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On the Radius of Convergence of Cascaded Analytic Nonlinear Systems

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***Abstract*— A complete analysis is presented of the radius of convergence of the cascade connection of two analytic nonlinear input-output systems represented as Fliess operators. Such operators are described by convergent functional series, which are indexed by words over a noncommutative alphabet. Their generating series are therefore specified in terms of noncommutative formal power series. Given growth conditions on the coefficients of the generating series for the component systems, the radius of convergence of the cascaded system is computed.**

I. INTRODUCTION

Most complex systems found in applications can be viewed as a set of interconnected subsystems. This paper focuses on the cascade connection of analytic nonlinear input-output operators represented as *Fliess operators* [7], [8]. Such operators are described by functional series indexed by the set of words X^* over the noncommutative alphabet $X = \{x_0, x_1, \ldots x_m\}$. Their generating series are therefore specified in terms of noncommutative formal power series, the set of which is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. Specifically, one can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Define iteratively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the Fliess operator

$$F_c[u](t) = \sum_{\eta\in X^*} (c, \eta)\, E_\eta[u](t, t_0),$$

where $(c, \eta) \in \mathbb{R}^\ell$, $\eta \in X^*$. If there exist real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \eta \in X^*, \tag{1}$$

then F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0,\, t_0 + T]$ into $B^\ell_\mathfrak{q}(S)[t_0,\, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [11]. (Here $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) The set of all such *locally convergent* series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. In particular, when $\mathfrak{p} = 1$, the series defining $y = F_c[u]$ converges if

$$\max\{R, T\} < \frac{1}{M_c(m + 1)} \tag{2}$$

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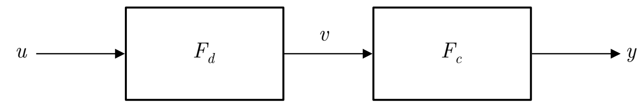

Fig. 1. The cascade connection of two Fliess operators.

[2], [3]. Let $\pi : \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}^+ \cup \{0\}$ take each series c to the *smallest* possible geometric growth constant M_c satisfying (1). In this case, $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ can be partitioned into equivalence classes, and the number $1/M_c(m + 1)$ will be referred to as the *radius of convergence* for the class $\pi^{-1}(M_c)$. This is in contrast to the usual situation where a radius of convergence is assigned to individual series. In practice, it is not difficult to estimate the minimal M_c for many series, in which case, the radius of convergence for $\pi^{-1}(M_c)$ provides an easily computed *lower bound* for the radius of convergence of c in the usual sense. Finally, when c satisfies the more stringent growth condition

$$|(c, \eta)| \leq K_c M_c^{|\eta|}, \quad \eta \in X^*, \tag{3}$$

then the series F_c defines an operator from the extended space $L^m_{\mathfrak{p},e}(t_0)$ into $C[t_0, \infty)$ [11]. The set of all such *globally convergent* series is designated by $\mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$.

The cascade connection of two Fliess operators as depicted in Fig. 1 always produces another input-output system with a Fliess operator representation [4], [5]. It was shown by Gray and Li in [9] that local convergence is preserved under composition, while global convergence in general is not preserved. For example, rational systems, which are always globally convergent, need not produce another rational system when cascaded [4], [5]. No claim was made in [9] that the growth constants derived there for the cascade connection constituted the radius of convergence, and, in fact, certain examples presented therein strongly suggested otherwise. Recent work, however, on self-excited feedback connected Fliess operators has produced some powerful new techniques for computing the radius of convergence of interconnected systems [10]. It will be shown in this paper that these methods can be applied to the cascade connection.

The remainder of the paper is organized as follows. In the next section, some mathematical preliminaries are presented to better frame the problem and establish the notation. In Section III the radius of convergence is computed for the case when the subsystems are both locally convergent. The global case is addressed in the subsequent section.

II. PRELIMINARIES

A. Formal Power Series

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η, while $|\eta|_{x_i}$ is the number of times the letter x_i appears in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is written as X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η). Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*}(c,\eta)\,\eta$. The notation $c \le d$ means that the component series satisfy $(c_i, \eta) \le (d_i, \eta)$ for all $\eta \in X^*$ and $i = 1, 2, \ldots, \ell$. The collection of all formal power series over X, $\mathbb{R}^\ell\langle\langle X\rangle\rangle$, forms an associative $\mathbb{R}$-algebra under the catenation product and a commutative $\mathbb{R}$-algebra under the shuffle product, denoted here by $\sqcup\!\sqcup$.

