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Singularly Perturbed Implicit Control Law for Linear Time Varying SISO System. Part II: State Observation

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***Abstract*— This paper considers the problem of stabilizing a single-input single-output linear time varying system using a low order controller and an equalizer filter. The closed loop is a linear singularly perturbed system with uniform asymptotic stability behavior. Following the results of Kokotović's book, we show how to design a control scheme, control law plus state observer, such that the system dynamics is assigned by a Hurwitz polynomial with constant coefficients. We calculate bounds, $\varepsilon \in (0, \varepsilon^*)$, for guaranteeing the uniform asymptotic stability of the singularly perturbed closed loop system.**

Notation

- $\underline{\chi}_k^i \in \mathbb{R}^k$ stands for the vector which the i-th entry is equal to 1 and the other ones are equal to 0. $\mathrm{I}_k \in \mathbb{R}^{k\times k}$ stands for the identity matrix of size k. $\mathrm{T}_u\{v^T\}$ stands for the upper triangular Toeplitz matrix, which first row is v^T. $\mathrm{T}_\ell\{v\}$ stands for the lower triangular Toeplitz matrix, which first column is v. $0_{\mu,\nu} \in \mathbb{R}^{\mu\times\nu}$ stands for the zero matrix, or simply 0_μ when $\mu = \nu$. And $\underline{0}_\nu \in \mathbb{R}^\nu$ stands for the zero vector. $BDM\{H_1, H_2, \ldots, H_n\}$ denotes a block diagonal matrix whose diagonal blocks are $\{H_1, H_2, \ldots, H_n\}$.
- Given a vector function $f(\cdot) \in \mathbb{R}^n$, $\|f(\cdot)\| = \|f(\cdot)\|_2$ and for a function matrix $A(\cdot) \in \mathbb{R}^{n\times n}$, $\|A(\cdot)\| = \|A(\cdot)\|_2$, see [2]. A vector function $f(\varepsilon, t) \in \mathbb{R}^n$ is said to be $\mathcal{O}(\varepsilon)$ over an interval $[t_1, t_2]$ if there exist positive constants K and ε^* such that $\|f(\varepsilon, t)\| \le K\varepsilon, \quad \forall\varepsilon \in [0, \varepsilon^*], \quad \forall t \in [t_1, t_2]$, see [4].

I. INTRODUCTION

This paper is a continuation of [14], where we have proposed a control law for SISO time varying systems, $\mathrm{d}\zeta/\mathrm{d}t = A(t)\zeta + B(t)u$, based on the singular perturbations approach [8], where the knowledge of the time varying parameters is not required, but only some bounds. The aim of such a control law is to approximately match the closed loop system to a given time-invariant linear state space system represented by $\mathrm{d}x_s/\mathrm{d}t = A_0x_s + B_1r$. We are now incorporating the observation problem to the control law proposed in [14], following also the singularly perturbed framework.

The synthesis of observers based on the singularly perturbed systems approach is not new, see for example [13], [5], [6]. With respect to the case of linear time varying systems two important papers are [7] and [11].

In [11] the synthesis of the state observer is realized by separating a classical full order Luenberger observer into the slow and fast subsystems, one observer is for the slow subsystem and the other observer is for the fast subsystem. For the stability analysis of the closed loop system, the knowledge of the parameters system is required.

In [7], the observer is only synthesized for the slow subsystem. For the stability analysis of the closed loop system, the knowledge of the parameters system is again required.

In this paper, we propose a high gain observer, for SISO time varying systems, based on the singular perturbations approach, where the knowledge of the parameters is not required. We only need the knowledge of some bounds on the parameters system. Based on these bounds, we give an upper bound, $\varepsilon^* > 0$, for the singularly perturber parameter, ε, such that for a positive ε smaller than ε^*, the closed loop stability is guaranteed.

The paper is organized as follows. The problem is first stated in section II. Next, in Section III we recall the singularly perturbed linear control law, proposed in [14], which aim is to lead the closed loop into the Kokotović's singularly perturbed system model. In Section IV we propose a high gain observer, by a proper approximation. In Section V we study the stability of the system and the observer together. Finally, in Section VI we give an academic example. All the proofs are sent to the Appendix.

II. System Definition and Representation Form

Let us consider a Linear Time Varying System (LTVS), which dynamics is represented by:

$$\frac{d^n}{dt^n}y + a_n(t)\frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_2(t)\frac{d}{dt}y + a_1(t)y = b(t)u \quad (1)$$

defined for $t \ge t_0 \ge 0$, with initial conditions: $y(t_0)$, $\mathrm{d}y(t_0)/\mathrm{d}t$, $\ldots$, $\mathrm{d}^{n-1}y(t_0)/\mathrm{d}t^{n-1}$, where $y \in \mathbb{R}$ is the

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dependent variable, $u \in \mathbb{R}$ is the input, at time $t \in J = [0, \infty)$. The coefficients, $a_i(t)$ and $b(t)$, are unknown and such that:[1]

- H1 $a_i(\cdot) \in \mathcal{C}^\infty(J, \mathbb{R})$, $\|a_i(t)\| \leq L_{0,a}$ and $\|\frac{\mathrm{d}}{\mathrm{d}t} a_i(t)\| \leq L_{1,a}$, $\forall\, i \in \{1, \ldots, n\}$, $\forall\, t \in J$.
- H2 $b(\cdot) \in \mathcal{C}^\infty(J, \mathbb{R})$, $0 < b_1 \leq b(t) \leq b_2$ and $\|\frac{\mathrm{d}}{\mathrm{d}t} b(t)\| \leq c$, $\forall\, t \in J$.

