.

III. Completions and Observability

First we examine how computing a completion affects observability. If computed from the derivative array equations as described later, the completion takes the form

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \sum_{i=0}^{k} \tilde{B}_i u^{(i)}(t) \tag{2}$$

independent of how the completion is computed. Here $u^{(i)}$ is the ith time derivative. From the theory of DAEs, there is a full row rank matrix M such that

$$M\tilde{x}(t) = \sum_{i=0}^{k} \hat{B}_i u^{(i)}(t) \tag{3}$$

defines the solution manifold of (1a). M is assumed without loss of generality to be full row rank and is determined while computing the completion. In [3] we use (3) to modify the reduced order observer approach of [14]. Any other completion computed from the derivative array is $\dot{\check{x}}(t) = (\tilde{A} + \Delta M)\check{x}(t) + \sum_{i=0}^{k} \check{B}_i u^{(i)}(t)$ for some matrix Δ [6].

Theorem 1: Let $\mathcal{O}_X$ be the observability matrix for the pair (X, C). Let $N(Y)$ denote the nullspace of Y. Then

$$N(M) \cap N(\mathcal{O}_{\tilde{A}}) = N(M) \cap N(\mathcal{O}_{\tilde{A}+\Delta M}). \tag{4}$$

In particular, each finite eigenvalue of the matrix pencil $sE + F$ is either observable or unobservable for all completions. That is, changing the completion does not alter which finite matrix pencil eigenvalues are observable.

Proof: $N(M)$ and $N(\mathcal{O}_{\tilde{A}})$ are $\tilde{A}$ invariant. $N(M)$ is invariant since $Mx(t) = 0$ describes the solutions of $E\dot{x}(t)+Fx(t) = 0$. If $\phi \in N(M)$, then $(\tilde{A} + \Delta M)\phi = \tilde{A}\phi$. But $\tilde{A}\phi$ is again in $N(M)$ by invariance. Hence $(\tilde{A}+\Delta M)^r\phi = \tilde{A}^r\phi$ for all integers $r \geq 0$. Theorem 1 now follows. ■

Let us examine the situation more carefully. Let V_1 and V_2 be subspaces such that $V_2 \oplus N(M) \cap N(\mathcal{O}_{\tilde{A}}) = N(M)$ and $V_1 \oplus N(M) = \mathbb{R}^n$. Assuming we pick a compatible set of coordinates, we get that

$$\tilde{A} = \begin{bmatrix} \mathrm{A}_1 & 0 & 0 \\ \mathrm{A}_2 & \mathrm{A}_3 & 0 \\ \mathrm{A}_4 & \mathrm{A}_5 & \mathrm{A}_6 \end{bmatrix}, \quad M = \begin{bmatrix} \mathrm{M}_1 & 0 & 0 \end{bmatrix},$$

and $C = \begin{bmatrix} \mathrm{C}_1 & \mathrm{C}_2 & 0 \end{bmatrix}$. Since $N(\mathcal{O}_{\tilde{A}}) \subset N(C)$ we know M_1 is invertible and C_2 is full column rank. Then

$$\tilde{A} + \Delta M = \begin{bmatrix} \mathrm{A}_1 + \Delta_1\mathrm{M}_1 & 0 & 0 \\ \mathrm{A}_2 + \Delta_2\mathrm{M}_1 & \mathrm{A}_3 & 0 \\ \mathrm{A}_4 + \Delta_3\mathrm{M}_1 & \mathrm{A}_5 & \mathrm{A}_6 \end{bmatrix}.$$

But for a given M_1, the matrix Δ can be chosen so the first block column of $\tilde{A} + \Delta M$ is anything. In particular, the rank of the observability matrix can vary from one completion to another, and thus, observability properties can vary between completions. The computational example will show that the rank of the observability matrix can differ even when working with the most widely used completions.

