2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Control of two-steering-wheels vehicles with the Transverse Function approach

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***Abstract*— The control of a wheeled vehicle with front and rear steering wheels is addressed. With respect to more classical car-like vehicles, an advantage of this type of mechanism is its enhanced maneuverability. The Transverse Function approach is used to derive feedback laws which ensure *practical* stabilization of arbitrary reference trajectories in the cartesian space, and *asymptotic* stabilization when the trajectory is feasible by the nonholonomic vehicle. Concerning this latter issue, previous results are extended to the case of transverse functions defined on the Special Orthogonal Group $\mathbb{SO}(3)$.**

I. Introduction

This study is about the control of ground vehicles with front and rear independent steering wheels. At the kinematic level, this system has three independent control inputs, namely the translational velocity along the direction joining the steering wheels' axles, and the steering-wheels' angular velocities. With respect to classical car-like vehicles with a single steering train, this type of vehicle provides superior maneuvering capabilities and the possibility of orienting the main vehicle's body independently of its translational motion (see, e.g., [1] and references therein). This can be used, for instance, to transport large payloads without changing the payload's orientation, thus minimizing energy consumption. From the control viewpoint, assuming that classical *rolling-without-slipping* nonholonomic constraints are satisfied at the wheel/ground contact level, the kinematic equations of this type of vehicle yield a locally controllable five-dimensional nonholonomic driftless system with $\mathbb{SE}(2) \times \mathbb{S}^1 \times \mathbb{S}^1$ as its configuration space. A complementary constraint is that singular kinematic configurations, when either the front steering wheel angle or the rear steering wheel angle is equal to $\pm\pi/2$, must be avoided whatever the desired gross displacement of the vehicle in the plane. This implies that some reference trajectories in $\mathbb{SE}(2)$, corresponding to the motion of a reference frame in the plane, can only be stabilized "practically" via maneuvers, alike the case of a car accomplishing sideways displacements. The Transverse Function approach [2] applies to this nonholonomic system the structure of which (unsurprisingly) presents similarities with the one of a car with two control inputs. In particular, it is also locally equivalent to a homogeneous (nilpotent) system which is invariant on a Lie group [3] [4]. However, its Lie Algebra is generated differently due to the third control input. In particular, only *first-order* Lie brackets of the control vector fields are needed to satisfy the Lie Algebra Rank Condition (LARC) –the local controllability condition– at any point, whereas a second-order Lie bracket is needed in the car case. This property, which reflects the symmetric steering action of the front and rear wheels, is of practical importance. In order to respect this symmetry at the control design level, one is led to consider transverse functions defined on the three-dimensional special orthogonal group $\mathbb{SO}(3)$, rather than on the two-dimensional torus –a solution used in the car case, for instance. Therefore, after the trident snake studied in [5], and the serial snake studied in [6], this is another example of a mechanical system for which the use of transverse functions defined on $\mathbb{SO}(3)$ is natural. Moreover, this example presents the complementary interest, and complication, of involving transverse functions defined on a manifold whose dimension (equal to three) is not minimal. The corresponding extra degree of freedom thus has to be taken into account at the control design level and, if possible, used effectively. For instance, a desirable feature is to ensure the asymptotic stabilization of *admissible* (or *feasible*) trajectories for which more classical control solutions, such as the Lyapunov-based nonlinear feedbacks proposed in [7], or linear feedbacks derived from linearized tracking error equations, apply. In the end one obtains a unique feedback control law which ensures the avoidance of kinematic singularities, the *practical* global stabilization of *any* (i.e. feasible or non-feasible) reference trajectory in $\mathbb{SE}(2)$, including fixed points, and the *asymptotic* stabilization of feasible reference trajectories for which this objective is achievable by using classical feedback control techniques –typically when adequate conditions of *persistent excitation* upon the reference translational velocity are satisfied.

The paper is organized as follows. The robot's kinematic model and associated controllability properties are presented in Section II. Transverse functions defined on $\mathbb{SO}(3)$ are derived in Section III. The control design is carried out in Section IV, and simulation results are given in Section V. Finally, the concluding Section VI points out a few research directions which could prolong the present study.

II. MODELING AND CONTROL PROBLEM STATEMENT

Figure 1 shows a schematized view from above of the system under consideration.

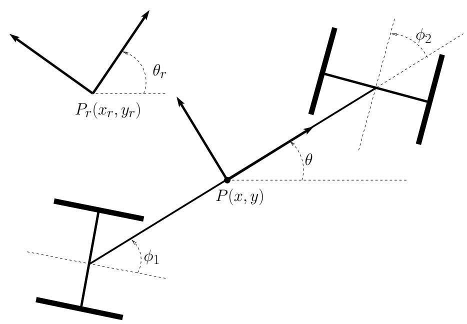

Fig. 1. Two-steering-wheels vehicle. View from above

The point P on the vehicle is located at mid-distance of the steering wheels' axles, and at the distance l of each axle. Consider an arbitrary fixed frame in the plane on which the vehicle moves. A mobile frame with origin P and orientation θ with respect to (w.r.t.) this fixed frame is attached to the vehicle's main body. By denoting the coordinates of P in the fixed frame as x and y, the vector $g := (x, y, \theta)'$, with the prime sign used for transpose, can be seen as an element of $\mathbb{SE}(2)$. Therefore any motion of this vehicle can be associated with a trajectory in $\mathbb{SE}(2)$. The desired motion of this mobile frame is specified by the motion of the reference frame with origin P_r whose position and orientation is given by $g_r := (x_r, y_r, \theta_r)'$. The control objective is to stabilize any trajectory of the reference frame, either practically (i.e. by ensuring small ultimate tracking errors) or asymptotically when this is possible, while avoiding the singular values $\pm\pi/2$ for the steering wheel angles $\phi_{1,2}$.