B. The Composition Product

To describe the generating series of the cascade connected system $F_c \circ F_d$, define the following family of mappings associated with $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$

$$D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup e),$$

where $i = 0, 1, \ldots, m$ and $d_0 := 1$. Assume $D_\emptyset$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Such maps can be composed in an obvious way so that $D_{x_i x_j} := D_{x_i} D_{x_j}$ provides an $\mathbb{R}$-algebra which is isomorphic to the usual $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ under the catenation product. The *composition product* of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as

$$(\underbrace{x_{i_k} x_{i_{k-1}} \cdots x_{i_1}}_{\eta}) \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ the definition is extended linearly as

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, D_\eta(1).$$

It was shown in [4], [5] that for any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the identity $F_c \circ F_d = F_{c\circ d}$ is satisfied. It is known in general that the composition product distributes to the left over the shuffle product. The following theorem states that local convergence is preserved under composition.

Theorem 1: [9] Suppose $c \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Then $c \circ d \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. Specifically,

$$|(c \circ d, \nu)| \le K_c((\phi(mK_d) + 1)M)^{|\nu|}(|\nu| + 1)!, \;\; \nu \in X^*,$$

where $\phi(x) := x/2 + \sqrt{x^2/4 + x}$ and $M = \max\{M_c, M_d\}$.

In light of (2) and the theorem above, a lower bound on the radius of convergence for $y = F_{c\circ d}[u]$ is $1/(\phi(mK_d) + 1)M(m + 1)$. No example has been presented to date for which the radius of convergence corresponds exactly to this bound.

Finally, in much of the analysis to follow, the subset of $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ described below will be useful.

Definition 1: [6], [7] A series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is said to be **exchangeable** if for arbitrary $\eta, \xi \in X^*$

$$|\eta|_{x_i} = |\xi|_{x_i},\; i = 0, 1, \ldots, m \;\;\Rightarrow\;\; (c, \eta) = (c, \xi).$$

Theorem 2: [10] If $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is an exchangeable series and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is arbitrary then the composition product can be written in the form

$$c \circ d = \sum_{k=0}^{\infty} \sum_{\substack{r_0,\ldots,r_m \ge 0 \\ r_0+\cdots+r_m=k}} (c, x_0^{r_0} \cdots x_m^{r_m}) \cdot D_{x_0}^{r_0}(1) \sqcup\!\sqcup \cdots \sqcup\!\sqcup D_{x_m}^{r_m}(1).$$

III. Locally Convergent Subsystems

The goal of this section is to calculate the smallest possible geometric growth constant for the cascade connection of two locally convergent Fliess operators, thereby producing the radius of convergence for the interconnection. The following theorem is a prerequisite for proving the main theorem of this section.

Theorem 3: Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c} \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ and $\bar{d} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$, where each component of $(\bar{c}, \eta) \in \mathbb{R}^\ell$ is $K_c M_c^{|\eta|}|\eta|!$, $\eta \in X^*$ with $K_c, M_c > 0$, and likewise, each component of $(\bar{d}, \eta) \in \mathbb{R}^m$ is $K_d M_d^{|\eta|}|\eta|!$, $\eta \in X^*$ with $K_d, M_d > 0$. If $\bar{b} = \bar{c} \circ \bar{d}$, then the sequence $(\bar{b}_i, x_0^k)$, $k \ge 0$ has the exponential generating function[1]

$$\begin{aligned} f(x_0) &:= \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k)\frac{x_0^k}{k!} \\ &= \frac{K_c}{1 - M_c x_0 + (mK_d M_c/M_d)\ln(1 - M_d x_0)} \end{aligned}$$

for any $i = 1, 2, \ldots, \ell$. Moreover, the smallest possible geometric growth constant for $\bar{b}$ is

$$M_{\bar{b}} = \frac{M_d}{1 - mK_d W\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mK_d M_c}\right)\right)},$$

where W denotes the Lambert W-function, namely, the inverse of the function

$$g(W) = W \exp(W)$$

[1].

Proof: There is no loss of generality in assuming $\ell = 1$. First observe that $\bar{c}$ is exchangeable, and thus, from Theorem 2 it follows that

$$\begin{aligned} \bar{b} &= \sum_{k=0}^{\infty} K_c M_c^k \sum_{\substack{r_0,\ldots,r_m \ge 0 \\ r_0+\cdots+r_m=k}} k!\, \frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \frac{(x_m \circ \bar{d})^{\sqcup\!\sqcup\, r_m}}{r_m!} \\ &= \sum_{k=0}^{\infty} K_c \left(M_c(x_0 + m x_0 \bar{d}_1)\right)^{\sqcup\!\sqcup\, k}. \end{aligned}$$

(Note that $\bar{d}_1 = \cdots = \bar{d}_m$.) Shuffling both sides of this equation by $M_c(x_0 + m x_0 \bar{d}_1)$ yields

$$\bar{b} \sqcup\!\sqcup M_c(x_0 + m x_0 \bar{d}_1) = \sum_{k=0}^{\infty} K_c(M_c(x_0 + m x_0 \bar{d}_1))^{\sqcup\!\sqcup\, k+1}.$$

Adding K_c to both sides gives

$$\bar{b} = K_c + M_c[\bar{b} \sqcup\!\sqcup (x_0 + m x_0 \bar{d}_1)]. \tag{4}$$

[1]The sequence $(\bar{b}_i, x_0^k)$, $k \ge 0$ and f are related by the formal Laplace-Borel transform as explained in [12].

