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Minimal realization of nonlinear MIMO equations in state-space form: polynomial approach

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Abstract— **The realization of nonlinear input-output equations in the classical state-space form can be studied by the polynomial approach in which the system is described by two polynomials from the non-commutative ring of skew polynomials. The aim of the present paper is to apply the polynomial methods to the realization problem. This allows to simplify the step-by-step algorithm based on certain sequences of subspaces of differential one-forms. The presented new formula allows to compute the differentials of the state coordinates directly from the polynomial description of the nonlinear system. This method is more clear, straight-forward and therefore better suited for implementation in different computer packages such as *Mathematica* or *Maple*. The developed theory and algorithm are illustrated by means of several examples.**

Index Terms— **nonlinear control system, continuous-time system, input-output model, polynomial method, state-space realization.**

I. INTRODUCTION

To describe the behavior of the real-life processes one frequently uses input-output (i/o) models. This allows to represent the object of practical interest in a compact and convenient form by means of differential equations. However, despite the simplicity of this approach, state-space description usually becomes the basis for analysis and control of nonlinear systems. Thus, the problem one encounters and the main goal of this paper is to bridge the gap between two modeling approaches and to present the algorithm allowing one to construct a minimal state-space model from an arbitrary set of nonlinear higher order i/o differential equations, whenever possible.

The state-space realization problem of nonlinear i/o models has a relatively long history. Some of the results on this subject may be found in [1], [2], [3], [4], [5] for continuous-time systems and in [6], [7], [8], [9], [10] for discrete-time systems, respectively. A great number of existing results have been obtained for single-input single-output (SISO) systems. However, multi-input multi-output (MIMO) case has not been left aside, what may be confirmed, for instance, by [5], [11], [12], [13], [14]. The comparison of different methods and the explicit relations between them have been reported in [3] and [12] for SISO and MIMO cases, respectively. One of the popular approaches is based on the algebraic formalism using the theory of differential one-forms [1]. The coordinate-free necessary and sufficient realizability conditions were formulated in terms of integrability of certain subspaces of one-forms. The algorithm for calculating the subspaces and the state coordinates was given as well. On the other hand, there exists a polynomial approach for the study of this problem in which the system is described by two polynomials from the non-commutative ring of skew polynomials [4], [15]. This technique represents the objects in a more compact form and allows to simplify and reduce the number of steps during calculations. Polynomial approach has been used so far to study the reduction of nonlinear i/o equations [16], the linear i/o [17] and transfer equivalence [18], controllability [19] and used also in introducing the concept of transfer function into the nonlinear domain [20], [21]. Thus, it has been already proved itself as a practical and reliable mathematical tool.

The aim of this paper is to use the straight-forward polynomial method in order to extend the realization algorithm presented in [4] to the MIMO case. The proposed algorithm combines well with the existing results for the reduction problem [16]. Both the results of [16] and this paper rely on system description in terms of two polynomial matrices. Moreover, it is known that if the system under consideration is not in the irreducible form, then the state-space realization is not minimal, i.e. accessible. Our result can be also understood as a generalization of the polynomial realization algorithm obtained in [22] for linear time-invariant systems. Apart from the fact that in [22] the kernel representation, without the inputs and outputs specified, is used as a starting point, the main difference comes from the fact that [22] deals with discrete-time systems. Thus, when adapting its results to the continuous-time case we had to replace the cut-and-shift operator, used to compute the state coordinates, by computation of the left quotients of two non-commutative polynomials. Finally, note that method may be easily implemented in any computer algebra system, for instance, in *Mathematica* or *Maple*.

The paper is organized as follows. Section II recalls the basic notions from the algebraic framework [1] and defines the realization problem studied in this paper. In the next section a polynomial system description is given. In Section IV the main result is presented, followed by the examples. Section V concludes the paper.

II. PROBLEM STATEMENT AND ALGEBRAIC FRAMEWORK

Consider a continuous-time MIMO nonlinear system, described by a set of higher order i/o differential equations,

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relating the inputs u_υ, $\upsilon = 1, \ldots, m$, the outputs y_ν, $\nu = 1, \ldots, p$ and a finite number of their time derivatives

$$y_i^{(n_i)} = \phi_i(y_\nu, \dot{y}_\nu, \ldots, y_\nu^{(n_{i\nu})}, \nu = 1, \ldots, p, \\ u_\upsilon, \dot{u}_\upsilon, \ldots, u_\upsilon^{(r_{i\upsilon})}, \upsilon = 1, \ldots, m) \quad (1)$$

for $i = 1, \ldots, p$. In (1) $u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m$, $y = [y_1, \ldots, y_p]^T \in \mathbb{R}^p$ and ϕ_i are real analytic functions. Notations $n := n_1 + \cdots + n_p$ and $r := \max\{r_{i\upsilon}, i = 1, \ldots, p, \upsilon = 1, \ldots, m\}$ are used below for system (1). Moreover, we assume that the following assumptions hold for system (1).

Assumption 1: System (1) is strictly proper, i.e. $r_{i\upsilon} < n_i$.

Assumption 2: System (1) is in the canonical form, which means that $n_{i\nu} < \min\{n_i, n_\nu\}$.

Note that if the system under consideration is not in form (1), then it can be transformed into (1) using the approach proposed in [23], at least locally.