Denote the velocity components of the point P, expressed in the mobile frame, as u_x and u_y, i.e. such that:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = R(\theta) \begin{pmatrix} u_x \\ u_y \end{pmatrix} \tag{1}$$

with $R(\theta)$ denoting the rotation matrix in the plane of angle θ. Define now

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} := \begin{pmatrix} \frac{\tan(\phi_1)-\tan(\phi_2)}{2} \\ \frac{\tan(\phi_1)+\tan(\phi_2)}{2l} \end{pmatrix}$$

One easily verifies that:

$$\begin{cases} u_y &= \eta_1 u_x \\ \dot{\theta} &= \eta_2 u_x \end{cases} \tag{2}$$

With u_x, the angular velocities $\dot{\phi}_1$ and $\dot{\phi}_2$ are the other two kinematic control inputs. Away from the steering wheels singular values $\pm\pi/2$, one can define the following change of control inputs:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} \frac{1}{\cos^2(\phi_1)} & -\frac{1}{\cos^2(\phi_2)} \\ \frac{1}{l\cos^2(\phi_1)} & \frac{1}{l\cos^2(\phi_2)} \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}$$

so that

$$\dot{\eta} = v \tag{3}$$

Define

$$\bar{R}(\theta) := \begin{pmatrix} R(\theta) & 0_{2\times 1} \\ 0_{1\times 2} & 1 \end{pmatrix}$$

with $0_{m\times n}$ denoting the $m \times n$ zero matrix, and note that the column vectors of $\bar{R}(.)$ form a basis of the Lie algebra of $\mathbb{SE}(2)$. By regrouping the equations (1)-(3) one obtains the following five-dimensional control system with three inputs:

$$\begin{cases} \dot{g} = \bar{R}(\theta)C(\eta)u_x \\ \dot{\eta} = v \end{cases} \tag{4}$$

with

$$C(\eta) := \begin{pmatrix} 1 \\ \eta \end{pmatrix}$$

One can remark that this system may also be written as:

$$\begin{cases} \dot{g} = X(g)C_g(\eta)w \\ \dot{\eta} = C_\eta w \end{cases} \tag{5}$$

with $X(g) = \bar{R}(\theta)$ and

$$w := \begin{pmatrix} u_x \\ v \end{pmatrix}, \; C_g(\eta) := \begin{pmatrix} C(\eta) & 0_{3\times 2} \end{pmatrix}, \; C_\eta := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a particular case of the class of systems described by the relation (13) in [3]. The control design proposed in this paper (cf. Propositions 3 and 4) thus applies to the present system, once a suitable transverse function has been determined. Concerning this latter issue, [3] focuses on the case of a motorized vehicle with trailers, each trailer having its hitch-point located on the axle of the preceding vehicle, and the proposed transverse function is derived from the one calculated for a locally equivalent chained systems with two control inputs. The existence of a third control input modifies this situation, since the system can no longer be equivalent to a chained system. In fact, it would be possible (and simple) to recover the car case by just maintaining one of the steering angles equal to a constant value, zero for instance. However, in doing so one looses the specific interest of the double steering train, namely the possibility of controlling the vehicle's orientation independently of the vehicle's translational motion. Moreover, the third input allows for the satisfaction of the Lie Algebra Rank Condition (LARC) at every regular point –this implies that the system is locally controllable at these points– by calculating *first-order* Lie brackets of the system's vector fields (v.f.) only. Indeed, in view of (4) the system's control v.f. are:

$$X_1(g, \eta) = \begin{pmatrix} \cos(\theta) - \sin(\theta)\eta_1 \\ \sin(\theta) + \cos(\theta)\eta_1 \\ \eta_2 \\ 0 \\ 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; X_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and the first-order Lie brackets $X_4 := [X_1, X_2]$ and $X_5 := [X_1, X_3]$ are given by

$$X_4(g) = \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; X_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

One easily verifies that, for every point (g,η), the matrix $\mathcal{C} = (X_1|X_2|X_3|X_4|X_5)(g,\eta)$ is invertible. This property is to be compared with the car case for which one has to go to the order two to satisfy this controllability condition. It is also related to the physical intuition that an extra actuated degree of freedom should facilitate the control of the system, just as in the case of controllable linear systems. It turns out that this difference in the generation of the corresponding Lie algebras has also consequences at the transverse function design level. This issue is addressed in the next section.

III. Design of transverse functions

Let $f\ :\ (\alpha,t)\mapsto f(\alpha,t)$ denote a smooth function from $K\times\mathbb{R}$ to $\mathbb{SE}(2)\times\mathbb{S}^1\times\mathbb{S}^1$, with K a compact manifold of dimension m (≥ 2). Along any smooth curve $\alpha(.)$

$$\dot{f}(\alpha,t) = d_\alpha f(\alpha,t)\dot{\alpha} + \partial^t f(\alpha,t)$$

with d_α (resp. ∂^t) the operator of differentiation w.r.t. α (resp. t). The time-derivative $\dot{\alpha}$ can itself be decomposed as

$$\dot{\alpha} = \sum_{i=1}^{m} Y_i(\alpha)\omega_{\alpha,i}$$

with $\{Y_{i=1\ldots m}\}$ a set of v.f. spanning the tangent space of K at α and $\omega_{\alpha,i=1\ldots m}$ the coefficients associated with this decomposition. From now on, we will assume that the set of v.f. Y_i has been chosen once for all and we will use the notation $\partial^\alpha f(\alpha,t) := d_\alpha f(\alpha,t)Y(\alpha)$ to simplify the writing of the derivative of f which, with this notation, is given by

$$\dot{f}(\alpha,t) = \partial^\alpha f(\alpha,t)\omega_\alpha + \partial^t f(\alpha,t)$$

with ω_α the m-dimensional vector of components $\omega_{\alpha,i=1,\ldots,m}$.