By inspection, $(\bar{b},\emptyset)=K_c$, $(\bar{b},x_0)=K_cM_c(1+mK_d)$ and $(\bar{b},x_i)=0$ for $i=1,2,\ldots,m$. Let $(\bar{b},\nu_n):=\max\{(\bar{b},\nu):\nu\in X^n\}$. For any $\nu\in X^n, n\geq 2$ it follows from (4) that

$$
\begin{aligned}
&(\bar{b},\nu)\\
&=M_c\sum_{i=0}^{n}\sum_{\substack{\eta\in X^i\\ \xi\in X^{n-i}}}(\bar{b},\eta)(x_0+mx_0\bar{d}_1,\xi)(\eta \sqcup\!\sqcup\, \xi,\nu)\\
&=M_c\sum_{i=0}^{n-1}\sum_{\substack{\eta\in X^i\\ \xi\in X^{n-i}}}(\bar{b},\eta)(x_0+mx_0\bar{d}_1,\xi)(\eta \sqcup\!\sqcup\, \xi,\nu)\\
&\leq M_c\sum_{i=0}^{n-1}(\bar{b},\nu_i)\sum_{\substack{\eta\in X^i\\ x_0\xi'\in X^{n-i}}}(x_0+mx_0\bar{d}_1,x_0\xi')\,(\eta \sqcup\!\sqcup\, x_0\xi',\nu)\\
&=M_c\sum_{i=0}^{n-2}(\bar{b},\nu_i)\sum_{\substack{\eta\in X^i\\ \xi'\in X^{n-i-1}}}(1+m\bar{d}_1,\xi')(\eta \sqcup\!\sqcup\, x_0\xi',\nu)+\\
&\qquad M_c(\bar{b},\nu_{n-1})\sum_{\eta\in X^{n-1}}(1+m\bar{d}_1,\emptyset)(\eta \sqcup\!\sqcup\, x_0,\nu).
\end{aligned}
$$

In the first summation directly above, note that $|\xi'|\geq 1$, and thus, $(1+m\bar{d}_1,\xi')=m(\bar{d}_1,\xi')$. Consequently,

$$
\begin{aligned}
(\bar{b},\nu)&\leq M_c\sum_{i=0}^{n-2}(\bar{b},\nu_i)\,mK_dM_d^{(n-i-1)}(n-i-1)!\cdot\\
&\qquad\sum_{\substack{\eta\in X^i\\ \xi'\in X^{n-i-1}}}(\eta \sqcup\!\sqcup\, x_0\xi',\nu)+(\bar{b},\nu_{n-1})M_c(1+mK_d)\cdot\\
&\qquad\sum_{\eta\in X^{n-1}}(\eta \sqcup\!\sqcup\, x_0,\nu)\\
&\leq M_c\sum_{i=0}^{n-2}(\bar{b},\nu_i)\,mK_dM_d^{(n-i-1)}(n-i-1)!\cdot\\
&\qquad\sum_{\substack{\eta\in X^i\\ \xi\in X^{n-i}}}(\eta \sqcup\!\sqcup\, \xi,\nu)+(\bar{b},\nu_{n-1})M_c(1+mK_d)\cdot\\
&\qquad\sum_{\substack{\eta\in X^{n-1}\\ \xi\in X}}(\eta \sqcup\!\sqcup\, \xi,\nu)\\
&=M_c\sum_{i=0}^{n-2}(\bar{b},\nu_i)\,mK_dM_d^{(n-i-1)}(n-i-1)!\binom{n}{i}+\\
&\qquad(\bar{b},\nu_{n-1})M_c(1+mK_d)n.
\end{aligned}
$$

Note that the inequality above still holds when the left-hand side is replaced with $(\bar{b},\nu_n)$. Now let $a_n,\ n\geq 0$ be the sequence satisfying the recurrence relation

$$
\begin{aligned}
a_n&=M_c\sum_{i=0}^{n-2}a_imK_dM_d^{(n-i-1)}(n-i-1)!\binom{n}{i}+\\
&\qquad a_{n-1}M_c(1+mK_d)n,\ n\geq 2,
\end{aligned}
$$

where $a_0=K_c$ and $a_1=K_cM_c(1+mK_d)$. Since the relation above involves only positive terms, it follows that $(\bar{b},\nu_n)\leq a_n$, $\forall n\geq 0$. It is easily verified that the sequence $a_n, n\geq 0$ has the exponential generating function

$$
f(x_0)=\frac{K_c}{1-M_cx_0+(mK_dM_c/M_d)\ln(1-M_dx_0)}. \tag{5}
$$

When all the growth constants and m are unity, $a_n,\ n\geq 0$ is the integer sequence number A052820 in the Online Encyclopedia of Integer Sequences (OEIS) [13].