The realization problem is defined as follows. Given a nonlinear system, described by the set of the i/o equations of the form (1), find, if possible, the state coordinates $x \in X \subseteq \mathbb{R}^n$, $x = \psi(y_\nu, \dot{y}_\nu, \ldots, y_\nu^{(n_i-1)}, u_\upsilon, \dot{u}_\upsilon, \ldots, u_\upsilon^{(r_{i\upsilon})})$ such that in these coordinates the system takes the classical state-space form

$$\begin{array}{rcl} \dot{x} & = & f(x, u) \\ y & = & h(x) \end{array} \quad (2)$$

such that $x(t) \in X \subset \mathbb{R}^n$, and the sequences $\{u(t), y(t), t \geq 0\}$, generated by (2) (for different initial states), coincide with those, satisfying equation (1). Then (2) will be called a realization of (1). A system (1) is said to be realizable if there exists a realization of the form (2) for it.

Next, we recall only the basic aspects of the algebraic formalism for nonlinear control systems, described in [1]. Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of the independent system variables

$$\mathcal{C} = \{y_i, \dot{y}_i, \ldots, y_i^{(n_i-1)}, i = 1, \ldots, p, \\ u_\upsilon^{(l_\upsilon)}, \upsilon = 1, \ldots, m, l_\upsilon \geq 0\}.$$

Let $s : \mathcal{K} \to \mathcal{K}$ denote the time derivative operator $\frac{\mathrm{d}}{\mathrm{d}t}$. Then the pair $(\mathcal{K}, s)$ is differential field [24]. Over the field $\mathcal{K}$ one can define a differential vector space, $\mathcal{E} := \mathrm{span}_\mathcal{K} \{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$ spanned by the differentials of the elements of $\mathcal{K}$. Consider a one-form $\omega \in \mathcal{E}$ such that $\omega = \sum_i \alpha_i \mathrm{d}\varphi_i$, $\alpha_i, \varphi_i \in \mathcal{K}$. Its derivative $\dot{\omega}$ is defined by $\dot{\omega} = \sum_i (\dot{\alpha}_i \mathrm{d}\varphi_i + \alpha_i \mathrm{d}\dot{\varphi}_i)$.

A sequence of subspaces $\{\mathcal{H}_k\}_{k=1}^\infty$ of $\mathcal{E}$ is defined by

$$\begin{array}{rcl} \mathcal{H}_1 & = & \mathrm{span}_\mathcal{K}\{\mathrm{d}y_i, \mathrm{d}\dot{y}_i, \ldots, \mathrm{d}y_i^{(n_i-1)}, i = 1, \ldots, p, \\ & & \mathrm{d}u_\upsilon, \mathrm{d}\dot{u}_\upsilon, \ldots, \mathrm{d}u_\upsilon^{(r)}, \upsilon = 1, \ldots, m\}, \\ \mathcal{H}_{k+1} & = & \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\}, k \geq 1. \end{array} \quad (3)$$

Note that there exists an integer k^* such that $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty$. Now, we assume that the i/o differential equation (1) is in the irreducible form. The latter means that $\mathcal{H}_\infty$ is trivial, i.e. $\mathcal{H}_\infty = \{0\}$. An nth order realization of equation (1) is accessible if and only if system (1) is irreducible, see [1].

System (2) is said to be single-experiment observable if the observability matrix has generically full rank

$$\mathrm{rank}_\mathcal{K} \frac{\partial(h(x), sh(x), \ldots, s^{n-1}h(x))}{\partial x} = n.$$

We say that $\omega \in \mathcal{E}$ is an *exact* one-form, if there exists $\xi \in \mathcal{K}$ such that $\mathrm{d}\xi = \omega$. A one-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed*. A subspace is integrable or closed, if it has a basis which consists only of closed one-forms. Integrability of the subspace of one-forms can be checked by the Frobenius theorem.

Theorem 1 ([25]): Let $\mathcal{V} = \mathrm{span}_\mathcal{K}\{\omega_1, \ldots, \omega_\kappa\}$ be a subspace of $\mathcal{E}$. $\mathcal{V}$ is closed if and only if

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_\kappa = 0 \quad (4)$$

for all $i = 1, \ldots, \kappa$.

Theorem 2 ([1]): The nonlinear system, described by the set of irreducible i/o differential equation (1), has an observable and accessible state-space realization iff for $1 \leq k \leq r + 2$ the subspaces $\mathcal{H}_k$ defined by (3) are completely integrable. Moreover, the state coordinates can be obtained by integrating the basis vectors of $\mathcal{H}_{r+2}$.

III. POLYNOMIAL SYSTEM DESCRIPTION

Polynomial framework is built upon the linear algebraic framework. The differential field $(\mathcal{K}, s)$ induces a ring of the left differential polynomials $\mathcal{K}[\partial; s]$. The elements of $\mathcal{K}[\partial; s]$ can be uniquely written in the form $a(\partial) = \sum_{i=0}^n a_i \partial^{n-i}$, $a_i \in \mathcal{K}$, where ∂ is a formal variable and $a(\partial) \neq 0$ if and only if at least one of the functions a_i, $i = 0, \ldots, n$ is nonzero. If $a_0 \not\equiv 0$, then the positive integer n is called the degree of the left polynomial $a(\partial)$ and may be denoted by $\deg a(\partial)$. The addition of the left polynomials is defined in the standard way. However, for $a \in \mathcal{K}$ the multiplication is defined by

$$\partial \cdot a := a \cdot \partial + s(a).$$

Lemma 1 ([18]): Let $a \in \mathcal{K}$. Then $\partial^n \cdot a \in \mathcal{K}[\partial; s]$, for $n \geq 0$, and $\partial^n \cdot a = \sum_{i=0}^n \binom{n}{i} s^{n-i}(a) \partial^i$.