We recall that the function f is said to be transverse to the control v.f. X_1, X_2, and X_3 of System 4 if the matrix

$$H(\alpha,t) := \big(X_1(f_g,f_\eta)|X_2|X_3| - \partial^\alpha f\big)(\alpha,t)$$

with f_g and f_η denoting the components of f in $\mathbb{SE}(2)$ and $\mathbb{S}^1\times\mathbb{S}^1$ respectively, has full rank (equal to five) $\forall(\alpha,t)\in K\times\mathbb{R}$. The local controllability of the system (4) ensures –and is in fact equivalent to– the existence of such a function [2]. In previous papers, the authors showed that there are multiple systematic ways of synthesizing transverse functions. The approach here retained for this task borrows the method from [3] which consists in working with a locally feedback-equivalent homogeneous system invariant on a Lie group for which the explicit calculation of transverse functions is simple.

A. *Locally feedback-equivalent homogeneous system*

Consider the control system

$$\begin{cases} \dot{\xi}_1 = u_1 \\ \dot{\xi}_2 = \xi_4 u_1 \\ \dot{\xi}_3 = \xi_5 u_1 \\ \dot{\xi}_4 = u_2 \\ \dot{\xi}_5 = u_3 \end{cases} \tag{6}$$

One verifies that, in the neighborhood of $(g,\eta) = (0,0)$, this system is feedback-equivalent to (4) via the changes of coordinates and inputs defined by

$$\Phi\ :\ (g,\eta)\mapsto \xi := \Phi(g,\eta) = \begin{pmatrix} g \\ \frac{\cos(\theta)\eta_1+\sin(\theta)}{d(\theta,\eta_1)} \\ \frac{\eta_2}{d(\theta,\eta_1)} \end{pmatrix} \tag{7}$$

and

$$\Psi\ :\ (g,\eta,u_x,v)\mapsto$$
$$u := \begin{pmatrix} d(\theta,\eta_1)u_x \\ \frac{v_1}{d(\theta,\eta_1)^2} + \frac{1+\eta_1^2}{d(\theta,\eta_1)^2}\eta_2 u_x \\ \frac{v_2}{d(\theta,\eta_1)} + \frac{\eta_2^2(\cos(\theta)\eta_1+\sin(\theta))u_x+\eta_2\sin(\theta)v_1}{d(\theta,\eta_1)^2} \end{pmatrix} \tag{8}$$

with $u = (u_1,u_2,u_3)'$ and $d(\theta,\eta_1) := \cos(\theta) - \sin(\theta)\eta_1$. In view of (6), the control v.f. of this system are

$$Z_1(\xi) = \begin{pmatrix} 1 \\ \xi_4 \\ \xi_5 \\ 0 \\ 0 \end{pmatrix},\ Z_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ Z_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and the only non-zero Lie brackets generated by these v.f. are $Z_4 := [Z_1,Z_2] = -e_2$ and $Z_5 := [Z_1,Z_3] = -e_3$, with e_i denoting the i-th canonical vector of $\mathbb{R}^5$. This system is thus nilpotent and, since the Lie algebra generated by its control v.f. is five-dimensional, i.e. of the same dimension as the system itself, it is left-invariant on $\mathbb{R}^5$ w.r.t. some group product which, as one can easily verify, is defined by

$$xy = \begin{pmatrix} x_1+y_1 \\ x_2+y_2+y_1x_4 \\ x_3+y_3+y_1x_5 \\ x_4+y_4 \\ x_5+y_5 \end{pmatrix} \tag{9}$$

The next step consists in determining transverse functions for this system. A possibility, pointed out in early papers on the transverse function approach [8], [2], consists in forming the ordered group product of two *elementary* exponential functions, each defined on $\mathbb{S}^1$ and involving a v.f. derived from the way the system's Lie Algebra is generated. This possibility yields transverse functions depending on a minimal number of variables, i.e. two, but it does not respect the symmetric role played by the generating v.f. Z_2 and Z_3 (see [9] for more details on this issue). Another possibility, which respects this symmetry, consists in the design of transverse functions defined on $\mathbb{SO}(3)$. This approach, first introduced in [5] for the control of the trident snake, is considered next.