A simple device can be used to recast this differential equation into the form of a linear state equation with input u and output y. Defining the state variables as $\zeta^T = \begin{bmatrix} y & \mathrm{d}y/\mathrm{d}t & \ldots & \mathrm{d}^{n-1}y/\mathrm{d}t^{n-1} \end{bmatrix}$, we then get time varying linear state equation:

$$\begin{gathered} \mathrm{d}\zeta/\mathrm{d}t = A(t)\zeta + B(t)u \\ A(t) = \left(\mathrm{T}_u\{(\underline{\chi}_n^2)^T\} - \underline{\chi}_n^n(\underline{\mathfrak{a}}_n(t))^T\right) \\ B(t) = b(t)\underline{\chi}_n^n \\ y(t) = C\zeta,\ C = (\underline{\chi}_n^1)^T, \end{gathered} \tag{2}$$

where: $\underline{\mathfrak{a}}_k(t) = \begin{bmatrix} a_1(t) & \cdots & a_k(t) \end{bmatrix}^T$, $k \in \{1, \ldots, n\}$; the initial state condition is: $\zeta(t_0) = \begin{bmatrix} y(t_0) \ \mathrm{d}y(t_0)/\mathrm{d}t \ \cdots \ \mathrm{d}^{n-1}y(t_0)/\mathrm{d}t^{n-1} \end{bmatrix}^T$.

III. Singularly Implicit Control Law

For the state space representation (2) we have proposed in [14] the following control law, composed by a singularly perturbed control law and an equalizer filter,

Singularly perturbed control law

$$\varepsilon u = -(\underline{\chi}_n^n)^T(\zeta + h) + (\underline{\chi}_2^1)^T \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} \tag{3}$$

Equalizer filter

$$\begin{aligned} \begin{bmatrix} \frac{\mathrm{d}x_n}{\mathrm{d}t} \\ \frac{\mathrm{d}x_{n+1}}{\mathrm{d}t} \end{bmatrix} &= \begin{bmatrix} -(\bar{\underline{\mathfrak{a}}}_n)^T + (1+\ell)(\underline{\chi}_n^n)^T \\ -(\beta - 1)(\underline{\chi}_n^n)^T \end{bmatrix}(\zeta + h) \\ &\quad + \begin{bmatrix} -(1+\ell) & -\ell \\ (\beta - 1) & -\beta \end{bmatrix}\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} + \underline{\chi}_2^1 r \end{aligned} \tag{4}$$

where: $\ell = 1/\tau - \beta$ and $\bar{\underline{\mathfrak{a}}}_k = \begin{bmatrix} \bar{a}_1 & \cdots & \bar{a}_k \end{bmatrix}^T$, with: $k \in \{1, \ldots, n\}$; β, τ and ε are positive parameters; and $\bar{a}_1, \ldots, \bar{a}_n$ are the coefficients of the Hurwitz polynomial $p(\lambda) = \lambda^n + \bar{a}_n\lambda^{n-1} + \cdots + \bar{a}_2\lambda + \bar{a}_1$. r is a signal reference and h is a perturbation, such that:

- H3 $r \in L^\infty \cap \mathcal{C}^\infty(J, \mathbb{R})$.
- H4 h is bounded continuous real function, which norm is of order ε.

The aim of the *equalizer filter* is:

1) To assign the closed loop dynamics at a time invariant linear system with the Hurwitz characteristic polynomial $p(\lambda)$.
2) To assign a rate of exponential convergence to the desired dynamics.

The aim of the *singularly perturbed control law* is:

1) To change the base representation system for obtaining a *singularly perturbed model*.
2) To close the desired dynamics by an ε order.

The perturbation signal h is considered, in order to take into account the effects of the high gain observer, which is considered in Section IV. The closed loop system is represented by the following singularly perturbed description:

$$\begin{aligned} \begin{bmatrix} \mathrm{d}x/\mathrm{d}t \\ \varepsilon\mathrm{d}z/\mathrm{d}t \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21}(\varepsilon, t) & A_{22}(\varepsilon, t) \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \\ &\quad + \begin{bmatrix} A_{13} \\ A_{23}(t) \end{bmatrix} h + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} r \end{aligned} \tag{5}$$

where $x = \begin{bmatrix} \zeta_1 & \cdots & \zeta_{n-1} & x_n & x_{n+1} \end{bmatrix}^T$, $z = \zeta_n$ and:

$$A_{11} = \left[\begin{array}{c|cc} \mathrm{T}_u\{(\underline{\chi}_{(n-1)}^2)^T\} & \underline{0}_{(n-1)} & \underline{0}_{(n-1)} \\ \hline -(\bar{\underline{\mathfrak{a}}}_{n-1})^T & -(1+\ell) & \ell \\ (\underline{0}_{(n-1)})^T & (\beta - 1) & -\beta \end{array}\right]$$

$$A_{12} = \left[\begin{array}{c} \underline{\chi}_{(n-1)}^{(n-1)} \\ \hline -\bar{a}_n + (1+\ell) \\ -(\beta - 1) \end{array}\right], \quad B_1 = \left[\begin{array}{c} \underline{0}_{(n-1)} \\ \hline \underline{\chi}_2^1 \end{array}\right],$$

$$A_{21}(\varepsilon, t) = \left[\begin{array}{c|cc} -\varepsilon(\underline{\mathfrak{a}}_{(n-1)}(t))^T & b(t) & 0 \end{array}\right]$$

$$A_{22}(\varepsilon, t) = [-\varepsilon a_n(t) - b(t)],$$

$$A_{13} = \left[\begin{array}{c|c} 0_{(n-1)} & \underline{0}_{(n-1)} \\ \hline -(\bar{\underline{\mathfrak{a}}}_{(n-1)})^T & -\bar{a}_n + (1+\ell) \\ (\underline{0}_{(n-1)})^T & -(\beta - 1) \end{array}\right]$$

$$A_{23}(t) = \left[\begin{array}{c|c} (\underline{0}_{(n-1)})^T & b(t) \end{array}\right]$$