Next it will be shown that $(\bar{c}\circ\bar{d},x_0^n)=a_n,\ n\geq 0$. It is sufficient to show that the zero-input response of the cascade system represented by the Fliess operator $F_{\bar{c}\circ\bar{d}}$ is equal to f. The generating series for $v_1=F_{\bar{d}_1}[0]$ is

$$
c_{v_1}=\bar{d}_1\circ 0=\sum_{k=0}^{\infty}K_dM_d^kk!\;x_0^k,
$$

and thus,

$$
v_1(t)=\sum_{k=0}^{\infty}K_dM_d^kt^k=\frac{K_d}{1-M_dt}.
$$

Now from (4) observe

$$
\bar{c}\circ\bar{d}=K_c+(\bar{c}\circ\bar{d})\sqcup\!\sqcup\, M_c(x_0+mx_0\bar{d}_1).
$$

Note that $x_0\bar{d}_1$ has the exponential generating function $\int_0^t v_1(\tau)\,d\tau$. Therefore,

$$
\begin{aligned}
y(t)&=F_{\bar{c}}[v](t)=F_{\bar{c}}[F_{\bar{d}}[0]](t)=F_{\bar{c}\circ\bar{d}}[0](t)\\
&=K_c+y(t)M_c\left(t+m\int_0^t v_1(\tau)\,d\tau\right)\\
&=\frac{K_c}{1-M_c\left(t+m\int_0^t v_1(\tau)\,d\tau\right)}\\
&=\frac{K_c}{1-M_ct+(mK_dM_c/M_d)\ln(1-M_dt)}\\
&=f(t).
\end{aligned}
$$

This proves that for every $n\geq 0$

$$
(\bar{b},\nu)\leq(\bar{b},\nu_n)\leq a_n=(\bar{b},x_0^n),\ \nu\in X^n.
$$

Since f is analytic at the origin, the smallest geometric growth constant is determined by the location of any singularity nearest to the origin in the complex plane, say x_0' [15, Theorem 2.4.3]. Specifically, $M_{\bar{b}}=1/|x_0'|$, where it is easily verified from (5) that x_0' is the positive real number

$$
x_0'=\frac{1}{M_d}\left[1-mK_dW\left(\frac{1}{mK_d}\exp\left(\frac{M_c-M_d}{mK_dM_c}\right)\right)\right].
$$

This proves the theorem. ∎

It is known that if u is analytic with generating series c_u, then $y=F_c[u]$ is also analytic [14], and its generating series is given by $c_y=c\circ c_u$ [9], [12]. In this situation, the following corollary is useful for estimating a lower bound on the interval of convergence for the output.

Corollary 1: Let $X=\{x_0,x_1,\ldots,x_m\}$ and $X_0=\{x_0\}$. Let $\bar{c}\in\mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $\bar{c}_u\in\mathbb{R}^{m}_{LC}[[X_0]]$, where each component of $(\bar{c},\eta)\in\mathbb{R}^{\ell}$ is $K_cM_c^{|\eta|}|\eta|!$, $\eta\in X^*$ with $K_c,M_c>0$, and likewise, each component of $(\bar{c}_u,x_0^n)\in\mathbb{R}^m$ is $K_{c_u}M_{c_u}^nn!$, $n\geq 0$ with $K_{c_u},M_{c_u}>0$. If $\bar{c}_y=\bar{c}\circ\bar{c}_u$, then the sequence $(\bar{c}_{y_i},x_0^k), k\geq 0$, has the exponential generating function

$$
f(x_0)=\frac{K_c}{1-M_cx_0+(mK_{c_u}M_c/M_{c_u})\ln(1-M_{c_u}x_0)}
$$

for any $i = 1, 2, \ldots, \ell$. Moreover, the smallest possible geometric growth constant for $\bar{c}_y$ is

$$M_{\bar{c}_y} = \frac{M_{c_u}}{1 - mK_{c_u} W\left(\frac{1}{mK_{c_u}} \exp\left(\frac{M_c - M_{c_u}}{mK_{c_u} M_c}\right)\right)}.$$

The following lemma is also needed to prove the main result.

Lemma 1: Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ such that $|c_i| \leq d_i$, $i = 1, 2, \ldots, \ell$, where $|c_i| := \sum_{\eta\in X^*} |(c_i, \eta)|\ \eta$. For $\xi \in X^*$ it follows that $|\xi \circ c| \leq \xi \circ d$.