Since the system's Lie Algebra can be generated by first-order Lie brackets only, it is known from [10] that tranverse functions defined on $\mathbb{SO}(m)$ –with m the number of control inputs– exist. In the present case $m=3$ and, using the fact that the Lie bracket $[Z_2,Z_3]$ is null, a possible transverse function is given by

$$\bar{f}(Q) = \exp\left(\varepsilon \sum_{i=1}^{3} a_i(Q)Z_i + \frac{\varepsilon^2}{2}(b_3(Q)Z_4 - b_2(Q)Z_5)\right)$$
$$= \begin{pmatrix} \varepsilon a_1(Q) \\ \frac{\varepsilon^2}{2}(a_1(Q)a_2(Q) - b_3(Q)) \\ \frac{\varepsilon^2}{2}(a_1(Q)a_3(Q) + b_2(Q)) \\ \varepsilon a_2(Q) \\ \varepsilon a_3(Q) \end{pmatrix}, \quad Q \in \mathbb{SO}(3) \tag{10}$$

with $\exp(Z)$ denoting the solution at time $t = 1$ of the system $\dot{x} = Z(x)$, starting from the neutral element of the group product (here equal to the null vector), and

$$\begin{array}{l} a = DQe_1, \ D = diag\{d_1, d_2, d_3\} \\ b = \bar{D}Qe_3, \ \bar{D} = diag\{d_2d_3, d_1d_3, d_1d_2\} \end{array}, \ d_{1,2,3} \in \mathbb{R}\backslash\{0\}$$

e_i ($i = 1, 2, 3$) the canonical basis of $\mathbb{R}^3$, and $a_{1,2,3}$ (resp. $b_{1,2,3}$) the components of the vector a (resp. b). The design parameters ε and $d_{1,2,3}$ allow one to modify the size of the transverse function. The property of transversality is ensured as soon as none of these parameters is equal to zero. A complication arises from the fact that this function is defined on a three-dimensional manifold, whereas the minimal dimension required to obtain transversality is equal to two. We will see further how this extra dimension gives rise to an extra control variable which has to be dealt with at the control design level.

At this point, let us recall that if $\bar{f}(.)$ is transverse to a set of v.f. which are left-invariant on a Lie group, then the left-translation of this function by any constant element, or by any smooth time-dependent function, is also transverse to this set. In particular, given a reference rotation matrix $Q^\star$, then the function defined by

$$\bar{\bar{f}}(Q) := \bar{f}(Q^\star)^{-1}\bar{f}(Q) \tag{11}$$

is transverse to the v.f. $Z_{1,2,3}$ provided that the corresponding function $\bar{f}(.)$ given by (10) is a transverse function. The reason for such a modified transverse function is to allow for the asymptotic tracking of feasible reference trajectories. More precisely, define the tracking error $\tilde{\xi} := \xi_r(t)^{-1}\xi$ with $\xi_r(t)$ a predefined reference trajectory, then it suffices to have $Q(t)$ converge to $Q^\star$ while the "error" $\tilde{\xi}\bar{\bar{f}}(.)^{-1}$ converges to the group's neutral element (equal to zero) to ensure that $\xi(t)$ converges to $\xi_r(t)$. The second condition is satisfied by a proper design of the control law. This is the core of the transverse function control approach and of the associated objective of *practical* stabilization of *any* reference trajectory. As for the convergence of $Q(t)$ to $Q^\star$, it depends on i) the "admissibility" of the reference trajectory, ii) a proper choice of $Q^\star$, and iii) classical "persistent excitation" properties of the reference trajectory that ensure the controllability of the linear approximation of the tracking error system.

B. Conditions for asymptotic stabilization of admissible reference trajectories

The choice of the reference matrix $Q^\star$ has to be made in combination with the monitoring of an extra control variable. Define the modified error vector $z := \tilde{\xi}\bar{\bar{f}}^{-1}$ and the extended control vector $\bar{u} = \begin{pmatrix} u \\ \omega \end{pmatrix}$, with $\omega \in \mathbb{R}^3$ the angular velocity vector associated with the variation of Q, i.e. $\dot{Q} = QS(\omega)$ with $S(.)$ the skew-symmetric matrix-valued function associated with the cross-product operation in $\mathbb{R}^3$, i.e. $S(a)b = a \times b$. Let $A(.)$ denote the 5×3 matrix-valued function such that

$$\dot{\bar{\bar{f}}}(Q) = Z(\bar{\bar{f}}(Q))A(Q)\omega$$

with $Z = (Z_1, \dots, Z_5)$, and let $A_1(Q)$ and $A_2(Q)$ denote the sub-matrices of $A(Q)$, of respective dimensions 3×3 and 2×3, such that $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. Define also

$$G := \begin{pmatrix} I_3 \\ 0_{2\times3} \end{pmatrix}, \ \bar{G}(Q) := (G|-A(Q)) = \begin{pmatrix} I_3 & -A_1(Q) \\ 0_{2\times3} & -A_2(Q) \end{pmatrix}$$

with I_n denoting the $n \times n$ identity matrix. Let Ad^Z denote the expression of the Ad operator in the basis Z, i.e. the matrix-valued function defined by $\text{Ad}(\xi)Z(e)v := Z(e)\text{Ad}^Z(\xi)v$. By using classical differential calculus on Lie groups, the time-derivative of z is given by (see also [4])

$$\dot{z} = Z(z)\text{Ad}^Z(\bar{\bar{f}}(Q))(\bar{G}(Q)\bar{u} - \text{Ad}^Z(\tilde{\xi}^{-1})v_r) \tag{12}$$

with v_r the 5-dimensional vector such that $\dot{\xi}_r = Z(\xi_r)v_r$. The transversality property of $\bar{\bar{f}}$ implies that the 5×6 matrix $\bar{G}(Q)$ is of full rank $\forall Q \in \mathbb{SO}(3)$. Therefore the rank of $A_2(Q)$ is equal to two. Let μ denote a smooth 3-dimensional vector-valued function such that $\mu(Q,t) \in Ker(A_2(Q))$, $\forall(Q,t)$. Take, for instance,

$$\mu(Q,t) = (I_3 - A_2(Q)^\dagger A_2(Q))\rho(Q,t) \tag{13}$$

with $A_2(Q)^\dagger$ a right inverse of $A_2(Q)$, and ρ denoting a "free" vector-valued function which will be specified further in order to obtain the desired stability result. Define

$$\bar{\mu}(Q,t) = \begin{pmatrix} A_1(Q) \\ I_3 \end{pmatrix} \mu(Q,t) \ \in Ker(\bar{G}(Q)) \tag{14}$$