Lemma 2 ([1]): Let $F \in \mathcal{K}$. Then $s(\mathrm{d}F) = \mathrm{d}(sF) = \mathrm{d}\dot{F}$.

A ring is called an *integral domain*, if it does not contain any zero divisors. This means that for any two elements a and b of the ring, $ab = 0$ implies either $a = 0$ or $b = 0$.

Proposition 1 ([26]): The ring $\mathcal{K}[\partial; s]$ is an integral domain.

Let us define $\partial^k \mathrm{d}y_\nu := \mathrm{d}(s^k y_\nu) = \mathrm{d}y_\nu^{(k)}$ and $\partial^l \mathrm{d}u_\upsilon := \mathrm{d}(s^l u_\upsilon) = \mathrm{d}u_\upsilon^{(l)}$, $k, \nu = 1, \ldots, p$, $\upsilon = 1, \ldots, m$ and $l \geq 0$ in the vector space $\mathcal{E}$. Since every one-form $\omega \in \mathcal{E}$ has the form

$$\omega = \sum_{\nu=1}^p \sum_{i=0}^{n-1} a_{\nu i} \mathrm{d}y_\nu^{(i)} + \sum_{\upsilon=1}^m \sum_{j=0}^k b_{\upsilon j} \mathrm{d}u_\upsilon^{(j)},$$

where $a_{\nu i}, b_{\upsilon j} \in \mathcal{K}$, so ω can be expressed in terms of the left differential polynomials in the following way

$$\omega = \sum_{\nu=1}^p \left(\sum_{i=0}^{n-1} a_{\nu i} \partial^i \right) \mathrm{d}y_\nu + \sum_{\upsilon=1}^m \left(\sum_{j=0}^k b_{\upsilon j} \partial^j \right) \mathrm{d}u_\upsilon.$$

A left differential polynomial can be considered as an operator acting on vectors $y = [y_1, \ldots, y_p]^T$ and $u = [u_1, \ldots, u_m]^T$ from $\mathcal{E}$ as $(\sum_{i=0}^{k} a_i \partial^i)(\alpha \mathrm{d}\zeta) := \sum_{i=0}^{k} a_i (\partial^i \cdot \alpha)\mathrm{d}\zeta$ with $a_i, \alpha \in \mathcal{K}$ and $\mathrm{d}\zeta \in \{\mathrm{d}y, \mathrm{d}u\}$, where by Lemma 1, $\partial^i \cdot \alpha = \sum_{k=0}^{i} \binom{i}{k} s^{i-k}(\alpha)\partial^k$. It is easy to note that $\partial(\omega) = s(\omega)$, for $\omega \in \mathcal{E}$.

The nonlinear system (1) can be represented in terms of two polynomial matrices. By differentiating (1) we obtain

$$\mathrm{d}y_i^{(n_i)} - \sum_{\nu=1}^{p} \sum_{\alpha=0}^{n_{i\nu}} \frac{\partial \phi_i}{\partial y_\nu^{(\alpha)}} \mathrm{d}y_\nu^{(\alpha)} - \\ - \sum_{\upsilon=1}^{m} \sum_{\beta=0}^{r_{i\upsilon}} \frac{\partial \phi_i}{\partial u_\upsilon^{(\beta)}} \mathrm{d}u_\upsilon^{(\beta)} = 0 \quad (5)$$

for $i = 1, \ldots, p$. Next, using relations $\partial^\alpha \mathrm{d}y_\nu := \mathrm{d}y_\nu^{(\alpha)}$, $\partial^\beta \mathrm{d}u_\upsilon := \mathrm{d}u_\upsilon^{(\beta)}$, we can rewrite (5) as

$$P(\partial)\mathrm{d}y + Q(\partial)\mathrm{d}u = 0, \quad (6)$$

where $P(\partial)$ and $Q(\partial)$ are $p \times p$ and $p \times m$-dimensional matrices respectively, whose elements $p_{i\nu}(\partial), q_{i\upsilon}(\partial) \in \mathcal{K}[\partial; s]$ and

$$p_{i\nu}(\partial) = \begin{cases} \partial^{n_i} - \sum_{\alpha=0}^{n_{i\nu}} p_{i\nu,\alpha}\partial^\alpha, \text{if } i = \nu, \\ - \sum_{\alpha=0}^{n_{i\nu}} p_{i\nu,\alpha}\partial^\alpha, \text{if } i \neq \nu, \end{cases}$$

$$q_{i\upsilon}(\partial) = - \sum_{\beta=0}^{r_{i\upsilon}} q_{i\upsilon,\beta}\partial^\beta,$$

whereas $p_{i\nu,\alpha} = \frac{\partial \phi_i}{\partial y_\nu^{(\alpha)}} \in \mathcal{K}$, $q_{i\upsilon,\beta} = \frac{\partial \phi_i}{\partial u_\upsilon^{(\beta)}} \in \mathcal{K}$. Equation (6) describes the behavior of system (1) in terms of two polynomial matrices $P(\partial), Q(\partial)$ in derivative operator $\partial := s$ over the differential field $\mathcal{K}$. Further, the notations $p_{i\cdot}(\partial) := [p_{i1}(\partial), \ldots, p_{ip}(\partial)]$ and $q_{i\cdot}(\partial) := [q_{i1}(\partial), \ldots, q_{im}(\partial)]$ are used for row vectors of $P(\partial)$ and $Q(\partial)$, respectively.