Proof: The proof is by induction on $k = |\xi| - |\xi|_{x_0}$. Let $\xi_0 = x_0^{n_0}$ and $\xi_k = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} \cdots x_{i_1} x_0^{n_0}$ for $k > 0$, where $1 \leq i_j \leq m$. For $k = 0$, the claim is trivial since

$$\xi_0 \circ c = x_0^{n_0} \circ c = x_0^{n_0} = x_0^{n_0} \circ d = \xi_0 \circ d.$$

Assume now that $|(\xi_k \circ c, \eta)| \leq (\xi_k \circ d, \eta)$ up to some fixed $k \geq 0$. Observe that

$$\begin{aligned}
\xi_{k+1} \circ c &= x_0^{n_{k+1}+1}(c_{i_{k+1}} ⧢ (\xi_k \circ c)) \\
(\xi_{k+1} \circ c, \eta) &= (c_{i_{k+1}} ⧢ (\xi_k \circ c), x_0^{-(n_{k+1}+1)}(\eta)) \\
&= \sum_{j=0}^{n} \sum_{\substack{\alpha\in X^j \\ \beta\in X^{n-j}}} (c_{i_{k+1}}, \alpha)(\xi_k \circ c, \beta) \cdot \\
&\qquad (\alpha ⧢ \beta, x_0^{-(n_{k+1}+1)}(\eta)),
\end{aligned}$$

where $x_0^{-i}(\cdot)$ denotes the left-shift operator $x_0^{-1}(\cdot)$ applied i times, and $n := |x_0^{-(n_{k+1}+1)}(\eta)| \geq 0$. Therefore,

$$\begin{aligned}
|(\xi_{k+1} \circ c, \eta)| &\leq \sum_{j=0}^{n} \sum_{\substack{\alpha\in X^j \\ \beta\in X^{n-j}}} \left|(c_{i_{k+1}}, \alpha)\right| |(\xi_k \circ c, \beta)| \cdot \\
&\qquad (\alpha ⧢ \beta, x_0^{-(n_{k+1}+1)}(\eta)) \\
&\leq \sum_{j=0}^{n} \sum_{\substack{\alpha\in X^j \\ \beta\in X^{n-j}}} (d_{i_{k+1}}, \alpha)(\xi_k \circ d, \beta) \cdot \\
&\qquad (\alpha ⧢ \beta, x_0^{-(n_{k+1}+1)}(\eta)) \\
&= (\xi_{k+1} \circ d, \eta).
\end{aligned}$$

Thus, the inequality holds for all $k \geq 0$, and the lemma is proved. ■

Finally, the main result of this subsection is presented.

Theorem 4: Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $b = c \circ d$ then

$$|(b, \nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{6}$$

for some $K_b > 0$, where

$$M_b = \frac{M_d}{1 - mK_d W\left(\frac{1}{mK_d} \exp\left(\frac{M_c - M_d}{mK_d M_c}\right)\right)}.$$

Furthermore, no smaller geometric growth constant can satisfy (6).

Proof: Since $|d| \leq \bar{d}$, it follows from Lemma 1 that for any $\nu \in X^*$

$$\begin{aligned}
|(b, \nu)| &\leq \sum_{\eta\in X^*} |(c, \eta)||(\eta \circ d, \nu)| \\
&\leq \sum_{\eta\in X^*} K_c M_c^{|\eta|} |\eta|!\ (\eta \circ \bar{d}, \nu) \\
&= (\bar{b}_i, \nu),
\end{aligned}$$

where $\bar{b} = \bar{c} \circ \bar{d}$ and $i = 1, 2, \ldots, \ell$. In light of Theorem 3, $(\bar{b}_i, \nu)$ is *asymptotically* bounded by $M_b^{|\nu|}|\nu|!$. Thus, some $K_b > 0$ can always be introduced such that

$$(\bar{b}_i, \nu) \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^*.$$

Furthermore, $(\bar{b}_i, x_0^n)$ is growing exactly at this rate. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ■

Example 1: Let $X = \{x_0, x_1\}$ and $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $M = M_c = M_d$. Then

$$\begin{aligned}
M_b &= \frac{M}{1 - K_d W(1/K_d)} = \left(\frac{3}{2} + K_d + O\left(\frac{1}{K_d}\right)\right) M \\
&\approx K_d M
\end{aligned}$$

when $K_d \gg 1$. This is consistent with Theorem 1. On the other hand, if $K_d = 1$ then $M_b = (1 - W(1))^{-1} M = 2.3102M$, which is *less* than the estimate $(\phi_g + 1)M = 2.6180M$ given by Theorem 1. □

Example 2: Suppose $X = \{x_0, x_1\}$ and $\bar{b} = \bar{c} \circ \bar{d}$ with $\bar{c} = \sum_{\eta\in X^*} K_c M_c^{|\eta|} |\eta|!\ \eta$ and $\bar{d} = \sum_{\eta\in X^*} K_d M_d^{|\eta|} |\eta|!\ \eta$. The output of the cascaded system as shown in Fig. 1 is described by the state space system

$$\begin{aligned}
\dot{z}_1 &= \frac{M_c}{K_c} z_1^2 (1 + z_2), \quad z_1(0) = K_c \\
\dot{z}_2 &= \frac{M_d}{K_d} z_2^2 (1 + u), \quad z_2(0) = K_d \\
y &= z_1.
\end{aligned}$$