Then, in view of (12), any feedback control in the form

$$\bar{u} = \bar{G}(Q)^\dagger(\text{Ad}^Z(\tilde{\xi}^{-1})v_r + (Z(z)\text{Ad}^Z(\bar{\bar{f}}))^{-1}Kz) + \bar{\mu}(Q,t) \tag{15}$$

with

$$\bar{G}(Q)^\dagger = \begin{pmatrix} I_3 & -A_1(Q)A_2(Q)^\dagger \\ 0_{3\times3} & -A_2(Q)^\dagger \end{pmatrix} \tag{16}$$

a right inverse of $\bar{G}(Q)$, yields the closed-loop equation $\dot{z} = Kz$. Choosing K as a constant Hurwitz matrix yields the exponential stabilization of $z = 0$. Note that the last term $\bar{\mu}$ in the control expression only arises when the dimension of the extended control vector is larger than the system's dimension. A contribution of the present study is to show how to combine this term with an adequately chosen rotation matrix $Q^\star$ in order to ensure the local asymptotic stability of this matrix on the zero dynamics $z = 0$, when the reference trajectory ξ_r is admissible, i.e. when $v_r = Gu_r$.

Proposition 1 *Let q denote a quaternion associated with the rotation matrix Q, and $Im(q)$ denote its imaginary part. Apply the control law (15) to the system (6) with $A_2(Q)^\dagger$ chosen as the Moore-Penrose pseudo-inverse of $A_2(Q)$, i.e. $A_2(Q)^\dagger = A_2(Q)'(A_2(Q)A_2(Q)')^{-1}$. Then, on the exponentially stabilized zero dynamics $z = 0$ the following choices for $Q^\star$, ρ, and the sign of ε:*

$$\begin{cases} Q^\star := I_3 \\ \rho(Q,t) := -k_\rho |u_{r,1}(t)| Im(q) \ , \ k_\rho > 0 \\ sign(\varepsilon) = -sign(u_{r,1}) sign(d_1) \end{cases} \tag{17}$$

make $\bar{\bar{f}}(Q) = 0$, and subsequently $\tilde{\xi} = 0$, locally exponentially stable provided that i) u_r is bounded and ii) there exist constants $T, \delta > 0$ such that

$$\forall t \in \mathbb{R}_+, \quad \int_t^{t+T} |u_{r,1}(s)|\, ds \geq \delta \tag{18}$$

The proof of this proposition can be found in [9].

Relation (18) is a *persistent excitation* condition whose satisfaction ensures the controllability of the linear approximation of the error system associated with the tracking error $\tilde{\xi}$. It is a classical condition when addressing the asymptotic stabilization of feasible reference trajectories for nonholonomic systems [11], [12].

C. Transverse functions for the original system

The control problem addressed in this paper is the *practical* stabilization of any reference trajectory $g_r(t) = (x_r, y_r, \theta_r)'(t)$ for the system (4). Let $w_r \in \mathbb{R}^3$ denote the associated reference velocity, i.e. the vector such that $\dot{g}_r = \bar{R}(\theta_r) w_r$. The reference trajectory is admissible (or feasible) if there exist functions η_r and $u_{x,r}$ such that $w_r(t) = C(\eta_r(t)) u_{x,r}(t)$, $\forall t$. These functions are given by $u_{x,r} = w_{r,1}$ and $\eta_r = (\frac{w_{r,2}}{w_{r,1}}, \frac{w_{r,3}}{w_{r,1}})'$ respectively. They are well defined and unique as long as $w_{r,1} \neq 0$. A fixed point, for which $w_r = 0$, is also an admissible trajectory, but the function η_r is not unique in this case. When the first component of w_r is equal to zero at some time instant, with one of the other two components different from zero, the trajectory is not admissible (feasible). For the control design we propose to use a smooth function $\bar{\eta}_r$ with the properties of being i) always well defined, ii) a "good" approximation of η_r when $w_{r,1}$ is not small, iii) equal to the null vector when $w_{r,1} = 0$, and iv) bounded by predefined arbitrary values. The idea for the first three properties is to make $\bar{\eta}_r(t)$ a "reasonable" reference trajectory for the "shape" vector η, independently of the admissibility of $g_r(t)$. As for the fourth property, its usefulness will be explained shortly thereafter in relation to the property of transversality. An example of such a function is

$$\bar{\eta}_{r,i} = \bar{\eta}_{i,max} \tanh\left(\frac{w_{r,1} w_{r,1+i}}{\bar{\eta}_{i,max}(w_{r,1}^2 + \nu)}\right) \ , \ i = 1,2 \tag{19}$$

with $\bar{\eta}_{i,max} > 0$ the upperbound of $|\bar{\eta}_{r,i}|$ and ν a small positive number. Define

$$\xi_r(t) := \begin{pmatrix} 0_{3\times 1} \\ \bar{\eta}_r(t) \end{pmatrix} \tag{20}$$