IV. Using Stabilized Differentiation

The first method of computing a completion uses stabilized differentiation and forms the stabilized LSC. Suppose that E, F, B, C, D are constants and that $\{E, F\}$ is a regular pencil. That is, $sE + F$ is square and is not identically singular as a function of s. Suppose the index is k and the desired convergence rate of the observer is δ. Take $\lambda > \delta$. Then the stabilized derivative array is formed by applying the differential polynomial $\frac{d}{dt} + \lambda$ to (1a) k times, giving

$$\mathcal{E}w(t) + \mathcal{F}x(t) = \mathcal{B}v(t), \tag{5}$$

the stabilized version of the derivative array equations where

$$\mathcal{F} = \begin{bmatrix} \lambda^0 F \\ \vdots \\ \lambda^k F \end{bmatrix}, w(t) = \begin{bmatrix} \dot{x} \\ \vdots \\ x^{(k+1)} \end{bmatrix}, v(t) = \begin{bmatrix} u \\ \vdots \\ u^{(k)} \end{bmatrix},$$

and $\mathcal{E}$, $\mathcal{B}$ respectively are the following $(k+1) \times (k+1)$ block matrices,

$$\begin{bmatrix} E & 0 & \dots & 0 \\ \lambda E + F & E & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & E \end{bmatrix}, \begin{bmatrix} B & 0 & \dots & 0 \\ \lambda B & B & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & B \end{bmatrix}.$$

In the LTI case the index k of the DAE may be taken as the smallest nonnegative integer such that $[\mathcal{E}, \mathcal{F}]$ is full row rank and the first $\dim(x)$ columns of $\mathcal{E}$ are independent of the other columns of $\mathcal{E}$ [4]. In the LTV case there is an additional condition that $\mathcal{E}$ has constant rank. In practice a subset of the equations (5) may suffice for carrying out subsequent calculations and would not affect our results. For instance the example (12) is a Hessenberg system [4] so that three differentiations of (12c), two differentiations of (12b), and one differentiation of (12a) could be used to build the derivative array (5).

Previous theory shows $[\mathcal{E}, \mathcal{F}]$ is full row rank but $\mathcal{E}$ is neither full row nor full column rank. Taking the least squares solution of (5) for $w(t)$ given x, u uniquely determines $\dot{x}(t)$ and results in the stabilized LSC with output

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \sum_{i=0}^{k} \tilde{B}_i u^{(i)}(t) \tag{6a}$$

$$\tilde{y}(t) = C\tilde{x}(t) + Du(t). \tag{6b}$$

The LSC comes from the first block row of $\mathcal{E}^\dagger [\mathcal{B} \ -\mathcal{F}]$ where $\mathcal{E}^\dagger(\mathcal{B}v(t) - \mathcal{F}x(t))$ is the least squares solution of (5) for $w(t)$ and $\mathcal{E}^\dagger$ is the Moore-Penrose inverse of $\mathcal{E}$ [8]. All solutions of (1a) are also solutions of (6a), which can be written as $\tilde{x}(t) = x(t) + \bar{x}(t)$. $x(t)$ satisfies (1a), and

$$\frac{d}{dt}(H\bar{x}(t)) = (-\lambda I + N)H\bar{x}(t) + g(t) \tag{7}$$

is satisfied by $H\bar{x}(t)$ for N a nilpotent matrix of index k and H an appropriate coordinate change. Thus $\bar{x}(t)$ is a linear combination of functions with form $t^j e^{-\lambda t}$ for $0 \leq j \leq k-1$.

