2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

An efficient approach to the design of observers for continuous-time systems with discrete-time measurements

Filippo Cacace, Valerio Cusimano, Alfredo Germani

Abstract— **This paper describes an efficient discretization approach for nonlinear continuous-time systems. A Carleman linearization approach is used to evaluate the exact coefficients of the Taylor-Lie expansion of the dynamics of the system. The resulting discretization scheme is used to build a discrete-time observer that displays good performance. The paper shows the advantages of using an integrated discretization - observation approach for large discretization intervals.**

I. INTRODUCTION

State estimation for continuous-time systems when only discrete-time measurements are available is a problem of the utmost importance in many applications. The problem is substantial when the discretization interval is large enough to prevent the use of observers for continuous-time systems. This happens frequently in applications fields such as Medicine, Biology and Economy. The most widely used approach to deal with this problem is to build an approximate discrete version of the system which is used to build a discrete-time observer. In this paper we adopt an integrated discretization-observer method, where the observer design is based on the structure of the discretized system.

Discretization methods for continuous-time systems have been widely investigated. Classical numerical techniques for the integration of nonlinear ODEs such as Euler, Runge-Kutta, etc are accurate only for very small values of the discretization interval. Approaches that generalize the notion of the convolution integral and that aim, in principle, at developing a nonlinear analogue of the discretization approach for linear systems have been presented in [19], [20]. In presence of the zero-order hold (ZOH) element the continuous-time system becomes autonomous over the sampling interval, and this enables the use of an immediate sampled representation of the original system within the context of the Taylor-Lie series theory [11], [14], [15]. The use of higher-order terms of the Taylor-Lie series makes the method very accurate, but the algorithms presented so far for the derivation of the exact expression of the Taylor coefficients of the discretized system are quite complicated (see for example [12]). Another approach to the discretization problem makes use of the Carleman linearization method, where the nonlinear system is approximated by a linear system with an extended state containing the vector of the original state variables together with its Kronecker powers. This method is used for example in [24], that uses a Carleman linearization with a fixed starting point. The drawback of this approach is that the system size increases very rapidly with the degree of desired accuracy.

Our approach is also based on a Carleman linearization, which is however used to derive the coefficients of the Taylor approximation in a simple way. The original system size is then preserved.

Many papers discuss the extension to discrete-time systems of observer design techniques devised and developed for continuous time systems. An approach widely investigated is to find a nonlinear change of coordinates that transform the system into some form suitable for observer design using linear methodologies [1], [16], [25], [26]. Another popular approach is to design the observer in the original coordinates and to use iterative algorithm, typically inspired by the Newton method, that asymptotically solve suitably defined observability maps [7], [8], [10], [21]. The use of the Extended Kalman Filter as a local observer for noise-free systems and its convergence properties have been investigated in [3], [4], [22], [23]. In [9] the use of the polynomial approximations of nonlinear discrete-time systems for solving the observation problem has been investigated, and conditions for the exponential convergence of the Polynomial Extended Kalman Filter, when used as an observer, are studied. Recently, hybrid approaches have been proposed for certain classes of systems [2], [13]. In the hybrid approach a continuous-time nonlinear observer is coupled with an inter-sample output predictor. Clearly, the method provides a continuous-time observer.

In this paper we use a variant of the Newton approach of [7], [8], which is however integrated in the discretization scheme to produce a simpler and efficient observer design.

II. DISCRETIZATION OF NONLINEAR SYSTEMS

The discretization approach that we use in this paper is known as *Carleman linearization* [17], and it consists of building an infinite dimensional linear system equivalent to the original nonlinear one. In the scheme that we propose here, the linear system is solved exactly and the solution is truncated at the approximation chosen. It turns out that the terms of the truncated approximation correspond to those of the Taylor expansion of the solution of the nonlinear differential equation. The scheme provides an efficient recursive procedure for the computation of the terms of the Taylor series of the solution, which is amenable of easy evaluation by means of sparse matrices.