and note that, in view of (7), $\Phi(\xi_r(t)) = \xi_r(t)$. Define also

$$\hat{f}(Q,t) := \xi_r(t) \bar{\bar{f}}(Q) \tag{21}$$

Setting $Q^\star = I_3$, and using (10) for the function $\bar{f}$ involved in the definition (11) of the function $\bar{\bar{f}}$, gives

$$\hat{f}(Q,t) = \begin{pmatrix} \varepsilon d_1 (q_{11} - 1) \\ \frac{\varepsilon^2}{2} d_1 d_2 (1 - q_{33} + q_{11} q_{21}) + \hat{f}_1 \bar{\eta}_{r,1} \\ \frac{\varepsilon^2}{2} d_1 d_3 (q_{11} q_{31} + q_{23}) + \hat{f}_1 \bar{\eta}_{r,2} \\ \varepsilon d_2 q_{21} + \bar{\eta}_{r,1} \\ \varepsilon d_3 q_{31} + \bar{\eta}_{r,2} \end{pmatrix} \tag{22}$$

with q_{ij} the element of Q at the crossing of the $i{-}th$ row and $j{-}th$ column. By application of Proposition 2 in [10], if $\bar{\bar{f}}$ is a transverse function for the homogeneous system (6), then

$$f(Q,t) := \Phi^{-1}(\hat{f}(Q,t)) \tag{23}$$

is a transverse function for the (feedback-equivalent) original system (4), provided that $\hat{f}(Q,t)$ remains inside the domain where Φ^{-1} is a diffeomorphism, i.e. provided that $\bar{d}(Q,t) := (\cos(\hat{f}_3) + \sin(\hat{f}_3)\hat{f}_4)(Q,t)$ never crosses zero. It thus suffices that $|\hat{f}_3(Q,t)| < \frac{\pi}{2}$ and $|\tan(\hat{f}_3)(Q,t)||\hat{f}_4(Q,t)| < 1$, $\forall (Q,t)$. Clearly, the satisfaction of these conditions set bounds upon i) the parameters ε and $d_{1,2,3}$ of the function (22), and ii) the components of $\bar{\eta}_r$. For instance, the inequality $|\hat{f}_3||\hat{f}_4| < 1$ is satisfied when $|\varepsilon d_1| [\frac{|\varepsilon d_2|}{2} (\frac{|\varepsilon d_3|(4+\sqrt{2})}{8} + 3|\bar{\eta}_{r,2}|) + (\frac{|\varepsilon d_3|}{4} + 2|\bar{\eta}_{r,2}|)|\bar{\eta}_{r,1}|] < 1$. Given *arbitrary* bounds on the components of $\bar{\eta}_r$, this inequality can be satisfied by choosing ε small enough. However, in practice, it matters to use parameters which are not too small, in order to limit the control amplitude and the frequency of maneuvers when tracking non-admissible trajectories.

IV. Control design

Consider a transverse function $f(Q,t)$ for the system (4), as defined by (23). The issue now is to synthesize control inputs u_x and v which practically stabilize any reference trajectory $g_r(t) = (x_r, y_r, \theta_r)'(t)$. As pointed out before, a possibility consists in applying the control design proposed in [3] which exploits the specific structure of the system and the possibility of controlling the shape vector η directly.

Let f_g and f_η denote the components of f such that $f = \begin{pmatrix} f_g \\ f_\eta \end{pmatrix}$, with $dim(f_g) = dim(g)$ and $dim(f_\eta) = dim(\eta)$. Set $z_\eta := \eta - f_\eta$, then

$$\dot{z}_\eta = v - \partial^Q f_\eta(Q,t)\omega - \partial^t f_\eta(Q,t)$$

with $\partial^t f_\eta(Q,t) = \frac{\partial f_\eta}{\partial \bar{\eta}_r}(Q,t)\dot{\bar{\eta}}_r(t)$. In order to exponentially stabilize $z_\eta = 0$ one can consider the control defined by

$$v = \partial^Q f_\eta(Q,t)\omega + \partial^t f_\eta(Q,t) - k_\eta z_\eta \ , \ k_\eta > 0 \tag{24}$$

which yields the closed-loop equation $\dot{z}_\eta = -k_\eta z_\eta$. This control can be computed once ω has been determined. Define the tracking error $\tilde{g} := g_r^{-1} \bullet g$, with $\bullet$ denoting the usual group product in $\mathbb{SE}(2)$, i.e.

$$g_1 \bullet g_2 = \begin{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + R(\theta_1) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \theta_1 + \theta_2 \end{pmatrix}$$

and g^{-1} the inverse of g, i.e. the element of $\mathbb{SE}(2)$ such that $g^{-1} \bullet g = 0$. One easily verifies that

$$\dot{\tilde{g}} = \bar{R}(\tilde{\theta})C(\eta)u_x + p(\tilde{g}, t)$$

with

$$p(\tilde{g}, t) := -\bar{R}(-\theta_r)\dot{g}_r + \begin{pmatrix} \tilde{g}_2 \\ -\tilde{g}_1 \\ 0 \end{pmatrix} \dot{\theta}_r$$

and $\tilde{\theta} = \theta - \theta_r$. Define also $z_g := \tilde{g} \bullet f_g^{-1}(Q, t)$, which may be viewed as the tracking error in $\mathbb{SE}(2)$ "modified" by the transverse function. One shows that