The actual outputs $y(t)$ from (1) only involve $x(t)$ that satisfy (1a). However, since (6) is constructed so $\bar{x}(t)$ goes to zero fast enough, it suffices to construct our observer for (1) based on (6). Using the usual techniques for observer design, an observer for (6) has the form

$$\dot{\hat{x}}(t) = \tilde{A}\hat{x}(t) + \sum_{i=0}^{k} \tilde{B}_i u^{(i)}(t) + L(y(t) - \hat{y}(t)), \tag{8a}$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t). \tag{8b}$$

The pair $(\tilde{A}, C)$ does not need to be observable, but all eigenvalues γ of $\tilde{A}$ with $Re(\gamma) \geq -\delta$ must be observable. We refer to this condition as strongly detectable. Additionally, L is chosen, using the usual algorithms, so that all eigenvalues ψ of $\tilde{A} - LC$ have $Re(\psi) < -\delta$.

The existence of derivatives of $u(t)$ in (8a) raises questions of both a computational and a practical nature, but it is important to note two things. First, $u(t)$ is a known input. In many problems it is possible to provide derivatives, at least to low order, of this input. Second, in particular problems many $\tilde{B}_i$ are often zero. This structure is illustrated in Section VI.

V. Using the Alternative Stabilization

An approach is introduced in [12] that numerically finds an index one DAE and integrates it with a BDF method. We modified this approach in [13] to get the alternative stabilized completion (ASC). The method of computing a completion using alternative stabilization is computationally more intricate to program but has a number of potential advantages in the generation of stabilized dynamics to be used in observers for LTV and nonlinear systems. In this section we review how the extra dynamics of the ASC have merely $e^{-\lambda t}$ terms rather than $t^j e^{-\lambda t}$ terms for LTI systems.

$\mathcal{E}$ is singular. A wide tilde on matrices $\mathcal{E}, \mathcal{F}, \mathcal{B}$ indicates the last block row has been deleted. These modified matrices include enough information from the derivative array equations to explicitly give the constraints defining the solution manifold as well as the continuous derivatives but not enough to determine all the derivatives of the algebraic variables.

Let Z_2 be a matrix whose columns form an orthonormal basis for $R(\widetilde{\mathcal{E}})^\perp$. Thus $Z_2^T\widetilde{\mathcal{E}} = 0$. Let $L = k-1$ and $Z_2^T = [Z_{2,0}^T, \ldots, Z_{2,L}^T]$. Let T_2 be a matrix whose columns form an orthonormal basis for $N(Z_2^T\widetilde{\mathcal{F}})$. Since $\widetilde{\mathcal{F}}^T = [F^T, 0, \ldots, 0]^T$, we have $Z_2^T\widetilde{\mathcal{F}} = Z_{2,0}^T F$. Let Z_1 be a matrix whose d columns form an orthonormal basis for $R(ET_2)$. The matrix $\begin{bmatrix} Z_1^T E \\ Z_{2,0}^T F \end{bmatrix}$ is square and invertible. Observe that Z_1, Z_2, T_2 are unique up to right multiplication by an orthogonal transformation. Now define

$$\Gamma = \begin{bmatrix} Z_1^T & 0_{d\times kn} \\ Z_2^T & 0_{a\times n} \\ 0_{a\times n} & Z_2^T \\ & Z_3^T \end{bmatrix} = \begin{bmatrix} Z_1^T & 0 & \cdots & 0 \\ Z_{2,0}^T & \cdots & Z_{2,L}^T & 0 \\ 0 & Z_{2,0}^T & \cdots & Z_{2,L}^T \\ Z_{3,0}^T & Z_{3,1}^T & \cdots & Z_{3,k}^T \end{bmatrix}.$$

The Z_3^T are extra orthonormal rows, orthogonal to the other rows, that make Γ invertible. Γ is conformal with $\mathcal{E}$ but not $\widetilde{\mathcal{E}}$ and is unique up to an orthogonal matrix on the left.