A MATLAB generated zero-input response is shown in Fig. 2 when $K_c = 1$, $M_c = 2$, $K_d = 3$ and $M_d = 4$. As expected from Theorem 3, the finite escape time of the output is $t_{esc} = 1/M_b = 0.1028$. The output responses corresponding to the analytic inputs $u_1(t) = 1/(1 - t)$ and $u_2(t) = 1/(1 - t^2)$, each having growth constants $K_{c_u} = M_{c_u} = 1$, are also shown in the figure. Their respective finite escape times are $t_{esc} = 0.08321$ and $t_{esc} = 0.08377$. Here u_1 has the shortest escape time since all the coefficients of its generating series are growing at the maximum rate, while u_2 has all its odd coefficients equal to zero. By Corollary 1, any finite escape time for the output corresponding to any analytic input with the given growth constants K_{c_u}, M_{c_u} must be at least as large as $T = 1/M_{\bar{c}_y} = 0.05073$. □

IV. Globally Convergent Subsystems

The goal of this section is to calculate the smallest possible geometric growth constant for the cascade connection of two globally convergent Fliess operators, thus producing the radius of convergence for the interconnection. The following two theorems are essential for proving the main theorem.

Theorem 5: Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c} \in \mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$ and $\bar{d} \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$, where each component of $(\bar{c}, \eta) \in \mathbb{R}^\ell$ is $K_c M_c^{|\eta|}$, $\eta \in X^*$ with $K_c, M_c > 0$, and likewise, each component of $(\bar{d}, \eta) \in \mathbb{R}^m$ is $K_d M_d^{|\eta|}$, $\eta \in$

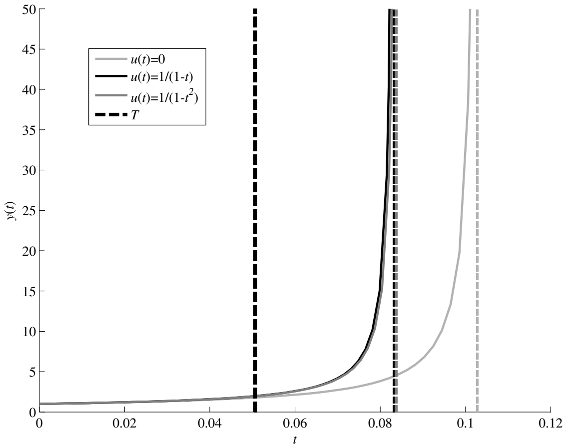

Fig. 2. Output responses of the cascade system $F_{\bar{c}\circ\bar{d}}$ to various analytic inputs in Example 2.

X^* with $K_d, M_d > 0$. If $\bar{b} = \bar{c}\circ\bar{d}$, then $(\bar{b}_i,\nu) \leq (\bar{b}_i, x_0^{|\nu|})$, $\nu\in X^*$, and the sequence $(\bar{b}_i, x_0^k)$, $k\geq 0$ has the exponential generating function

$$f(x_0) = K_c \exp\left(\frac{mK_d\exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c}\right)$$

for any $i = 1, 2, \ldots, \ell$.

Proof: As in the local case, there is no loss of generality in assuming $\ell = 1$. Using Theorem 2, observe that

$$\begin{aligned}\bar{b} &= \sum_{k=0}^{\infty}\frac{K_c M_c^k}{k!}\sum_{\substack{r_0,\ldots,r_m\geq 0\\ r_0+\cdots+r_m=k}} k!\,\frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!}\sqcup\!\sqcup\cdots\sqcup\!\sqcup\frac{(x_m\circ\bar{d})^{\sqcup\!\sqcup\, r_m}}{r_m!}\\ &= K_c\sum_{k=0}^{\infty}\frac{\left(M_c(x_0+mx_0\bar{d}_1)\right)^{\sqcup\!\sqcup\, k}}{k!}.\end{aligned}$$

Therefore, $(\bar{b},\emptyset) = K_c$ and

$$\begin{aligned}&x_0^{-1}(\bar{b})\\ &= K_c\sum_{k=1}^{\infty}\frac{\left(M_c(x_0+mx_0\bar{d}_1)\right)^{\sqcup\!\sqcup\, k-1}}{(k-1)!}\sqcup\!\sqcup\, M_c(1+m\bar{d}_1)\\ &= \bar{b}\sqcup\!\sqcup\, M_c(1+m\bar{d}_1).\end{aligned} \tag{7}$$

By inspection,

$$\begin{aligned}(x_0^{-1}(\bar{b}),\emptyset) &= K_cM_c(1+mK_d)\\ (x_0^{-1}(\bar{b}),x_0) &= K_cM_cmK_dM_d + K_c(M_c(1+mK_d))^2\\ (x_0^{-1}(\bar{b}),x_i) &= K_cM_cmK_dM_d,\ i=1,2,\ldots,m.\end{aligned}$$