Since $\mathcal{K}[\partial; s]$ is an Ore ring, one can construct the division ring of fractions. If $p(\partial) = p_1(\partial)p_2(\partial)$ and $\deg(p_1(\partial)) > 0$, then $p_1(\partial)$ is called a *left divisor* of $p(\partial)$.

To find the left divisor one can use the *left Euclidean division algorithm*, see [27]. Note that, in order to perform the left Euclidean division algorithm, it is sufficient that s be an automorphism. The main idea behind this algorithm is that for given two polynomials $p_1(\partial)$ and $p_2(\partial)$ with $\deg(p_1(\partial)) > \deg(p_2(\partial))$, there exists a unique polynomial $\gamma(\partial)$ and a unique left remainder polynomial $\rho(\partial)$ such that $p_1(\partial) = p_2(\partial)\gamma(\partial) + \rho(\partial)$ and $\deg(\rho(\partial)) < \deg(p_2(\partial))$.

IV. REALIZATION

Now, we introduce the certain one-forms in terms of which our main result will be formulated. Let

$$\omega_{i,l} = \begin{bmatrix} p_{i\cdot,l}(\partial) & q_{i\cdot,l}(\partial) \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix} \quad (7)$$

for $i = 1, \ldots, p$, $l = 1, \ldots, n_i$, where $p_{i\cdot,l}(\partial)$ and $q_{i\cdot,l}(\partial)$ are Ore polynomials, which can be recursively calculated from the equalities

$$\begin{aligned} p_{i\cdot,l-1}(\partial) &= \partial \cdot p_{i\cdot,l}(\partial) + \xi_{i\cdot,l}, \quad \deg \xi_{i\cdot,l} = 0, \\ q_{i\cdot,l-1}(\partial) &= \partial \cdot q_{i\cdot,l}(\partial) + \gamma_{i\cdot,l}, \quad \deg \gamma_{i\cdot,l} = 0 \end{aligned} \quad (8)$$

with the initial polynomials $p_{i\cdot,0}(\partial) := p_{i\cdot}(\partial)$ and $q_{i\cdot,0}(\partial) := q_{i\cdot}(\partial)$.

Theorem 3: For the input-output model (1), the subspaces $\mathcal{H}_k$ for $k = 1, \ldots, r+2$ can be calculated as

$$\mathcal{H}_k = \mathrm{span}_{\mathcal{K}}\{\omega_{i,l},\ i = 1, \ldots, p,\ l = 1, \ldots, n_i, \\ \mathrm{d}u_\upsilon, \ldots, \mathrm{d}u_\upsilon^{(r-k+1)}, \upsilon = 1, \ldots, m\}. \quad (9)$$

Proof: The proof is by mathematical induction. Throughout the proof we assume that $i, \nu = 1, \ldots, p$ and $\upsilon = 1, \ldots, m$.

First, we show that formula (9) holds for $k = 1$. Taking $k = 1$ in (9), yields

$$\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}}\{\omega_{i,l},\ l = 1, \ldots, n_i, \mathrm{d}u_\upsilon, \ldots, \mathrm{d}u_\upsilon^{(r)}\}.$$

In order to simplify the following discussion note that (8) can be rewritten as

$$\begin{aligned} p_{i\cdot,0}(\partial) &= \partial^l \cdot p_{i\cdot,l}(\partial) + \Xi_{i\cdot,l}(\partial), \quad \deg \Xi_{i\cdot,l}(\partial) < l, \\ q_{i\cdot,0}(\partial) &= \partial^l \cdot q_{i\cdot,l}(\partial) + \Gamma_{i\cdot,l}(\partial), \quad \deg \Gamma_{i\cdot,l}(\partial) < l. \end{aligned} \quad (10)$$

Suppose $l = n_i$. According to (7), $\omega_{i,n_i} = p_{i\cdot,n_i}(\partial)\mathrm{d}y + q_{i\cdot,n_i}(\partial)\mathrm{d}u$. Due to the structure of the i/o equations, $\deg(p_{ii}(\partial)) = n_i$ and $p_{ii}(\partial)$ is monic. It follows from (10) that $p_{ii,n_i}(\partial)$ is a left quotient of $p_{ii}(\partial)$ and ∂^{n_i}, thus $p_{ii,n_i}(\partial) = 1$. Degrees of all other polynomials $\{p_{i\nu}(\partial), i \neq \nu, q_{i\cdot}(\partial)\}$ are strictly less than n_i, which means that quotient of any polynomial $\{p_{i\nu}(\partial), i \neq \nu, q_{i\cdot}(\partial)\}$ and ∂^{n_i} is zero. Consequently, $\omega_{i,n_i} = \mathrm{d}y_i$. Next, we take $l = n_i - 1$ and compute ω_{i,n_i-1}. Since $\deg(p_{ii}(\partial)) = n_i$ and $p_{ii}(\partial)$ is monic it follows from (10) that $p_{ii,n_i-1}(\partial)$ is the first degree polynomial in the form $\partial + \alpha$, $\alpha \in \mathcal{K}$. At the same time the left quotients $\{p_{i\nu,n_i-1}(\partial), i \neq \nu,\ q_{i\cdot,n_i-1}(\partial)\}$ are just functions from $\mathcal{K}$. Thus,