Instead of solving $\mathcal{E}w(t) = \mathcal{B}v(t) - \mathcal{F}x(t)$ in the least squares sense, we shall solve $\Gamma\mathcal{E}w(t) = \Gamma(\mathcal{B}v(t) - \mathcal{F}x(t))$ using least squares. That is, $\bar{w}(t) = (\Gamma\mathcal{E})^\dagger\Gamma(\mathcal{B}v(t) - \mathcal{F}x(t))$. This answer is unique and independent of how the Z_1, Z_2, T_2 are constructed numerically. While the intermediate calculations are done pointwise and may not be smooth in t, the final answer is smooth in t. For the remainder of this section, the t dependence has been omitted and $\widetilde{\mathbf{f}}'$ rather than $\dot{\widetilde{\mathbf{f}}}$ has been used to improve readability. The augmented form of the derivative array equations $[\mathcal{E}||\mathcal{F}|\mathbf{f}]$ is $[\Gamma\mathcal{E}||\Gamma\mathcal{F}|\Gamma\mathbf{f}]$, where $\mathbf{f} = \mathcal{B}v$. This system is

$$\left[\begin{array}{c|cc||c|c} Z_1^T E & 0 & \cdots & -Z_1^T F & Z_1^T Bu \\ 0 & 0 & \cdots & -Z_{2,0}^T F & Z_2^T\widetilde{\mathbf{f}} \\ Z_{2,0}^T F & 0 & \cdots & 0 & Z_2^T\widetilde{\mathbf{f}}' \\ \hline W_0 & J_1 & \cdots & W_1 & W_2 \end{array}\right]. \tag{9}$$

The third row of (9) is the derivative of the second row. We replace (9) with its stabilized differentiation version

$$\left[\begin{array}{c|cc||c|c} Z_1^T E & 0 & \cdots & -Z_1^T F & Z_1^T Bu \\ 0 & 0 & \cdots & -Z_{2,0}^T F & Z_2^T\widetilde{\mathbf{f}} \\ Z_{2,0}^T F & 0 & \cdots & -\lambda Z_{2,0}^T F & Z_2^T\widetilde{\mathbf{f}}' + \lambda Z_2^T\widetilde{\mathbf{f}} \\ \hline W_0 & J_1 & \cdots & W_1 & W_2 \end{array}\right]. \tag{10}$$

But $\begin{bmatrix} Z_1^T E \\ Z_{2,0}^T F \end{bmatrix}$ is invertible and the J block is full row rank. In this special circumstance the Moore-Penrose inverse of the block lower triangular matrix to the left of || in (10) is also a block lower triangular matrix [8]. Thus the dynamics of this new completion are given by

$$\dot{\tilde{x}} = \begin{bmatrix} Z_1^T E \\ Z_{2,0}^T F \end{bmatrix}^{-1} \left(\begin{bmatrix} Z_1^T Bu \\ Z_2^T\widetilde{\mathbf{f}}' + \lambda Z_2^T\widetilde{\mathbf{f}} \end{bmatrix} - \begin{bmatrix} Z_1^T F \\ \lambda Z_{2,0}^T F \end{bmatrix} \tilde{x} \right). \tag{11}$$

The ASC (11) contains all the solutions of the original DAE. In addition, the free response (when $\mathbf{f} = 0$) has a space of additional solutions $ce^{-\lambda t}$, where c is a constant, of dimension equal to the solution manifold.

VI. Computational Example

To illustrate the above algorithms we consider

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), && \text{(12a)} \\ \dot{x}_2(t) &= Kx_1(t) + Sx_2(t) + H^T x_3(t) + Qu_1(t), && \text{(12b)} \\ 0 &= Hx_1(t) + u_2(t), && \text{(12c)} \\ y(t) &= Cx(t), && \text{(12d)} \end{aligned}$$

and $u = [u_1, u_2]^T$. We take $D = 0$ since D does not affect the observer design. (12) is in the general form of a constrained mechanical system. Here $x_1(t)$ is position, $x_2(t)$ is velocity, (12c) is a physical constraint, and $H^T x_3(t)$ can be thought of as the force caused by the constraint. $Qu_1(t)$, if present, is an applied force, and $u_2(t)$ allows for adjusting the constraint. Depending on the problem, $u_2(t)$ can be another control or known input. H is full row rank.