Note that the matrix $A_{22}(0, t)$ satisfies for all $t \in J$:

$$\begin{aligned} &\|A_{22}(0, t)\|_2 \leq b_2 \\ &\|\tfrac{\mathrm{d}}{\mathrm{d}t} A_{22}(0, t)\|_2 \leq c \ \text{ and} \\ &-b_2 \leq \Re e\,\lambda(A_{22}(0, t)) \leq -b_1 \end{aligned} \tag{6}$$

In [14], we have obtained the following particularization of Theorem 4.1, Lemma 4.1 and Theorem 6.1 in [8] for our case study:

Theorem 1: Given the matrix $A_{22}(\varepsilon, t)$, the properties (6) and $\|h\| = \mathcal{O}(\varepsilon)$. If:

$$\begin{aligned} &\bar{a}_n + \beta > 1 \\ &\tau < 1/(\bar{a}_n + \beta - 1) \text{ and} \\ &\varepsilon_1^* = \tfrac{\beta b_1}{\beta \bar{M}_2 + K\bar{M}_1\bar{M}_3}, \end{aligned} \tag{7}$$

where:

$$\begin{aligned} \bar{M}_1 &= \sqrt{1 + (\bar{a}_1 - \tfrac{1}{\tau})^2 + 2(1-\beta)^2}, \\ \bar{M}_2 &= \left|\bar{a}_1 + \beta - \tfrac{1}{\tau} - 1\right| \\ \bar{M}_3 &= \sqrt{\textstyle\sum_{i=1}^n \bar{a}_i^2 + (\beta - \tfrac{1}{\tau})^2} \end{aligned}$$

are satisfied, then the singularly perturbed description (5) is uniformly asymptotically stable for $\varepsilon \in (0, \varepsilon_1^*)$. Moreover, for all $\varepsilon \in (0, \varepsilon_1^*)$ the following expressions hold uniformly on $t \in [t_0, t_f]$:

$$\begin{aligned} &x(t) = x_s(t) + \mathcal{O}(\varepsilon) \\ &\text{and} \\ &z(t) = -A_{22}^{-1}(t)A_{21}(t)x_s(t) + z_f\left((t - t_0)/\varepsilon\right) + \mathcal{O}(\varepsilon) \end{aligned} \tag{8}$$

where x_s is solution of the slow system, $\mathrm{d}x(t)/\mathrm{d}t = A_{11}x + A_{12}z$ and $\varepsilon\mathrm{d}z/\mathrm{d}t = A_{21}(\varepsilon, t)x + A_{22}(\varepsilon, t)z$,

[1] For simplicity, in this paper we only consider functions of class $\mathcal{C}^\infty(J, \mathbb{R})$. But it could be considered functions of class $\mathcal{C}^k$, where k is a sufficiently positive large integer such that the derivability conditions were fulfilled. See also Corollary 2.4.12 of [12].

and z_f is solution of the fast system, $\mathrm{d}z_f(\tau)/\mathrm{d}\tau = A_{22}(0,t_0)z_f(\tau) + A_{23}(t_0+\varepsilon\tau)h(t_0+\varepsilon\tau)$, with the initial conditions: $x_s(t_0) = x_0$ and $z_f(0) = A_{22}^{-1}(0,t_0)A_{21}(t_0)x_0 + z_0$.

IV. Singularly Perturbed Observer

In this section we are going to assume that the state is no longer available, so we have to observe it. Since the signals, $a_i(t)$, $i \in \{1,\dots,n\}$, are unknown (see Hypothesis H1), we have to use a high gain observer. We use in fact, the proper approximation of a non proper system proposed by [9]. Indeed, let us consider the ideal observer (*c.f.* (2)):

$$N\mathrm{d}w/\mathrm{d}t = w(t) - \Gamma y \;\; ; \;\; \bar{y} = (\underline{\chi}^{(\bar{n}+1)}_{(\bar{n}+1)})^T w \qquad (9)$$

where for our case: $N = \mathrm{T}_\ell\{\underline{\chi}^2_{(\bar{n}+1)}\}$ and $\Gamma = \underline{\chi}^1_{(\bar{n}+1)}$. y, $\bar{y}$ and w are the input, the output and the descriptor variables, respectively, and $\bar{n} = n-1$.

In [9], the authors proposed the following singularly perturbed proper approximation:

$$\begin{bmatrix} \frac{\mathrm{d}x_f}{\mathrm{d}t} \\ \varepsilon\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\beta_f & -\varepsilon^{\bar{n}+1}(\underline{\chi}^1_{\bar{n}})^T \\ \underline{\chi}^{\bar{n}}_{\bar{n}} & -(M_{\bar{n}} - U_{\bar{n}}) \end{bmatrix} \begin{bmatrix} x_f \\ \bar{z} \end{bmatrix} + \begin{bmatrix} \varepsilon^{\bar{n}} q_{(1,2)} \\ -Q_0\underline{\chi}^1_{(\bar{n}+1)} \end{bmatrix} y$$
$$y_f = (\underline{\chi}^1_{\bar{n}})^T\bar{z} - \frac{1}{\varepsilon}q_{(1,2)}y \qquad (10)$$

where $x_f \in \mathbb{R}^1$, $\bar{z} \in \mathbb{R}^{\bar{n}}$, and $y_f \in \mathbb{R}^1$. β_f and ε are two positive real numbers. $U_{\bar{n}}$ and $M_{\bar{n}}$ are the Butterworth filter's matrices [1], namely:

$$U_{\bar{n}} = \mathrm{T}_u\{(\underline{\chi}^2_{\bar{n}})^T\},$$
$$M_{\bar{n}} = \begin{cases} BDM\{M_1,\dots,M_{\bar{n}/2}\} & \text{, for } \bar{n} \text{ even} \\ BDM\{M_1,\dots,M_{(\bar{n}-1)/2},1\} & \text{, for } \bar{n} \text{ odd} \end{cases}, \qquad (11)$$