$$\omega_{i,n_i-1} = \mathrm{d}\dot{y}_i + \alpha \mathrm{d}y_i + \sum_{\substack{\nu=1, \\ \nu \neq i}}^{n} p_{i\nu,n_i-1}\mathrm{d}y_\nu + \sum_{\upsilon=1}^{m} q_{i\upsilon,n_i-1}\mathrm{d}u_\upsilon.$$

Since we have represented ω_{i,n_i-1} as a linear combination of $\mathrm{d}\dot{y}_i$, $\mathrm{d}y_\nu, \mathrm{d}u_\upsilon \in \mathcal{H}_1$, we can replace ω_{i,n_i-1} by $\mathrm{d}\dot{y}_i$. Continuing in this fashion it is possible to show that $\mathcal{H}_1$ agrees with (3).

Assume next that formula (9) holds for k and we prove it to be valid for $k+1$. The proof is based on the definition of the subspaces $\mathcal{H}_k$. We have to prove that

$$\mathcal{H}_{k+1} = \mathrm{span}_{\mathcal{K}}\{\omega_{i,l}, \mathrm{d}u, \ldots, \mathrm{d}u^{(r-k)}\}, \quad (11)$$

calculated according to formula (9), satisfies condition (3).

First, we will show that the one-forms $\omega_{i,l}, \mathrm{d}u, \ldots, \mathrm{d}u^{(r-k)}$ are in $\mathcal{H}_k$. It is obvious, since we have assumed formula (9) to hold for k.

Second, we have to prove that the derivatives of the basis one-forms of (11) belong to $\mathcal{H}_k$. By (7), we have

$$\dot{\omega}_{i,l} = \begin{bmatrix} \partial \cdot p_{i\cdot,l}(\partial) & \partial \cdot q_{i\cdot,l}(\partial) \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix}.$$

Using relations (8) yields

$$\dot{\omega}_{i,l} = \begin{bmatrix} p_{i\cdot,l-1}(\partial) - \xi_{i\cdot,l} & q_{i\cdot,l-1}(\partial) - \gamma_{i\cdot,l} \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix}. \quad (12)$$

After reordering terms in (12) we get

$$\dot{\omega}_{i,l} = \begin{bmatrix} p_{i\cdot,l-1}(\partial) & q_{i\cdot,l-1}(\partial) \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix} - \\ - \begin{bmatrix} \xi_{i\cdot,l} & \gamma_{i\cdot,l} \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix}. \quad (13)$$

The one-form $\dot{\omega}_{i,l}$ is now represented as a sum of two terms. To deal with the first term, we consider two separate cases. In case $l = 1$ the first term yields $p_{i\cdot,0}(\partial)\mathrm{d}y + q_{i\cdot,0}(\partial)\mathrm{d}u = p_{i\cdot}(\partial)\mathrm{d}y + q_{i\cdot}(\partial)\mathrm{d}u = 0$ due to polynomial system description (6). In case $l = 2, \ldots, n_i$, the first term of (13) equals to $\omega_{i,l-1}$ by (7), thus it is in $\mathcal{H}_k$. As for the second term of (13), it is a linear combination of $\mathrm{d}y_i, \mathrm{d}u_v \in \mathcal{H}_k$, since the elements of $\xi_{i\cdot,l}$ and $\gamma_{i\cdot,l}$ are functions from $\mathcal{K}$. Therefore, $\dot{\omega}_{i,l} \in \mathcal{H}_k$ for $l = 1, \ldots, n_i$. Finally, we observe that the derivatives of the rest of the basis one-forms in (11) are $\mathrm{d}\dot{u}_v, \ldots, \mathrm{d}u_v^{(r-k+1)}$, which are also in $\mathcal{H}_k$. Thus, we have proved that $\mathcal{H}_{k+1}$, computed according to (9), agrees with definition (3). ∎

Remark 1: The differentials of the state coordinates can be found from the subspace $\mathcal{H}_{r+2}$, see Theorem 2. Though in case of the realizable i/o equation, $\mathcal{H}_{r+2}$, defined by (9), is completely integrable, the one-forms $\omega_{i,l}$ for $i = 1, \ldots, p$, $l = 1, \ldots, n_i$, are not necessarily always exact. Therefore, one has to find for $\mathcal{H}_{r+2}$ a new integrable basis, using the linear transformations.