When implementing an observer for a DAE, it is unnecessary to numerically integrate (1a) since $y(t)$ is a measured output of the system. However, in running an example to illustrate and test our observer design, it is necessary to produce an $x(t)$ whose values lie on the solution manifold. System (12) has the structure of a Hessenberg system [4] so a consistent initial condition $x(0)$ is given by

$$\begin{aligned} x_1(0) &= -H^\dagger u_2(0), && \text{(13a)} \\ x_2(0) &= -H^\dagger \dot{u}_2(0), && \text{(13b)} \\ x_3(0) &= -H^\dagger\left[Kx_1(0) + Sx_2(0) + Qu_1(0)\right] \\ &\quad -(HH^T)^{-1}\ddot{u}_2(0). && \text{(13c)} \end{aligned}$$

Once the consistent initial condition has been calculated, we integrate the stabilized completion, which includes the corresponding solution of the DAE, so we can compare the system's solution with the observer.

As a specific example let $K = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $S = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $u(t) =$

$[\sin(t)\ \sin(t)]^T$. S is chosen $S > 0$ to keep the free response from being damped and posing an additional challenge for the observer. We consider three illustrative choices of C:

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (14)$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \quad (15)$$

$$C_3 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}. \quad (16)$$

System (12) is index three, so we take $k = 3$. The finite eigenvalues are $0.1250 \pm 0.9922i$ for the matrix pencil

$$\left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, - \begin{bmatrix} 0 & I & 0 \\ K & S & H^T \\ H & 0 & 0 \end{bmatrix} \right\},$$

indicating the free response is a slowly growing oscillation and the solution manifold is two dimensional (see Figure 1). Please note in the following subsections, space prohibits the inclusion of all computed quantities but they are available from the authors upon request.

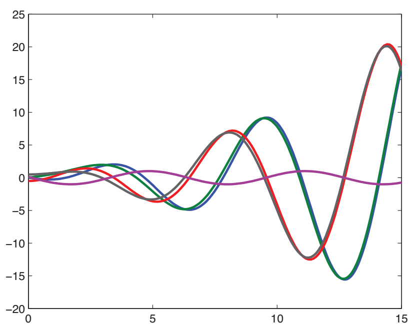

Fig. 1. Trajectories of the DAE (12) on [0 15].

A. Stabilized Least Squares Completion Results

We compute the stabilized LSC with $\lambda = 2$ to find $\tilde{A}$ and $\tilde{B}$. The eigenvalues of $\tilde{A}$ are numerically $0.1250 \pm 0.9922i, -2, -2, -2$. As expected [13], these eigenvalues include the two matrix pencil eigenvalues and three eigenvalues equal to -2 provided by this completion. The matrix $\tilde{B} = [\tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3]$ is

$$\left[\begin{array}{cc|cc|cc|cc} 0.0 & -0.783 & 0 & -0.565 & 0 & -0.087 & 0 & 0.0 \\ 0.0 & 0.783 & 0 & 0.565 & 0 & 0.087 & 0 & 0.0 \\ 1.0 & -0.174 & 0 & -0.348 & 0 & -0.130 & 0 & 0.0 \\ 1.0 & 0.174 & 0 & 0.348 & 0 & 0.130 & 0 & 0.0 \\ 0.0 & -2.130 & 0 & -3.261 & 0 & -2.098 & 0 & -0.5 \end{array} \right].$$

If $u_2(t) = 0$, we may drop the second column of each $\tilde{B}_i$ leaving only $\tilde{B}_0$ to be nonzero. The situation in which many of the higher derivatives of $u(t)$ do not appear in (6a) is common. Controls frequently do not enter through the constraints but act more like $u_1(t)$ than $u_2(t)$ since it is easier to apply a time varying force than a time varying position.