$$M_j = (\sin\theta_j)\mathrm{I}_2 + \mathrm{T}_\ell\{(\cos^2\theta_j)\underline{\chi}^2_2\}\;,\quad \theta_1 = \pi/(2\bar{n}),$$
$$\theta_{j+1} = \theta_j + \Delta\theta\;,\quad \Delta\theta = \pi/\bar{n}\;,\quad j \in \{1,\dots,\bar{n}-1\}. \qquad (12)$$

$$\det(\lambda\mathrm{I}_{\bar{n}} + (M_{\bar{n}} - U_{\bar{n}})) =$$
$$\begin{cases} \prod_{i=1}^{\bar{n}/2}\left((\lambda+\sin\theta_i)^2 + \cos^2\theta_i\right), \text{ for } \bar{n} \text{ even} \\ (\lambda+1)\prod_{i=1}^{(\bar{n}-1)/2}\left((\lambda+\sin\theta_i)^2 + \cos^2\theta_i\right), \text{ for } \bar{n} \text{ odd} \end{cases} \qquad (13)$$

The matrix $Q_0 \in \mathbb{R}^{\bar{n}\times(\bar{n}+1)}$ is obtained by solving the following algebraic system equations:[2]

$$Q_0\left(\frac{1}{\varepsilon}N\right) + (M_{\bar{n}} - U_{\bar{n}})^{-1}Q_0 = -(M_{\bar{n}} - U_{\bar{n}})^{-1}\underline{\chi}^{\bar{n}}_{\bar{n}}(\underline{\chi}^{\bar{n}+1}_{\bar{n}+1})^T \qquad (14)$$

$$R_0 = -\frac{1}{\varepsilon}Q_0N \qquad (15)$$

[2]Let us recall that the eigenvalues of the Butterworth filter are all different, and placed over the semi-circle of radius $1/\varepsilon$, on the left-half complex plane; thus Spectrum(N) $\cap$ Spectrum$(M_{\bar{n}} - U_{\bar{n}})^{-1}$ $= \emptyset$, hence there exists a unique solution for these equations [3].

And the number $q_{(1,2)}$ corresponds to entry $(1,2)$ of matrix Q_0.

In [9] is proved the following Theorem (*c.f.* Theorem 5.1 in [8]):

Theorem 2 ([9]): Let us consider the following Butterworth Filter:

$$\begin{bmatrix} \frac{\mathrm{d}x_f}{\mathrm{d}t} \\ \varepsilon\frac{\mathrm{d}z_f}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\beta_f & -\varepsilon^{\bar{n}+1}(\underline{\chi}^1_{\bar{n}})^T \\ \underline{\chi}^{\bar{n}}_{\bar{n}} & -(M_{\bar{n}} - U_{\bar{n}}) \end{bmatrix} \begin{bmatrix} x_f \\ z_f \end{bmatrix} + \begin{bmatrix} 0 \\ \underline{\chi}^{\bar{n}}_{\bar{n}} \end{bmatrix}\bar{y}$$
$$y_f = (\underline{\chi}^1_{\bar{n}})^T z_f \qquad (16)$$

with the initial conditions: $x_f(0) \in \mathbb{R}^1$ and $z_f(0) \in \mathbb{R}^{\bar{n}}$. Then there exists $\varepsilon^* \in (0,1)$, such that for any $\varepsilon \in (0,\varepsilon^*)$:

1) The cascade formed by (9) and (16) is externally equivalent to (10) [3].
2) The output, y_f, of system (10), satisfies:

$$\begin{aligned} y_f(t) &= \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}y(t) + \mathrm{e}^{-(\beta+\varepsilon^n)t}x_f(0) + \mathcal{O}(\sqrt{\varepsilon}) \\ &= \zeta_n(t) + \mathrm{e}^{-(\beta+\varepsilon^n)t}x_f(0) + \mathcal{O}(\sqrt{\varepsilon}) \quad \forall\; t \geq t^* \end{aligned} \qquad (17)$$

where $t^* = \mathcal{O}(\frac{\varepsilon}{\sin\theta_1 - \sqrt{2}\varepsilon^n}\ln(1/\sqrt{\varepsilon}))$.

In the Appendix -A we show the key points of the proof of this Theorem, which enable us to built the matrices Q_0 and R_0.

Corollary 1: There exist matrices, D_1 and D_2,

$$D_1 = -\widehat{Q}_0^{-1}(M_{\bar{n}} - U_{\bar{n}}) \quad \text{and} \quad D_2 = -\widehat{Q}_0^{-1}\underline{q}_1, \qquad (18)$$

where: $\widehat{Q}_0 = [\;\underline{q}_2 \;\; \underline{q}_3 \;\; \cdots \;\; \underline{q}_{\bar{n}+1}\;]$, and the $\underline{q}_i$ are the column vectors of matrix Q_0 for $i \in \{1,\dots,\bar{n}+1\}$, such that:

$$\begin{aligned} \zeta_f(t) &= \zeta(t) + h(t) = D_1\bar{z}(t) + D_2y(t) \\ h(t) &= \mathcal{O}(\varepsilon), \quad \forall\; t \geq t". \end{aligned} \qquad (19)$$

Furthermore: $\lim_{\varepsilon\to 0}\zeta_f(t) = \zeta(t)$.

V. Closed Loop System

Applying the control scheme composed by: the singularly perturbed control law (3), the equalizer filter (4), and the singularly perturbed observer, (10) and (19), we get the closed loop system described by (5).

Now, in view of Corollary 1, Theorems 1 and 2 we conclude the stability of the closed loop system (5).

In order to show how to synthesize the results of this paper, we consider the following illustrative example.