Example 1: Consider the system

$$\begin{array}{rcl} \ddot{y}_1 & = & u_2\dot{y}_1 + \dot{u}_1 y_2 \\ y_2^{(3)} & = & -u_1\dot{y}_1 + y_1\dot{y}_2 - \ddot{u}_2 \end{array} \quad (14)$$

that can be described by two polynomial matrices in the following way

$$P(\partial) = \begin{pmatrix} \partial^2 - u_2\partial & -\dot{u}_1 \\ u_1\partial - \dot{y}_2 & \partial^3 - y_1\partial \end{pmatrix}$$

and

$$Q(\partial) = \begin{pmatrix} -y_2\partial & -\dot{y}_1 \\ \dot{y}_1 & \partial^2 \end{pmatrix}.$$

From (14) one can get that $n_1 = 2, n_2 = 3, n_{11} = 1, n_{12} = 0, n_{21} = 1, n_{22} = 1$ and $r_{11} = 1, r_{12} = 0, r_{21} = 0, r_{22} = 2$. Thus, for system (14), $n = n_1 + n_2 = 5$ and $r = \max\{r_{11}, r_{12}, r_{21}, r_{22}\} = 2$. Next compute, according to (8), the left quotients of the elements in matrices $P(\partial)$ and $Q(\partial)$ as

$$\begin{aligned} \begin{bmatrix} p_{1\cdot,0}(\partial) & q_{1\cdot,0}(\partial) \end{bmatrix} &= \begin{bmatrix} \partial^2 - u_2\partial & -\dot{u}_1 & -y_2\partial & -\dot{y}_1 \end{bmatrix}, \\ \begin{bmatrix} p_{1\cdot,1}(\partial) & q_{1\cdot,1}(\partial) \end{bmatrix} &= \begin{bmatrix} \partial - u_2 & 0 & -y_2 & 0 \end{bmatrix}, \\ \begin{bmatrix} p_{1\cdot,2}(\partial) & q_{1\cdot,2}(\partial) \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} p_{2\cdot,0}(\partial) & q_{2\cdot,0}(\partial) \end{bmatrix} &= \begin{bmatrix} u_1\partial - \dot{y}_2 & \partial^3 - y_1\partial & \dot{y}_1 & \partial^2 \end{bmatrix}, \\ \begin{bmatrix} p_{2\cdot,1}(\partial) & q_{2\cdot,1}(\partial) \end{bmatrix} &= \begin{bmatrix} u_1 & \partial^2 - y_1 & 0 & \partial \end{bmatrix}, \\ \begin{bmatrix} p_{2\cdot,2}(\partial) & q_{2\cdot,2}(\partial) \end{bmatrix} &= \begin{bmatrix} 0 & \partial & 0 & 1 \end{bmatrix}, \\ \begin{bmatrix} p_{2\cdot,3}(\partial) & q_{2\cdot,3}(\partial) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Further, recall that $\mathrm{d}y = [\mathrm{d}y_1, \mathrm{d}y_2]^T$, $\mathrm{d}u = [\mathrm{d}u_1, \mathrm{d}u_2]^T$. Since $r = 2$, according to Remark 1 and using (7), the last subspace of the one-forms $\mathcal{H}_{r+2} = \mathcal{H}_4 = \mathrm{span}_{\mathcal{K}}\{\omega_{i,j}\}$ for $i = 1, 2$, $j = 1, \ldots, n_i$ can be represented in the following form

$$\begin{aligned} \omega_{1,1} &= \begin{bmatrix} \partial - u_2 & 0 & -y_2 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix} = \mathrm{d}\dot{y}_1 - u_2\mathrm{d}y_1 - \\ &\quad - y_2\mathrm{d}u_1, \\ \omega_{1,2} &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix} = \mathrm{d}y_1, \\ \omega_{2,1} &= \begin{bmatrix} u_1 & \partial^2 - y_1 & 0 & \partial \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix} = \\ &= u_1\mathrm{d}y_1 + \mathrm{d}\ddot{y}_2 - y_1\mathrm{d}y_2 + \mathrm{d}\dot{u}_2, \\ \omega_{2,2} &= \begin{bmatrix} 0 & \partial & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix} = \mathrm{d}\dot{y}_2 + \mathrm{d}u_2, \\ \omega_{2,3} &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix} = \mathrm{d}y_2. \end{aligned}$$

Though the subspace $\mathcal{H}_4$ is completely integrable, $\omega_{1,1}$ and $\omega_{2,1}$ are not exact and we have to replace them by integrable linear combinations of one-forms from $\mathcal{H}_4$ to obtain the differentials of the state coordinates

$$\begin{array}{rclcl} \mathrm{d}x_1 & = & \omega_{1,2} & = & \mathrm{d}y_1 \\ \mathrm{d}x_2 & = & \omega_{2,3} & = & \mathrm{d}y_2 \\ \mathrm{d}x_3 & = & \omega_{1,1} + u_2\omega_{1,2} - u_1\omega_{2,3} & = & \mathrm{d}(\dot{y}_1 - u_1 y_2) \\ \mathrm{d}x_4 & = & \omega_{2,2} & = & \mathrm{d}(\dot{y}_2 + u_2) \\ \mathrm{d}x_5 & = & \omega_{2,1} - u_1\omega_{1,2} + y_1\omega_{2,3} & = & \mathrm{d}(\ddot{y}_2 + \dot{u}_2) \end{array}$$