For each of the output matrices C_i, $i = 1, 2, 3$, we checked the observability of $(\tilde{A}, C_i)$ by computing the rank of the observability matrix. When the observability matrix was not full column rank, we determined the observable modes by computing the standard form for unobservable systems. We found that the observability matrix of $(\tilde{A}, C_1)$ has rank 5. Since each eigenvalue is observable, we can construct an observer for this case. For C_2 the observability matrix has rank 3. The two unobservable eigenvalues are the matrix pencil eigenvalues, and since they are unstable, it is not possible to construct an observer for any completion. The observability matrix of $(\tilde{A}, C_3)$ has rank 2. This calculation was numerically more delicate than the previous two, so we used an SVD to make the rank determination. The matrix pencil eigenvalues are the two observable eigenvalues, indicating $(\tilde{A}, C_3)$ is strongly detectable. Thus, an observer can be constructed for this case.

Subsection VI.B discusses the ASC results, but from Theorem 1 we already know for which C_i the matrix pencil eigenvalues are observable. For any completion of (12), Theorem 1 tells us the matrix pencil eigenvalues will be unobservable for C_2 and observable for C_1 or C_3. However, using different completions can impact observer design and performance, as illustrated in Subsection VI.C.

B. Alternative Stabilized Completion Results

We compute the ASC with $\lambda = 2$. The eigenvalues of $\tilde{A}$ are -2 with an algebraic multiplicity of three and $0.1250 \pm 0.9922i$. These eigenvalues are the same as before but instead of -2 going with a 3×3 Jordan block from (7), there are now three eigenvectors for -2. For C_1 the rank of the observability matrix has changed from 5 to 3. The matrix pencil eigenvalues remain observable, as expected from Theorem 1, but two eigenvalues provided by the completion are now unobservable. Since these unobservable eigenvalues are stable, we can again construct an observer for this case. For C_3 neither the rank of the observability matrix nor the observability of the matrix pencil eigenvalues changes.

C. Observer Results

To illustrate the behavior of the two observers based on Sections VI.A and VI.B we take $\hat{x}(0) = [6\ 7\ 8\ 9\ 10]^T$. Each observer places the observable eigenvalues at ρ.

1) Observers for C_1: There is a dramatic difference in the behavior of the observers constructed from the two completions when considering output matrix C_1 and $\rho = -1$. Figures 2 and 3 graph the observer estimation error when using the stabilized LSC and the ASC, respectively. The error converges to zero faster for the observer constructed from the ASC. Also the error for the observer constructed from the stabilized LSC is initially much larger and oscillates more. As predicted by the theory, the estimation error is of the form $t^j e^{-\lambda t}$ for the observer using the stabilized LSC while it is of the form $e^{-\lambda t}$ for the observer using the ASC.

2) Observers for C_3: Figure 4 shows the observer estimation error for both observers when considering C_3 and $\rho = -1$. The solid lines are for the observer constructed from the ASC and the dashed lines are for the observer constructed from the stabilized LSC. Unlike the observers for C_1, the behavior of these observers constructed from the two completions for C_3 is similar.

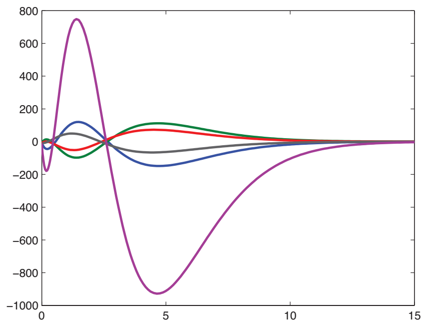

Fig. 2. Estimation error with $\rho = -1$ using the stabilized LSC with output matrix C_1.

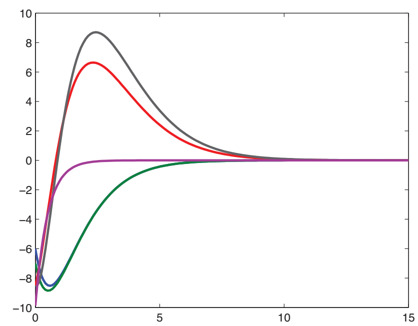

Fig. 3. Estimation error with $\rho = -1$ using the ASC with output matrix C_1.