VI. Illustrative Example

Let us consider a LTVS represented by (1) with $n = 3$ and with parameters:

$$\begin{cases} a_1(t) = \sum_{j=1}^{5}\frac{1}{2j-1}\sin((2j-1)t) \\ a_2(t) = \sum_{j=1}^{5}\frac{(-1)^{j-1}}{j}\sin\left(\frac{jt}{2}\right) \\ a_3(t) = \sum_{j=1}^{5}\frac{j}{(2j-1)(2j+1)}\sin(4jt) \\ b(t) = 1 + 0.568\,(\sin(t) + \sin(2t)) \end{cases} \qquad (20)$$

[3]That is to say, the representation, (9), (11), (12) and (16), and the representation, (10), (11), (12), (14) and (15), have the same input-output trajectories (see for example [12]).

Note that $b_1 = 0.5$. The state representation is given by (2) with:

$$\underline{\mathfrak{a}}_3(t) = \begin{bmatrix} a_1(t) & a_2(t) & a_3(t) \end{bmatrix}^T \tag{21}$$

The parameters $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}_3$, of the equalizer filter (4), are the coefficients of the following Hurwitz polynomial $p(\lambda) = \lambda^3 + 0.92\lambda^2 + 0.25\lambda + 0.02$, namely:

$$(\bar{a}_1,\, \bar{a}_2,\, \bar{a}_3) = (0.02,\, 0.25,\, 0.92) \tag{22}$$

Since $\bar{a}_3 = 0.92$, then the selection, $\beta = 10$ and $\tau = 0.1$, satisfies inequalities (7). From (7), we get: $\varepsilon_1^* = 0.18$, then $\varepsilon \in (0, 0.18)$; let us take $\varepsilon = 0.09$, namely:

$$(\varepsilon,\, \beta,\, \tau) = (0.09,\, 10,\, 0.1) \tag{23}$$

Since $\bar{n} = n - 1 = 2$, then $\theta_1 = \pi/4$ (see (12)). And the other matrix Q_0 of the observer (10) is (see (29)): $Q_0 = \begin{bmatrix} \sqrt{2}/\varepsilon^2 & -1/\varepsilon & 0 \\ 0 & 1/(\sqrt{2}\varepsilon) & -1 \end{bmatrix}$. Then: $\underline{q}_1 = \begin{bmatrix} \sqrt{2}/\varepsilon^2 \\ 0 \end{bmatrix}$, $q_{(1,2)} = -1/\varepsilon$, $\widehat{Q}_0 = \begin{bmatrix} -1/\varepsilon & 0 \\ 1/(\sqrt{2}\varepsilon) & -1 \end{bmatrix}$ and $\widehat{Q}_0^{-1} = \begin{bmatrix} -\varepsilon & 0 \\ -1/\sqrt{2} & -1 \end{bmatrix}$. Based on the proof of Corollary 1, we get:[4] $\beta_f = 10$. Thus the singularly perturbed observer (10) takes the following form:

$$\begin{aligned} \mathrm{d}x_f/\mathrm{d}t &= -10x_f - (0.09)^3\begin{bmatrix} 1 & 0 \end{bmatrix}\bar{z} - (0.09)y \\ \mathrm{d}\bar{z}/\mathrm{d}t &= \tfrac{1}{0.09}\begin{bmatrix} 0 \\ 1 \end{bmatrix} x_f - \tfrac{1}{0.09}\begin{bmatrix} 1/\sqrt{2} & -1 \\ 1/2 & 1/\sqrt{2} \end{bmatrix}\bar{z} \\ &\quad - \tfrac{1}{0.09}\begin{bmatrix} \sqrt{2}/(0.09)^2 \\ 0 \end{bmatrix} y \\ y_f &= \begin{bmatrix} 1 & 0 \end{bmatrix}\bar{z} + (1/(0.09)^2)y \end{aligned} \tag{24}$$

From (18) and (19) in Corollary 1, we can compute ζ_f,

$$\zeta_f = \begin{bmatrix} \varepsilon/\sqrt{2} & -\varepsilon \\ 1 & 0 \end{bmatrix}\bar{z} + \begin{bmatrix} \sqrt{2}/\varepsilon \\ 1/\varepsilon^2 \end{bmatrix} y \tag{25}$$

The ideal model to match is:

$$\mathrm{d}^3y^*/\mathrm{d}t^3 + 0.92\mathrm{d}^2y^*/\mathrm{d}t^2 + 0.25\mathrm{d}y^*/\mathrm{d}t + 0.02y^* = r.$$

In order to satisfy H3, $r \in \mathcal{C}^\infty(J, \mathbb{R})$, the reference r has been chosen as follows:

$$r(t) = \frac{10}{2.75}\int_0^t \varphi(\sigma)\mathrm{d}\sigma, \quad t \in [0,\, 100]$$

where:[5] $\varphi(t) = \mathrm{e}^{-\frac{1}{1-(t')^2}}$, with $t' = (12/75)t - 1$.

A MATLAB® numerical simulation was performed with the solver settings: *"Start time"* = 0.0, *"Stop time"* = 100, *"Type"* = *"Fixed–Step"*, *"Solver"* = *"ode45 Runge–Kutta"*, *"Fixed–step size"* = 0.04, *"Periodic sample time constraint"* = *"Unconstrained"*, *"Tasking mode for periodic sample times"* = *"auto"*. In Fig. 1, we show the behavior of system, (2), (20) and (21), controlled by (3), (4), (22) and (23), with the state observer (24) and (25). Comparison can be done with the simulations given in [14].

[4] From (40), we have: $A_0 = -\beta - \varepsilon^3$, then from (41) we get: $\beta_f = \beta$.

[5] The function φ is taken from Definition 2.4.5 in [12].