F. Cacace, f.cacace@unicampus.it, Valerio Cusimano v.cusimano@unicampus.it and A. Germani are with Università Campus Bio-Medico di Roma, A. Germani is also with Dipartimento di Ingegneria Elettrica e dell'Informazione, Università degli Studi dell'Aquila, Poggio di Roio, 67040 L'Aquila, Italy, germani@ing.univaq.it

978-1-61284-799-3/11/$26.00 ©2011 IEEE

Let us consider the continuous-time nonlinear system

$$\begin{aligned}\dot{x}(t) =& f\left(x(t),\ u(t)\right)\\ y(t) =& h\left(x(t)\right),\end{aligned} \tag{1}$$

where $f : I\!R^n \times I\!R^p \to I\!R^n$ and $h : I\!R^n \to I\!R$ are analytic functions in all compact sets of $I\!R^n$. Given a discretization interval Δ, we assume $u(t)$ to be constant in $[k\Delta,\ (k+1)\Delta)$. The symbol $\otimes$ denotes the Kronecker matrix product, the notation $A^{[i]}$ is used for the Kronecker power of matrix A (that is $A \otimes A \otimes \cdots \otimes A$, repeated i times). The standard Jacobian of f can be formally written as $\nabla_x \otimes f$, where ∇_x denotes the operator $[\partial/\partial x_1 \ldots \partial/\partial x_n]$. Higher-order Jacobians of f can be expressed as $\nabla_x^{[i]} \otimes f = \nabla_x \otimes \left(\nabla_x^{[i-1]} \otimes f\right)$.

In the time interval $[k\Delta,\ (k+1)\Delta)$ $u(t)$ is constant and system (1) is autonomous,

$$\dot{x}(t) = f\left(x(t),\ u(k\Delta)\right), \quad t \in [k\Delta,\ (k+1)\Delta). \tag{2}$$

In this time interval we define the function $\varphi_k(t) = x(t) - x(k\Delta)$, such that $\varphi_k(k\Delta) = 0$, $x((k+1)\Delta) = x(k\Delta) + \varphi_k((k+1)\Delta)$, and $\dot{\varphi}_k(t) = \dot{x}(t)$. Under standard analiticity hypothesis, the Taylor expansion of $\dot{\varphi}_k(t)$ yields

$$\begin{aligned}\dot{\varphi}_k(t) &= \sum_{i=0}^{\infty} \frac{\nabla^{[i]} \otimes f(x)}{i!}\Big|_{x=x(k\Delta)} {\varphi_k}^{[i]}(t)\\ &= \sum_{i=0}^{\infty} A_i^1\left(x(k\Delta)\right) {\varphi_k}^{[i]}(t),\end{aligned} \tag{3}$$

where $A_0^1(x) = f(x,u)$, $A_1^1(x)$ is the standard Jacobian of $f(x,u)$, and $A_j^1(x) \in I\!R^{n \times n^j}$. Clearly, even if the $A_j^1(x)$ depend only on x within each discretization interval, in general they depend also on the value of $u(k\Delta)$. This has been omitted in the notation for the sake of clarity.

The Carleman linearization procedure is obtained by extending the system (3) with the time derivatives of the Kronecker powers $\varphi_k^{[i]}(t)$. It is known [17] that

$$\frac{d}{dt}\varphi_k^{[i]}(t) = \sum_{j=0}^{\infty} A_j^i(x(k\Delta)) {\varphi_k}^{[j+i-1]}(t), \tag{4}$$

where the coefficients $A_j^i(x) \in I\!R^{n^i \times n^{j+i-1}}$ can be easily computed from the $A_j^1(x)$ through the recursive expression

$$A_j^i = A_j^1 \otimes I_n^{[i-1]} + I_n \otimes A_j^{i-1}, \tag{5}$$

where I_n is the identity matrix of dimension n.