From the theory of numerical methods for DAEs [4], convergence rates for numerical integrators can be different for differential and algebraic variables. This different behavior between the variables is presented in Figures 5 to 7, which plot the estimation error for three state vector components using $\rho = -1, -2, -3, -4, -5$ and the stabilized LSC. In Figure 5, as ρ is decreased, the interval on which the estimation error in the second component of $\hat{x}_1$ is greater than a given nonzero tolerance becomes shorter but the magnitude of the estimation error on that interval becomes larger. Alternatively, Figures 6 and 7 show that for the state estimation error in $\hat{x}_2$, the interval on which the estimation error is greater than a given nonzero tolerance not only shrinks with decreasing ρ but the magnitude of the estimation error on that interval either stays bounded or goes to zero.

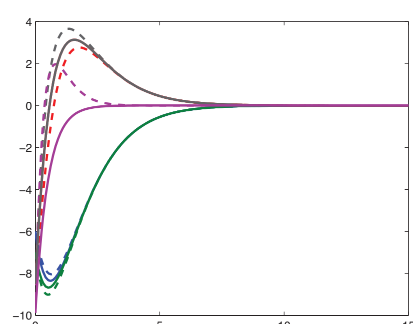

Fig. 4. Estimation error with C_3 using both observers with $\rho = -1$.

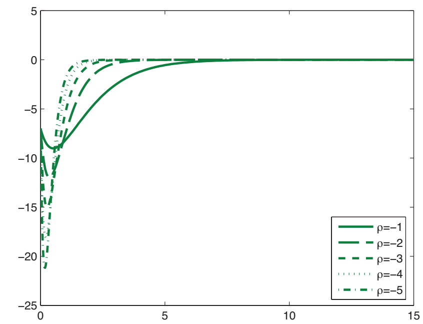

Fig. 5. Estimation error in the second component of $\hat{x}_1$ with $\rho = -1, -2, -3, -4, -5$ using C_3 and the stabilized LSC.

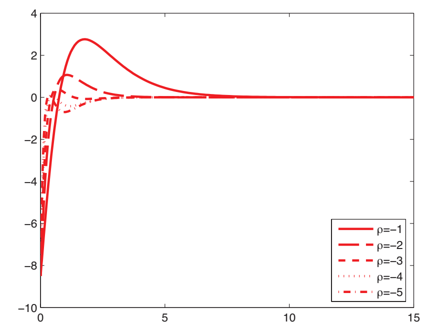

Fig. 6. Estimation error in the first component of $\hat{x}_2$ with $\rho = -1, -2, -3, -4, -5$ using C_3 and the stabilized LSC.

The greatest difference between the estimation errors of the two observers is the unobservable estimation error in $\hat{x}_3$, whose dynamics are set by the stabilized completion. Figure 8 reveals the estimation error in $\hat{x}_3$ is independent of the value of ρ. In both observers the rate of convergence of $\hat{x}_3$ is determined by λ, the parameter used in generating the stabilized completion. However, Figure 8 (right) shows simple exponential decay in contrast to what is seen in Figure 8 (left), which is due to the larger Jordan blocks present in the stabilized LSC.

In the study of DAEs, variables are sometimes described as being of a particular index. A variable is index 0 if it is given by an ODE and index k with $k \geq 1$ if it depends algebraically on $k - 1$ derivatives of a general input. If

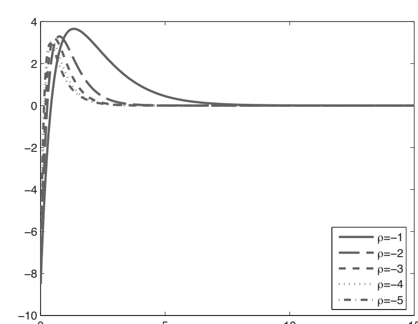

Fig. 7. Estimation error in the second component of $\hat{x}_2$ with $\rho = -1, -2, -3, -4, -5$ using C_3 and the stabilized LSC.