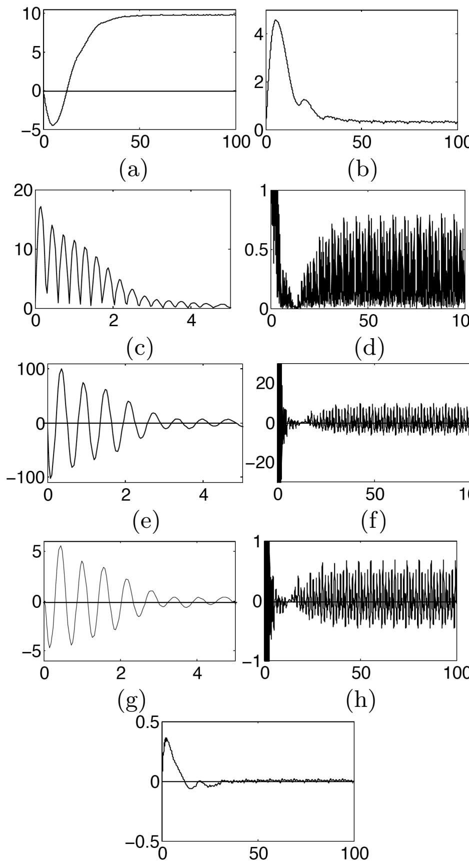

Fig. 1. Control variables: (a) output y, (b) matching model error $|y - y^*|$, (c)–(d) observation error $\|\zeta - \zeta_f\|$, (e)–(f) control law u, (g)–(h) equalizer filter signal x_4, and (i) equalizer filter signal x_3.

VII. Conclusion

In this paper, we have proposed a singular implicit control scheme for LTVS SISO systems. The control scheme is composed by the *singularly perturbed control law* (3) and the *equalizer filter* (4).

The aim of the *equalizer filter* is to assign the closed loop dynamics, and to assign a rate of exponential convergence. The aim of the *singularly perturbed control law* is to bring the system into a *singularly perturbed model*, and to get a desired dynamics by an ε order.

The parameters, β and τ, enable us to compute a sufficiently small ε such that the uniform asymptotic stability of the *singularly perturbed model* is guaranteed.

We have considered the perturbation signal h in order to take into account the effects of high gain observer. We have proposed a singularly perturbed observer which aim is to obtain the observed state $\zeta_f = \zeta + h$, where the vector function h is $\mathcal{O}(\varepsilon)$.

Appendix

A. *Key points of the proof of Theorem 2*

Putting the ideal non proper filter, (9), together with the Butterworth filter, (16), we get the global singularly perturbed system:

$$\underbrace{\begin{bmatrix} N & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \mathrm{I}_{\bar{n}} \end{bmatrix}}_{\mathbb{E}} \begin{bmatrix} \frac{dw}{dt} \\ \frac{dx_f}{dt} \\ \varepsilon \frac{dz_f}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathrm{I}_{\bar{n}+1} & 0 & 0 \\ 0 & -\beta_f & -\varepsilon^{\bar{n}+1}(\underline{\chi}^1_{\bar{n}})^T \\ \underline{\chi}^{\bar{n}}_{\bar{n}}(\underline{\chi}^{\bar{n}+1}_{\bar{n}+1})^T & \underline{\chi}^{\bar{n}}_{\bar{n}} & -(M_{\bar{n}} - U_{\bar{n}}) \end{bmatrix}}_{\mathbb{A}} \begin{bmatrix} w \\ x_f \\ z_f \end{bmatrix} + \underbrace{\begin{bmatrix} -\underline{\chi}^1_{(\bar{n}+1)} \\ 0 \\ 0 \end{bmatrix}}_{\mathbb{B}} y \tag{26}$$

$$y_f(t) = \underbrace{\begin{bmatrix} 0 & 0 & (\underline{\chi}^1_{\bar{n}})^T \end{bmatrix}}_{\mathbb{C}} \begin{bmatrix} w^T & x_f & z_f^T \end{bmatrix}^T$$

In order to prove the external equivalence between (26) and (10), in [9], the authors have followed the procedure shown hereafter:

1) Let us first define two invertible matrices: $Q = \begin{bmatrix} \mathrm{I}_{\bar{n}+1} & 0 & 0 \\ 0 & 1 & 0 \\ Q_0 & 0 & \mathrm{I}_{\bar{n}} \end{bmatrix}$ and $R = \begin{bmatrix} \mathrm{I}_{\bar{n}+1} & 0 & 0 \\ 0 & 1 & 0 \\ R_0 & 0 & \mathrm{I}_{\bar{n}} \end{bmatrix}$, where Q_0 satisfy (14) and (15).
2) Let us next note that: Spectrum$(N) \cap$ Spectrum$(M_{\bar{n}} - U_{\bar{n}})^{-1} = \emptyset$, then (14) has a unique solution (see for example Chapter 8 in [3]).
3) Let us now apply matrices Q and R to the matrices of (26): $Q\mathbb{E}R = \begin{bmatrix} N & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \mathrm{I}_{\bar{n}} \end{bmatrix}$,
$Q\mathbb{A}R = \begin{bmatrix} \mathrm{I}_{\bar{n}+1} & 0 & 0 \\ -\varepsilon^{\bar{n}+1}(\underline{\chi}^1_{\bar{n}})^T R_0 & -\beta_f & -\varepsilon^{\bar{n}+1}(\underline{\chi}^1_{\bar{n}})^T \\ 0 & \underline{\chi}^{\bar{n}}_{\bar{n}} & -(M_{\bar{n}} - U_{\bar{n}}) \end{bmatrix}$,
$Q\mathbb{B} = -\begin{bmatrix} \underline{\chi}^1_{(\bar{n}+1)} \\ 0 \\ Q_0 \underline{\chi}^1_{(\bar{n}+1)} \end{bmatrix}$, $\mathbb{C}R = \begin{bmatrix} R_0^T \underline{\chi}^1_{\bar{n}} \\ 0 \\ \underline{\chi}^1_{\bar{n}} \end{bmatrix}^T$.
4) Let us finally do the change of variable:

$$\bar{z}(t) = z_f(t) - R_0 w(t) \tag{27}$$

Adding its third row with the pre-multiplication of its first row by Q_0, we get the *externally equivalent* proper system (10).