For any $\nu\in X^n$, $n\geq 2$, it follows that

$$\begin{aligned}&(x_0^{-1}(\bar{b}),\nu)\\ &= M_c\sum_{i=0}^{n}\sum_{\substack{\eta\in X^i\\ \xi\in X^{n-i}}}(\bar{b},\eta)(1+m\bar{d}_1,\xi)(\eta\sqcup\!\sqcup\,\xi,\nu)\\ &= M_c\sum_{i=1}^{n-1}\sum_{\substack{x_0\eta'\in X^i\\ \xi\in X^{n-i}}}(\bar{b},x_0\eta')(1+m\bar{d}_1,\xi)(x_0\eta'\sqcup\!\sqcup\,\xi,\nu)+\\ &\quad M_c\sum_{x_0\eta'\in X^n}(\bar{b},x_0\eta')(1+m\bar{d}_1,\emptyset)(x_0\eta',\nu)+\\ &\quad M_c\sum_{\xi\in X^n}(\bar{b},\emptyset)(1+m\bar{d}_1,\xi)(\xi,\nu)\\ &= M_c\sum_{i=1}^{n-1}\sum_{\substack{\eta'\in X^{i-1}\\ \xi\in X^{n-i}}}(x_0^{-1}(\bar{b}),\eta')(1+m\bar{d}_1,\xi)(x_0\eta'\sqcup\!\sqcup\,\xi,\nu)+\\ &\quad M_c\sum_{\eta'\in X^{n-1}}(x_0^{-1}(\bar{b}),\eta')(1+m\bar{d}_1,\emptyset)(x_0\eta',\nu)+\\ &\quad M_c(\bar{b},\emptyset)m(\bar{d}_1,\nu).\end{aligned}$$

Therefore,

$$\begin{aligned}&(x_0^{-1}(\bar{b}),\nu)\\ &\leq M_c\sum_{i=1}^{n-1}(x_0^{-1}(\bar{b}),\eta_{i-1})mK_dM_d^{n-i}\sum_{\substack{\eta\in X^i\\ \xi\in X^{n-i}}}(\eta\sqcup\!\sqcup\,\xi,\nu)+\\ &\qquad (x_0^{-1}(\bar{b}),\eta_{n-1})M_c(1+mK_d)+K_cM_cmK_dM_d^n\\ &= M_c\sum_{i=1}^{n-1}(x_0^{-1}(\bar{b}),\eta_{i-1})mK_dM_d^{n-i}\binom{n}{i}+\\ &\qquad (x_0^{-1}(\bar{b}),\eta_{n-1})M_c(1+mK_d)+K_cM_cmK_dM_d^n.\end{aligned}$$

Similar to the analysis in the previous section, let $a_n,\ n\geq 0$ be the sequence satisfying the recurrence relation

$$\begin{aligned}a_n &= M_c\sum_{i=1}^{n-1}a_{i-1}mK_dM_d^{n-i}\binom{n}{i}+\\ &\qquad a_{n-1}M_c(1+mK_d)+K_cM_cmK_dM_d^n,\ n\geq 2,\end{aligned}$$

where $a_0 = K_cM_c(1+mK_d)$ and $a_1 = K_cM_cmK_dM_d + K_c(M_c(1+mK_d))^2$. It follows that $(x_0^{-1}(\bar{b}),\nu_n)\leq a_n,\ \forall n\geq 0$, and thus, $(\bar{b},\nu_n)\leq b_n,\ \forall n\geq 0$, where $b_n = a_{n-1}$ and $b_0 = K_c$. It is easily verified that the sequence $b_n, n\geq 0$ has the exponential generating function

$$f(x_0) = K_c \exp\left(\frac{mK_d\exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c}\right).$$

When all the growth constants and m are unity, b_n, $n\geq 0$ is the integer sequence number A000110 (shifted one position to the left) in the OEIS. These integers are called the *Bell numbers*.

Next it will be shown that $(\bar{c}\circ\bar{d}, x_0^n) = b_n$, $n\geq 0$. It is sufficient to show that the zero-input response of the cascade system represented by the Fliess operator $F_{\bar{c}\circ\bar{d}}$ is equal to f. The generating series for $v_1 = F_{\bar{d}_1}[0]$ is

$$c_{v_1} = \bar{d}_1\circ 0 = \sum_{k=0}^{\infty}K_dM_d^k\,x_0^k,$$

and thus,

$$v_1(t) = \sum_{k=0}^{\infty}K_dM_d^k\frac{t^k}{k!} = K_d\exp(M_dt).$$

From (7) and the fact that $x_i^{-1}(\bar{b}) = 0$, $i = 1, 2, \ldots, m$, it follows that