$$\begin{aligned} \dot{z}_g &= D(z_g, f_g)\bar{R}(\tilde{\theta}) \left(C(\eta)u_x - \bar{R}(-f_{g,3})\dot{f}_g + \bar{R}(-\tilde{\theta})p \right) \\ &= D(z_g, f_g)\bar{R}(\tilde{\theta}) \left(H(Q, t) \begin{pmatrix} u_x \\ \omega \end{pmatrix} + \Delta u_x + \bar{p} \right) \end{aligned}$$

with

$$\begin{aligned} D(z_g, f_g) &= \begin{pmatrix} I_2 & -R(z_{g,3}) \begin{pmatrix} -f_{g,2} \\ f_{g,1} \end{pmatrix} \\ 0_{1\times 2} & 1 \end{pmatrix} \\ H(Q, t) &= \begin{pmatrix} C(f_\eta(Q, t)) & -E(Q, t) \end{pmatrix} \\ E(Q, t) &= \bar{R}(-f_{g,3}(Q, t))\partial^Q f_g(Q, t) \\ \bar{p}(z_g, Q, t) &= \bar{R}(-\tilde{\theta})p(\tilde{g}, t) - \bar{R}(-f_{g,3}(Q, t))\tfrac{\partial f_g}{\partial \bar{\eta}_r}(Q, t)\dot{\bar{\eta}}_r \\ \Delta(z_\eta) &= C(\eta) - C(f_\eta) = \begin{pmatrix} 0 \\ z_\eta \end{pmatrix} \end{aligned}$$

It is simple to verify that the property of transversality of f implies that the 3×4 matrix $H(Q, t)$ is of full rank $\forall (Q, t)$. Therefore, using the fact that Δ exponentially converges to zero when v is given by (24), any control in the form

$$\begin{aligned} \begin{pmatrix} u_x \\ \omega \end{pmatrix} = H^\dagger(Q, t) &\left(-\bar{p} + \bar{R}(-\tilde{\theta})D^{-1}(z_g, f_g)K_g z_g \right) \\ &+ \bar{\mu}(Q, t) \end{aligned} \quad (25)$$

with

- $H^\dagger$ a right inverse of H,
- K_g a 3×3 Hurwitz matrix
- $\bar{\mu}$ a vector-valued function belonging to the kernel of H, i.e. such that $H(Q, t)\bar{\mu}(Q, t) = 0$, $\forall (Q, t)$,

yields the exponential convergence of z_g to zero. It follows that the feedback control law defined by (24) and (25) globally exponentially stabilizes $(z_g, z_\eta) = (0, 0)$. Since f_g is a bounded function the size of which can be rendered arbitrarily small via the choice of its parameters, $|\tilde{g}|$ is itself ultimately bounded by an arbitrarily small value. It is in this sense that the tracking error is "practically" stabilized.

Let us now focus on the complementary control term $\bar{\mu}$. As pointed out in Section III-B, the role of this term is to ensure the asymptotic stabilization of admissible trajectories, given an adequate value of the matrix $Q^\star$ involved in the transverse function. In view of the expression of H, i.e.

$$H(Q, t) = \begin{pmatrix} 1 & -E_1(Q, t) \\ f_\eta(Q, t) & -E_2(Q, t) \end{pmatrix}$$

one easily verifies that

$$H^\dagger := \begin{pmatrix} 1 & E_1 \\ O_{3\times 1} & I_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{E}_2^\dagger f_\eta & -\bar{E}_2^\dagger \end{pmatrix}$$

with $\bar{E}_2 := E_2 - f_\eta E_1$ and $\bar{E}_2^\dagger = \bar{E}_2'(\bar{E}_2\bar{E}_2')^{-1}$ –the Moore-Penrose right pseudo-inverse of E_2–, is a right pseudo-inverse of H, and that any function defined by

$$\bar{\mu} := \begin{pmatrix} E_1 \\ I_3 \end{pmatrix} (I_3 - \bar{E}_2^\dagger \bar{E}_2)\rho \quad (26)$$

with ρ any 3-dimensional vector-valued function, belongs to the kernel of H. By analogy with the problem treated in Section III-B, suitable choices for $Q^\star$, ρ, and the sign of the parameter ε involved in the transverse function expression, are provided by Proposition 1 with $w_{r,1}$ –the first component of $\bar{R}(-\theta_r)\dot{g}_r$– playing the role of $u_{r,1}$.

V. Simulation results

The simulation results reported next have been obtained with the transverse function (23) and the feedback control (24), (25), (26), with gains $K_g = -k_g I_3$ ($k_g = 1$), $k_\eta = 2$, and $k_\rho = 3$. The sign of ε, $Q^\star$, and ρ are specified in Proposition 1. The following transverse function parameters have been used: $|\varepsilon| = 0.2$, $d_1 = 0.5$, $d_2 = d_3 = 10$, with $\bar{\eta}_r$ as specified in (19), $\bar{\eta}_{1,max} = 1$, $\bar{\eta}_{2,max} = 1.5$, and $\nu = 0.01$.

The following table indicates the time history of the reference frame velocity vector $\dot{g}_r(t)$. Discontinuities at several time instants have been introduced purposefully in order to periodically re-initialize the tracking errors in the shape variables η and test the control performance during transient convergence phases.