One way for obtaining the matrices Q_0 and R_0 is the following (see also [10]):

1) Let us denote by $\underline{r}_i$ and $\underline{q}_i$ the column vectors of matrices R_0 and Q_0, respectively. Thus from (15), we get:

$$\begin{bmatrix} \underline{r}_1 & \cdots & \underline{r}_{\bar{n}+1} \end{bmatrix} = -\tfrac{1}{\varepsilon} \begin{bmatrix} \underline{q}_1 & \cdots & \underline{q}_{\bar{n}+1} \end{bmatrix} \begin{bmatrix} \underline{\chi}^2_{\bar{n}+1} & \cdots & \underline{\chi}^{\bar{n}+1}_{\bar{n}+1} & \underline{0}_{\bar{n}+1} \end{bmatrix} = -\tfrac{1}{\varepsilon} \begin{bmatrix} \underline{q}_2 & \cdots & \underline{q}_{\bar{n}+1} & \underline{0}_{\bar{n}} \end{bmatrix} \tag{28}$$

2) Then, from (14), (15) and (28), we get:

$$\begin{bmatrix} \underline{q}_1 & \cdots & \underline{q}_{\bar{n}} & \underline{q}_{\bar{n}+1} \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & \underline{\chi}^{\bar{n}}_{\bar{n}} \end{bmatrix} = -\tfrac{1}{\varepsilon}(M_{\bar{n}} - U_{\bar{n}}) \begin{bmatrix} \underline{q}_2 & \cdots & \underline{q}_{\bar{n}} & \underline{q}_{\bar{n}+1} & \underline{0}_{\bar{n}} \end{bmatrix} \tag{29}$$

3) For solving (29), the columns have to be equated from the last to the first.
4) Observe that: $(\underline{\chi}^1_{\bar{n}})^T R_0 = -\frac{1}{\varepsilon}\underline{q}_{(1,2)}(\underline{\chi}^1_{\bar{n}})^T$, where $q_{(i,j)}$ is the entry (i,j) of matrix Q_0.

B. Proof of Corollary 1

We first consider the case $\varepsilon = 0$, and then we analyze the case $\varepsilon > 0$.

1) Let us consider for a while that $\varepsilon = 0$, then from (16) and (9), we get:[6]

$$(M_{\bar{n}} - U_{\bar{n}})\, z_{f_0}(t) = \left(\underline{\chi}^{\bar{n}}_{\bar{n}}\right) \bar{y}(t) = \left(\underline{\chi}^{\bar{n}}_{\bar{n}}\right) (\underline{\chi}^{\bar{n}+1}_{\bar{n}+1})^T w(t) \tag{30}$$

Substituting (14) and (15) into (30), we obtain the equation:

$$(M_{\bar{n}} - U_{\bar{n}})\, z_{f_0}(t) = [(M_{\bar{n}} - U_{\bar{n}})\, R_0 - Q_0]\, w(t) \tag{31}$$

Using (27) and (31), the last equation can be written as:

$$(M_{\bar{n}} - U_{\bar{n}})\, \bar{z}_0(t) = -Q_0 w(t) \tag{32}$$

Because (9) is an ideal observed, the following identity holds:

$$\begin{aligned} -\widehat{Q}_0^{-1} (M_{\bar{n}} - U_{\bar{n}})\, \bar{z}_0(t) &= \left[\begin{array}{c|c} \widehat{Q}_0^{-1}\underline{q}_1 & \mathrm{I}_{\bar{n}} \end{array}\right] \zeta(t) \\ &= \left[\begin{array}{c|c} \widehat{Q}_0^{-1}\underline{q}_1 & 0_{\bar{n}} \end{array}\right] \zeta(t) + \left[\begin{array}{c|c} \underline{0}_{\bar{n}} & \mathrm{I}_{\bar{n}} \end{array}\right] \zeta(t) \end{aligned} \tag{33}$$

Finally we get:

$$-\widehat{Q}_0^{-1} (M_{\bar{n}} - U_{\bar{n}})\, \bar{z}_0(t) - \widehat{Q}_0^{-1}\underline{q}_1 y(t) = \begin{bmatrix} \mathrm{d}y/\mathrm{d}t & \cdots & \mathrm{d}^{n-1}y/\mathrm{d}t^{n-1} \end{bmatrix}^T \tag{34}$$

2) Let us now consider the case, $\varepsilon > 0$, which is based on the particular case $\varepsilon = 0$.
From (16) and (9) we get:

$$\begin{aligned} (\mathrm{d}/\mathrm{d}t + \beta_f)\, x_f &= -\varepsilon^{\bar{n}+1} \left(\underline{\chi}^1_{\bar{n}}\right)^T z_f \\ (\varepsilon \mathrm{I}_{\bar{n}} \mathrm{d}/\mathrm{d}t + (M_{\bar{n}} - U_{\bar{n}}))\, z_f &= \underline{\chi}^{\bar{n}}_{\bar{n}} x_f + \underline{\chi}^{\bar{n}}_{\bar{n}} \left(\underline{\chi}^{\bar{n}+1}_{\bar{n}+1}\right)^T w \end{aligned} \tag{35}$$