$$y'(t) = M_cy(t)(1+mK_d\exp(M_dt)),\ \ y(0) = K_c.$$

Solving this differential equation yields

$$y(t) = K_c \exp\left(\frac{mK_d \exp(M_d t) + M_d t - mK_d}{M_d/M_c}\right).$$

Thus, for every $n \geq 0$

$$(\bar{b}, \nu) \leq (\bar{b}, \nu_n) \leq b_n = (\bar{b}, x_0^n),\ \nu \in X^n,$$

and the theorem is proved. ■

Theorem 6: Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Assume $\bar{c}$ and $\bar{d}$ are defined as in Theorem 5. If $b = c \circ d$ and $\bar{b} = \bar{c} \circ \bar{d}$ then

$$|(b, \nu)| \leq (\bar{b}_i, x_0^{|\nu|}),\ \nu \in X^*,\ i = 1, 2, \ldots, \ell,$$

where the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ has the exponential generating function

$$f(x_0) = K_c \exp\left(\frac{mK_d \exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c}\right).$$

Proof: Again from Lemma 1, it follows that for any $\nu \in X^*$

$$\begin{aligned} |(b, \nu)| &\leq \sum_{\eta \in X^*} |(c, \eta)||(\eta \circ d, \nu)| \\ &\leq \sum_{\eta \in X^*} K_c M_c^{|\eta|}\ (\eta \circ \bar{d}, \nu) \\ &= (\bar{b}_i, \nu). \end{aligned}$$

By Theorem 5, $(\bar{b}_i, \nu)$ is bounded by $(\bar{b}_i, x_0^{|\nu|})$, which has the exponential generating function $f(x_0)$. Thus, the theorem is proved. ■

It is worth noting that the Bell numbers (without any left shift), B_n, have the exponential generating function $e^{e^x - 1}$. Their asymptotic behavior is

$$B_n \sim n^{-\frac{1}{2}} (\lambda(n))^{n+\frac{1}{2}} e^{\lambda(n)-n-1},$$

where $\lambda(n) = n/W(n)$. Thus, the Lambert W-function appears to also play a role in the global problem. More importantly, since the double exponential appearing in Theorem 5 has no finite singularities, as appeared in the local analysis in Section III, the following main result is immediate.

Theorem 7: The cascade connection of two globally convergent Fliess operators has a radius of convergence equal to infinity. Therefore, the output of such a system is always well defined over any finite interval of time when $u \in L^m_{1,e}(t_0)$.

It is important to understand that this theorem is *not* saying that the composite system has a globally convergent generating series in the sense of (3). If this were the case, then it would be possible to bound $y(t) = F_{c \circ d}[0]$ by a single exponential function rather than a double exponential function (see [11, Theorem 3.1]). Thus, the fastest possible growth rate for the coefficients of a cascade connection involving components with globally convergent generating series falls somewhere strictly *in between* the local growth condition (1) and the global growth condition (3).

Example 3: Suppose $X = \{x_0, x_1\}$ and $\bar{b} = \bar{c} \circ \bar{d}$ with $\bar{c} = \sum_{\eta \in X^*} K_c M_c^{|\eta|}\, \eta$ and $\bar{d} = \sum_{\eta \in X^*} K_d M_d^{|\eta|}\, \eta$. The cascaded system is described by the state space realization

$$\begin{aligned} \dot{z}_1 &= M_c z_1 (1 + z_2), \quad z_1(0) = K_c \\ \dot{z}_2 &= M_d z_2 (1 + u), \quad z_2(0) = K_d \\ y &= z_1. \end{aligned}$$

A MATLAB generated zero-input response of this system is shown on a double logarithmic scale in Fig. 3 when $K_c = M_c = K_d = M_d = 1$. As expected from Theorem 5, this plot asymptotically approaches that of $\tilde{y}(t) = t$ as $t \to \infty$. □

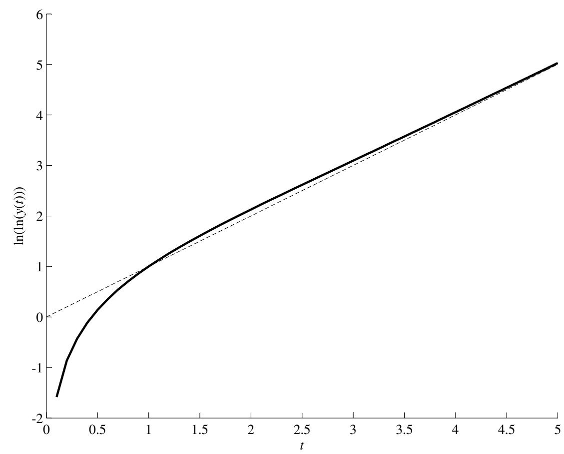

Fig. 3. Zero-input response of the cascade system $F_{\bar{c} \circ \bar{d}}$ in Example 3 on a double logarithmic scale and the function $\tilde{y}(t) = t$ (dashed line).

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