In these coordinates the system has the classical state-space form

$$\begin{array}{rcl} \dot{x}_1 & = & u_1 x_2 + x_3 \\ \dot{x}_2 & = & x_4 - u_2 \\ \dot{x}_3 & = & u_2 x_3 + u_1(u_2(x_2 + 1) - x_4) \\ \dot{x}_4 & = & x_5 \\ \dot{x}_5 & = & x_1(x_4 - u_2) - u_1(u_1 x_2 + x_3) \\ y_1 & = & x_1, \\ y_2 & = & x_2 \end{array}$$

Example 2: Consider a hopping robot, consisting of a body and a single leg, that can be described by the i/o equations as [1]

$$\begin{array}{rcl} \ddot{y}_1 & = & \frac{u_2}{m} + y_1\dot{y}_3^2 \\ \dot{y}_2 & = & -\frac{m}{J}y_1^2\dot{y}_3 \\ \ddot{y}_3 & = & -\frac{u_1 + 2my_1\dot{y}_1\dot{y}_3}{my_1^2} \end{array} \quad (15)$$

where m is the mass of the leg, J the inertia momentum of the body, y_1 denotes the length of the leg, y_2 the angular position of the body, and y_3 the angular position of the leg. Moreover, u_1 and u_2 control the orientation of the body with respect to the leg and the length of the leg, respectively.

Like in the previous example, (15) can be described by two polynomial matrices as follows

$$P(\partial)=\begin{pmatrix} \partial^2-\dot y_3^2 & 0 & -2y_1\dot y_3\partial\\ \frac{2my_1\dot y_3}{J} & \partial & \frac{my_1^2}{J}\partial\\ \frac{2\dot y_3}{y_1}\partial-\frac{2(u_1+my_1\dot y_1\dot y_3)}{my_1^3} & 0 & \partial^2+\frac{2\dot y_1}{y_1}\partial\end{pmatrix}$$

and

$$Q(\partial)=\begin{pmatrix} 0 & \frac{1}{m}\\ 0 & 0\\ -\frac{1}{my_1^2} & 0\end{pmatrix}.$$

From (15), $n=5$ and $r=0$. Next, compute, according to (8), the left quotients of the elements in matrices $P(\partial)$ and $Q(\partial)$ as

$$\begin{aligned}
\begin{bmatrix}p_{1\cdot,1}(\partial) & q_{1\cdot,1}(\partial)\end{bmatrix} &= \begin{bmatrix}\partial & 0 & -2y_1\dot y_3 & 0 & 0\end{bmatrix}\\
\begin{bmatrix}p_{1\cdot,2}(\partial) & q_{1\cdot,2}(\partial)\end{bmatrix} &= \begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix},\\
\begin{bmatrix}p_{2\cdot,1}(\partial) & q_{2\cdot,1}(\partial)\end{bmatrix} &= \begin{bmatrix}0 & 1 & \frac{my_1^2}{J} & 0 & 0\end{bmatrix},\\
\begin{bmatrix}p_{3\cdot,1}(\partial) & q_{3\cdot,1}(\partial)\end{bmatrix} &= \begin{bmatrix}\frac{2\dot y_3}{y_1} & 0 & \partial+\frac{2\dot y_1}{y_1} & 0 & 0\end{bmatrix},\\
\begin{bmatrix}p_{3\cdot,2}(\partial) & q_{3\cdot,2}(\partial)\end{bmatrix} &= \begin{bmatrix}0 & 0 & 1 & 0 & 0\end{bmatrix}.
\end{aligned}$$

Further, recall that $\mathrm{d}y=[\mathrm{d}y_1,\mathrm{d}y_2,\mathrm{d}y_3]^T$, $\mathrm{d}u=[\mathrm{d}u_1,\mathrm{d}u_2]^T$. By (7), we get the following basis one-forms of the last subspace $\mathcal{H}_2$

$$\begin{aligned}
\omega_{1,1} &= \begin{bmatrix}\partial & 0 & -2y_1\dot y_3 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix}=\\
&= \mathrm{d}\dot y_1-2y_1\dot y_3\mathrm{d}y_3,\\
\omega_{1,2} &= \begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix}=\mathrm{d}y_1,\\
\omega_{2,1} &= \begin{bmatrix}0 & 1 & \frac{my_1^2}{J} & 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix}=\\
&= \mathrm{d}y_2+\frac{my_1^2}{J}\mathrm{d}y_3,\\
\omega_{3,1} &= \begin{bmatrix}\frac{2\dot y_3}{y_1} & 0 & \partial+\frac{2\dot y_1}{y_1} & 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix}=\\
&= \frac{2\dot y_3}{y_1}\mathrm{d}y_1+\mathrm{d}\dot y_3+\frac{2\dot y_1}{y_1}\mathrm{d}y_3.\\
\omega_{3,2} &= \begin{bmatrix}0 & 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix}=\mathrm{d}y_3.
\end{aligned}$$