To obtain the infinite dimensional linear system associated to (2) we introduce the *extended state* $\Phi(t) = [\Phi_1^T(t),\ \Phi_2^T(t), \ldots]^T$, where $\Phi_i(t) = \varphi_k^{[i]}(t)$. From (4) we have

$$\dot{\Phi}(t) = L_k + M_k \Phi(t), \quad k\Delta \le t < (k+1)\Delta, \tag{6}$$

where the infinite dimensional matrices L_k and M_k have the block structure

$$L_k = \begin{bmatrix} A_0^1 \\ 0_{n^2\times 1} \\ 0_{n^3\times 1} \\ \cdots \end{bmatrix}, \quad M_k = \begin{bmatrix} A_1^1 & A_2^1 & A_3^1 & \cdots \\ A_0^2 & A_1^2 & A_2^2 & \cdots \\ 0_{n^3\times 1} & A_0^3 & A_1^3 & \cdots \\ 0_{n^4\times 1} & 0_{n^3\times n} & A_0^4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{7}$$

A solution of (6) can be obtained for $t \in [k\Delta, (k+1)\Delta)$ with initial condition $\Phi(k\Delta) = [0\ 0\ \ldots]^T$, as

$$\begin{aligned}\Phi(t) &= \int_{k\Delta}^{t} e^{M_k(t-\tau)} L_k d\tau = M_k^{-1}\left(e^{M_k(t-k\Delta)} - I\right) L_k\\ &= \sum_{j=1}^{\infty} M_k^{j-1} L_k \frac{t^j}{j!}, \quad k\Delta \le t < (k+1)\Delta.\end{aligned} \tag{8}$$

we are interested in $\varphi_k(t) = \Phi_1(t) = [I_n\ 0_{n\times n^2}\ \ldots]\Phi(t)$, that is used to compute the value $x((k+1)\Delta)$. This yields

$$x((k+1)\Delta) = x(k\Delta) + \sum_{j=1}^{\infty}\left([I_n\ 0_{n\times n^2}\ \ldots] M_k^{j-1} L_k\right)\frac{\Delta^j}{j!}. \tag{9}$$

It is easy to recognize that (9) is the Taylor series for the solution of (2) in the prescribed time interval. An approximate solution is obviously obtained by choosing a bound ν for the index of the summation. It is interesting to notice that, even if M_k and L_k are matrices with infinite dimensions, for any j the product $M_k^j L_k$ has only a finite number of non-zero blocks. The coefficients of the Taylor expansion of the solution of the nonlinear system coincide with the first block of $M_k^j L_k$, which is always a vector of size $n \times 1$.

The discretization procedure can be readily obtained by (9). Given Δ and a bound ν, in the first step we compute the coefficients $A_j^1 = \nabla^{[j]} \otimes f(x)$, $j = 0, \ldots, \nu$. Then, starting with $P_0 = L_k$, we compute the sequence $P_j = M_k P_{j-1}$. This requires the computation of the first j block-rows and columns of M_k, that are obtained by means of (5) from the elements of A_j^1. This step is trivial, since we only need to perform Kronecker products with the identity matrix. The expression of the discretization step becomes now

$$x((k+1)\Delta) = x(k\Delta) + \sum_{j=0}^{\nu}[I_n\ Z] P_j(x(k\Delta))\frac{\Delta^{j+1}}{(j+1)!}, \tag{10}$$

where and Z is a zero matrix of size $n \times \left((n^\nu - 1)/(n-1) - n\right)$. Notice that $[I_n\ Z]P_j(x)$ are the exact coefficients of the Taylor expansion of the solution. These coefficients are listed in Table I for $j = 0, \ldots, 3$. At each discretization step it is sufficient to evaluate these coefficients in the new point $x((k+1)\Delta)$.