If we apply the operator $(\mathrm{d}/\mathrm{d}t + \beta_f)$ to the second row of the last equation, we then have (recall that (9) is an ideal observer):

$$\begin{aligned} &(M_{\bar{n}} - U_{\bar{n}}) \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + \beta_f\right) z_f = \underline{\chi}^{\bar{n}}_{\bar{n}} \left(\underline{\chi}^{\bar{n}+1}_{\bar{n}+1}\right)^T \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + \beta_f\right) \zeta \\ &-\varepsilon \left(\mathrm{I}_{\bar{n}} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + \beta_f\right) \tfrac{\mathrm{d}}{\mathrm{d}t} + \varepsilon^{\bar{n}} \underline{\chi}^{\bar{n}}_{\bar{n}} \left(\underline{\chi}^1_{\bar{n}}\right)^T\right) z_f \end{aligned} \tag{36}$$

taking into account (14), (15) and (27), we have:

$$\begin{aligned} &\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + \beta_f\right) ((M_{\bar{n}} - U_{\bar{n}})\, \bar{z} + Q_0 \zeta) = \\ &-\varepsilon \left(\mathrm{I}_{\bar{n}} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + \beta_f\right) \tfrac{\mathrm{d}}{\mathrm{d}t} + \varepsilon^{\bar{n}} \underline{\chi}^{\bar{n}}_{\bar{n}} \left(\underline{\chi}^1_{\bar{n}}\right)^T\right) z_f \end{aligned} \tag{37}$$

[6] We write, z_{f_0} and $\bar{z}_0$, instead of, z_f and $\bar{z}$, for emphasizing that we are considering the case $\varepsilon = 0$.

This last equation is equivalent to:

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+\beta_f\right)\widehat{Q}_0\left[\widehat{Q}_0^{-1}\left(M_{\bar{n}}-U_{\bar{n}}\right)\bar{z}+\widehat{Q}_0^{-1}Q_0\zeta\right] = -\varepsilon\left(\mathrm{I}_{\bar{n}}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+\beta_f\right)\tfrac{\mathrm{d}}{\mathrm{d}t}+\varepsilon^{\bar{n}}\underline{\chi}_{\bar{n}}^{\bar{n}}\left(\underline{\chi}_{\bar{n}}^{1}\right)^T\right)z_f \quad (38)$$

Let us rewrite this last equation in the same form as (33) and (34):

$$\begin{array}{l}\widehat{Q}_0\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+\beta_f\right)h=\varepsilon\bar{h}\\ \widehat{Q}_0^{-1}\left(M_{\bar{n}}-U_{\bar{n}}\right)\bar{z}+\widehat{Q}_0^{-1}\underline{q}_1 y\\ +\left[\begin{array}{ccc}\mathrm{d}y/\mathrm{d}t & \cdots & \mathrm{d}^{n-1}y/\mathrm{d}t^{n-1}\end{array}\right]^T=-h\\ \bar{h}=\left(\mathrm{I}_{\bar{n}}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+\beta_f\right)\tfrac{\mathrm{d}}{\mathrm{d}t}+\varepsilon^{\bar{n}}\underline{\chi}_{\bar{n}}^{\bar{n}}\left(\underline{\chi}_{\bar{n}}^{1}\right)^T\right)z_f\end{array} \quad (39)$$

We can check that system (16) satisfies the invertibility condition of Theorem 3.1 in Chapter 2 in [8]. Indeed, let us first define the matrices:

$$\begin{array}{l}A_0=-\beta-\varepsilon^{\bar{n}+1}\left(\underline{\chi}_{\bar{n}}^{1}\right)^T\left(M_{\bar{n}}-U_{\bar{n}}\right)^{-1}\underline{\chi}_{\bar{n}}^{\bar{n}}\\ \text{and}\\ A_{22}=\left(M_{\bar{n}}-U_{\bar{n}}\right)\end{array} \quad (40)$$

From (13), we get: $\det\left(M_{\bar{n}}-U_{\bar{n}}\right)=1$; in [9], the matrix inverse of $\left(M_{\bar{n}}-U_{\bar{n}}\right)$ is computed. Furthermore, we can see from (13) that the $\bar{n}$ eigenvalues are distinct. Then Theorem 3.1 states that the eigenvalues of (16) are approximated as follows:

$$\begin{array}{l}\lambda_1=\lambda_1\left(A_0\right)+\mathcal{O}(\varepsilon)=-\beta_f+\mathcal{O}(\varepsilon)\\ \text{and}\\ \lambda_i=\frac{1}{\varepsilon}\left(\lambda_1\left(A_{22}\right)+\mathcal{O}(\varepsilon)\right),\ i\in\{1,\ldots,\bar{n}\}\end{array} \quad (41)$$

Since matrices A_0 and A_{22} are Hurwitz, then Corollary 3.1 in Chapter 2 in [8] implies that there exists an $\varepsilon^*>0$, such that (16) is asymptotically stable for all $\varepsilon\in(0,\varepsilon^*]$.

Now, since the input $\bar{y}$ of the filter (16) is obtained by means of the ideal observer (9), and since we have assumed conditions insuring differentiability and boundedness of the related signals (see assumptions H1–H4), it follows that $\bar{h}(t)$ is also a bounded vector function.

Finally, from equation (39) we have:

$$h(t)=\widehat{Q}_0^{-1}\mathrm{e}^{-\beta_f t}h(0)+\varepsilon\widehat{Q}_0^{-1}\int_0^t \mathrm{e}^{-\beta_f(t-\tau)}\bar{h}(\tau)\mathrm{d}\tau$$

then the vector function $h(t)$ tends exponentially to 0 when ε tends to 0. Therefore, from (39) we get:

$$\begin{array}{l}-\widehat{Q}_0^{-1}\left(M_{\bar{n}}-U_{\bar{n}}\right)\bar{z}-\widehat{Q}_0^{-1}\underline{q}_1 y(t)=\\ \left[\begin{array}{ccc}\mathrm{d}y/\mathrm{d}t & \cdots & \mathrm{d}^{n-1}y/\mathrm{d}t^{n-1}\end{array}\right]^T.\end{array} \quad (42)$$

This concludes the proof. □

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