Finally, we get $\mathcal{H}_2=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\dot y_1-2y_1\dot y_3\mathrm{d}y_3, \mathrm{d}y_1, \mathrm{d}y_2+\frac{my_1^2}{J}\mathrm{d}y_3, \frac{2\dot y_3}{y_1}\mathrm{d}y_1+\mathrm{d}\dot y_3+\frac{2\dot y_1}{y_1}\mathrm{d}y_3, \mathrm{d}y_3\}$. Simplifying the basis one-forms, the subspace can be rewritten as $\mathcal{H}_2=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1,\mathrm{d}\dot y_1,\mathrm{d}y_2,\mathrm{d}y_3,\mathrm{d}\dot y_3\}$, which is closed. Therefore, the state equations are

$$\begin{aligned}
\dot x_1 &= x_2\\
\dot x_2 &= \tfrac{u_2}{m}+x_1x_5^2\\
\dot x_3 &= -\tfrac{m}{J}x_1^2x_5\\
\dot x_4 &= x_5\\
\dot x_5 &= -\tfrac{u_1+2mx_1x_2x_5}{mx_1^2}\\
y_1 &= x_1\\
y_2 &= x_3\\
y_3 &= x_4
\end{aligned}$$

It should be mentioned that since equations (15) do not include derivatives of the control variables u_1, u_2, we need to integrate the elements of the subspace $\mathcal{H}_2$, which according to (3) is always in this form, see [1] for details. In fact, we can skip intermediate computations and directly write out the state space realization of i/o equations (15); however, we decided to show them to illustrate the theory presented above.

Example 3: Consider the "ball and beam" system [28], whose input is the angle and whose output is the ball position. The input-output equation of the system is

$$\ddot y=\frac{mR^2(y\dot u^2-g\sin(u))}{J+mR^2}, \tag{16}$$

where the constant parameters J, R, m represent, respectively, the inertia, radius and mass of the ball, and g is the gravitational constant.

Equation (16) can be described by two polynomials $p(\partial)=\partial^2-\frac{mR^2\dot u^2}{J+mR^2}$ and $q(\partial)=-\frac{2mR^2y\dot u}{J+mR^2}\partial+\frac{gmR^2\cos(u)}{J+mR^2}$. From (16), $n=2$ and $r=1$. Note that due to the fact that (16) is a single-input single-output system with $p=m=1$, one can simplify notation as $p_{11,0}=p_0, q_{11,0}=q_0$, etc. Next compute, according to (8), the left quotients of the polynomials $p(\partial)$ and $q(\partial)$ as

$$\begin{aligned}
p_1(\partial) &= \partial, & q_1(\partial) &= -\tfrac{2mR^2y\dot u}{J+mR^2},\\
p_2(\partial) &= 1, & q_2(\partial) &= 0.
\end{aligned}$$

Since $r=1$, according to Remark 1 and using (7), the elements of the last subspace $\mathcal{H}_3$ can be represented in the following form

$$\begin{aligned}
\omega_1 &= p_1(\partial)\mathrm{d}y+q_1(\partial)\mathrm{d}u=\mathrm{d}\dot y-\frac{2mR^2y\dot u}{J+mR^2}\mathrm{d}u,\\
\omega_2 &= p_2(\partial)\mathrm{d}y+q_2(\partial)\mathrm{d}u=\mathrm{d}y.
\end{aligned}$$

Finally, we get $\mathcal{H}_3=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y,\mathrm{d}\dot y-\frac{2mR^2y\dot u}{J+mR^2}\mathrm{d}u\}$. By Frobenius condition (4) the subspace $\mathcal{H}_3$ is not closed, and therefore, the i/o equations (16) do not admit the classical state-space realization.

V. CONCLUSIONS AND FUTURE WORKS

In this paper the minimal (accessible and observable) realization problem of nonlinear MIMO systems described by the set of i/o differential equations is addressed. The explicit formulas for deriving the differentials of the state coordinates were proposed, based on the polynomial representation of the system. Note that from the computational point of view the approach of this paper if compared to the earlier results has a number of advantages. First, it is straight-forward, meaning that there is no need to compute step-by-step all the $\mathcal{H}_k$ subspaces in order to find $\mathcal{H}_{r+2}$ as was proposed in [1], or the sequence of distributions $\mathcal{S}_k$ as in [14], or iterative Lie brackets of the vector fields as in [11]. In other words, using the polynomial representation of the system, one can immediately find the last subspace $\mathcal{H}_{r+2}$ with one-forms defining the differentials of the state coordinates. In addition, we have implemented the algorithm from [1] and the one presented in this paper in *Mathematica* package

NLControl [29], [30], and conclude that the program code of the introduced algorithm is shorter and more compact compared to the previous methods. The method suggested in this paper can be easily implemented within any symbolic programming language.

The possible direction for the future extension of this work is to construct the polynomial realization method for i/o equations defined on time scale, i.e. unifying the study of differential and difference equations, as well as for nonlinear system where the system variables are not partitioned into inputs and outputs, i.e. for the model used in the so-called behavioral approach, see, for example, [31] and [22]. Moreover, comparison of our results with those in the recent paper [5] needs detailed study at least by five reasons. First, the different mathematical tools are used. Second, whereas we are looking for generic solution valid everywhere except perhaps on a manifold of zero measure, the paper [5] is searching the solution around a specific (fixed) equilibrium (working) point. Third, in our paper like in the linear case, the minimality of the realization is defined as being accessible and observable whereas in [5] it is defined by minimality of the state dimension. Fourth, the starting point of the realization in the MIMO case is a canonical form of the set of i/o equations. This paper and [5] consider different canonical forms, respectively the extensions of Popov and Hermite canonical forms. Finally, this paper does not consider the case when the system is represented by the implicit set of differential equations, since the problem of transforming this set into the Popov form was studied in [14].

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