$t \in (s)$	$\dot{g}_r =$ (m/s, m/s, rad/s)$'$
$[0, 10)$	$(0, 0, 0)'$
$[10, 15)$	$(1, 0, 0)'$
$[15, 20)$	$(0, 0, -\frac{\pi}{10})'$
$[20, 30)$	$(0, 0.5, 0.5\pi\cos(\pi(t-20)))'$
$[30, 35)$	$(0, 0, \frac{\pi}{10})'$
$[35, 40)$	$(-\cos(\frac{\pi}{5}(t-35), \sin(\frac{\pi}{5}(t-35)), 0)'$
$[40, 45)$	$(2, 0, -2\sin(\frac{\pi}{3}(t-40)))'$
$[45, 50)$	$(0, -1, -\frac{\pi}{10})'$
$[50, 55)$	$(1.3, 1, \sin(3(t-50)))'$
$[55, 60)$	$(0, 0, 0)'$

The tracking of the reference frame starts after the first five seconds during which all velocities are kept equal to zero. The reference trajectory has been chosen so as to illustrate various control modes: *i)* fixed-point stabilization, when $t \in [5, 10) \cup [55, 60)$, *ii)* asymptotic tracking of admissible trajectories, when $t \in [10, 15) \cup [20, 30) \cup [35, 37.5-\varepsilon) \cup [37.5+\varepsilon, 40) \cup [40, 45)$, with a singularity avoidance at $t = 37.5$ when perfect tracking requires both steering wheel angles to be equal to $\pm\frac{\pi}{2}$, *iii)* practical stabilization of non-admissible trajectories, when $t \in [15, 20) \cup [30, 35) \cup [45, 50)$. Fig. 2 shows the (x, y) trajectories of the origin of the reference frame (dotted line) and of the origin of the frame attached to the vehicle (dashed line). It also shows superposed snapshots, taken at various time instants, of the wheeled vehicle and of the reference frame that it is tracking. The principle of

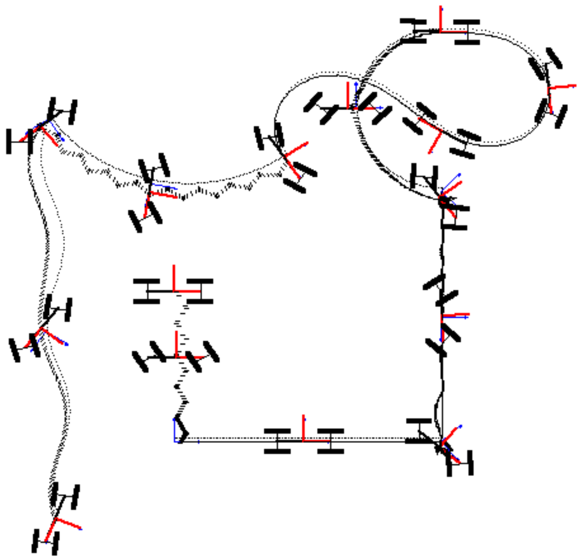

Fig. 2. Tracking of a reference frame

practical tracking is well illustrated by this figure. However, only a video of the simulation can qualitatively report of the "natural" character of the vehicle's motion.

Fig. 3 shows the exponential stabilization of $|z| = (|z_g|^2 + |z_\eta|^2)^{0.5}$ with respect to time, with re-initialization time instants corresponding to reference velocity discontinuities.

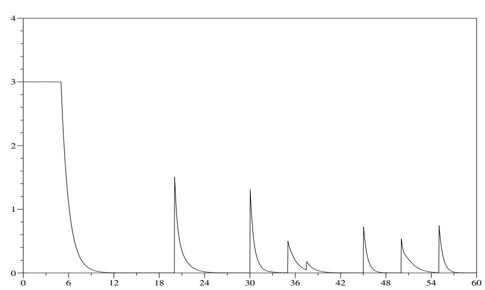

Fig. 3. $(z_g^2 + z_\eta^2)^{0.5}$ vs. time (s)

Finally, Fig. 4 shows the time evolution of the three components of the imaginary part of the quarternion associated with the matrix Q on which the task function depends. Perfect tracking occurs when all components are equal to zero. Imperfect tracking during phases when the reference trajectory is admissible is due to the non-equality between the vector $\bar{\eta}_r(t)$ defined by (19), which is used in the transverse function, and the reference steering angle vector $\eta_r(t)$.

VI. Conclusion and Research Directions

Extensions to the present work are multiple. For instance, several issues related to the choice and properties of adequate transverse functions have been pointed out in the core of the paper. Studying these issues will participate in the development of a methodology for the generation of

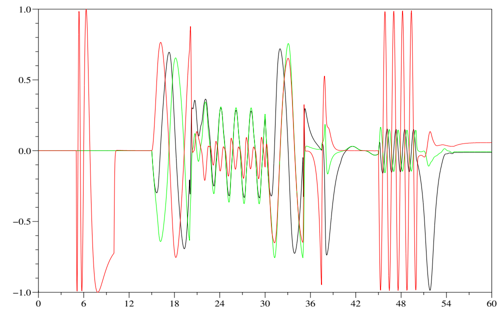

Fig. 4. $Im(q)$ vs. time (s)

transverse functions best adapted to control purposes. An extension, related to the use of "symmetrical" transverse functions defined on $\mathbb{SO}(n)$ and to our recent work [6] on snake-like wheeled mechanism, concerns the control of snake-like wheeled mechanisms *with orientable wheels* which facilitate the maneuvering of the system. Another extension of particular interest from both theoretical and practical standpoints concerns the control of the snakeboard [13], which may be viewed as an underactuated dynamical version of a two-steering-wheels vehicle.

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