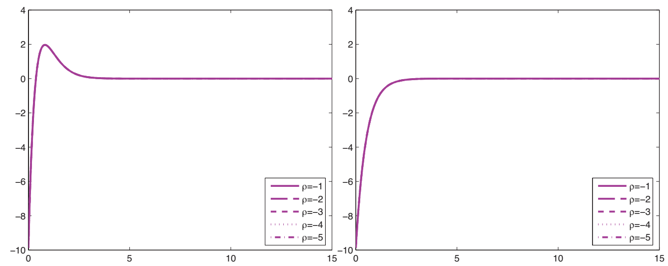

Fig. 8. Estimation error in $\hat{x}_3$ with $\rho = -1, -2, -3, -4, -5$ using C_3 and the stabilized LSC (left) and the ASC (right).

we transform (12) into canonical form, we discover two index 0 variables correspond with the finite modes, x_3 is an index three variable, part of x_1 is an index two variable, and part of x_2 is an index one variable. These higher index variables are the ones that get the stabilized dynamics when we form a stabilized LSC. Note that the independence of the convergence rate from the value of ρ in Figure 8 occurs with the higher index variable.

VII. Linear Time Varying DAEs

One advantage of the approach for full order observer design presented in this paper is that it has great potential for LTV DAEs. The application of this design to LTV DAEs is under development but is not as straightforward as the LTI case. We are able to compute (6a) and (11) for LTV DAEs so the numerical calculations at each time t result in a completion that is continuous in t. Coefficient continuity is one of the advantages of both the stabilized LSC and the ASC. However, the time dependence means the usual characterizations of observability cannot be used immediately since they involve derivatives of the completions' state coefficient matrices. It is possible to get these derivatives by working with larger derivative arrays, but a better way is to try and compute the information from the original derivative array.

The key to being able to compute the needed information from the derivative array comes from the work of Terrell [15], [16]. He provides a characterization of smooth observability in terms of a derivative array based on the original coefficients and the output matrix $C(t)$. In addition Terrell shows that under reasonable assumptions there exist smooth projections that pick out the unobservable part, the output nulling part, and an observable part and that these projections are the solutions of differential equations. Thus the projections can be generated with a numerical simulation.

The next piece of information that is needed to carry out the LTV observer design is on the smallest singular value of $M(t)$, where $M(t)$ describes the solution manifold. This information about $M(t)$ is needed to ensure that $M(t)x(t) - q(t)$ going to zero is providing an estimate of a part of $x(t)$ whose error is also going to zero.

Given an LTV system in the form of (1a) with sufficiently smooth coefficients we can test smooth observability using the techniques of [15], [16]. If the system is observable, we can then generate a stabilized completion. In principle an observer can now be constructed. However, doing so in a robust numerical manner requires some care and will be reported on later.

VIII. Conclusion

Recent results on the numerical generation of stabilized completions have provided new tools that can be used in the construction of observers for general unstructured DAEs. This paper has introduced some of the key ideas and illustrated their application on LTI DAEs. Two completion algorithms are presented and a computational example is worked. Both observers work well, but for this example, the ASC exhibits some advantages. The advantages of the ASC are expected to be even greater when the theory is worked out for the more general case of LTV DAEs.

Unlike traditional methods for constructing observers for LTI DAEs, our approach has considerable potential for the construction of observers for LTV and nonlinear DAEs since it does not require any specific structure to the DAE nor does it fundamentally rely on techniques restricted to LTI systems. Most importantly, there exist a number of useful results and numerical techniques for working with LTV and nonlinear versions of the derivative array that were developed as part of the work on general DAE integrators.

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