III. Discretization-based observer

A. Observability map

The discretization approach described in the previous section yields a discrete-time system of the form

$$\begin{aligned}x_{k+1} &= F_\Delta(x_k, u_k)\\ y_k &= h(x_k),\end{aligned} \tag{11}$$

where $F_\Delta(x_k, u_k)$ is in the form (10),

$$x_{k+1} = F_\Delta(x_k, u_k) = x_k + \sum_{j=1}^{\nu} C_j(x_k)\frac{\Delta^j}{j!}, \tag{12}$$

with $C_j(x) = [I_n\ Z]P_{j-1}(x)$.

j	$[I_n \ Z]P_j(x)$
0	$A_0^1 \quad = f(x)$
1	$A_1^1 A_0^1 \quad = J(x)f(x)$
2	$\left(A_1^1\right)^2 A_0^1 + 2A_2^1 \left(A_0^1\right)^{[2]}$
3	$\left(A_1^1\right)^3 A_0^1 + \left(2A_1^1 A_2^1 + 3A_2^1 A_1^2\right)\left(A_0^1\right)^{[2]} + 6A_3^1 \left(A_0^1\right)^{[3]}$

TABLE I
COEFFICIENTS OF THE TAYLOR EXPANSION OF THE DISCRETIZATION SCHEME

It is useful to notice that it is easy to invert $F_\Delta(x_k, u_k)$ using the Taylor expansion with a backward time step,

$$x_k = F_{-\Delta}(x_{k+1}, u_k) = x_{k+1} + \sum_{j=1}^{\nu} C_j(x_{k+1}) \frac{(-\Delta)^j}{j!}. \quad (13)$$

We follow an approach similar to [7] and [8], where the observer design is based on a nonlinear observability map on $\mathbb{R}^n$ $z_k = \Psi_{u_{[k]}}(x_k)$, that is, from the state variables to the last n samples of the scalar output variable,

$$z_k = \begin{bmatrix} y_k \\ y_{k-1} \\ \dots \\ y_{k-n+1} \end{bmatrix} = \begin{bmatrix} h(x_k) \\ h \circ F_{-\Delta}(x_k, u_{k-1}) \\ \dots \\ h \circ F_{-\Delta}^{(n-1)}(x_k, u_{k-n+1}) \end{bmatrix} = \Psi_{u_{[k]}}(x_k) \quad (14)$$

where $h \circ F_{-\Delta}(x) = h(F_{-\Delta}(x))$, and $F_{-\Delta}^{(i)}(x)$ denotes the application of $F_{-\Delta}$ iterated i times. This map $\Psi_{u_{[k]}}$ is parametrized by the vector $u_{[k]} = [u_{k-1} \ u_{k-2} \ \dots \ u_{k-n+1}] \in \mathbb{R}^{n-1}$. The difference with the approach of [7] and [8] is that here we use an observability map from x_k to z_k, whereas in the cited papers the observability map is from x_{k-n+1} to z_k. Obviously, this is due to availability of $F_{-\Delta}$, provided by the discretization scheme.

Definition 1: Given a C^∞ function $h : \mathbb{R}^n \to \mathbb{R}$ and the u-parametrized C^∞ vector field $F_{-\Delta}$ on $\mathbb{R}^n$, let $H^i(x, u) = h \circ F_{-\Delta}^{(i)}(x, u)$ and $H^0(x) = h(x)$. The observability matrix $Q_{u_{[k]}}(x)$ is defined as

$$Q_{u_{[k]}}(x_k) = \frac{d}{dx} \begin{bmatrix} H^0(x_k) \\ H^1(x_k, u_{k-1}) \\ \dots \\ H^{n-1}(x_k, u_{k-n+1}) \end{bmatrix} = \frac{d}{dx} \Psi_{u_{[k]}}(x_k). \quad (15)$$

Definition 2: The nonlinear system (11) is said to be *observable* in an open subset Ω of $\mathbb{R}^n$ if its observability map (14) is invertible in Ω.

Definition 3: The nonlinear system (11) is said to be *uniformly Lipschitz observable* in an open subset Ω of $\mathbb{R}^n$ if it is observable and if both maps $\Psi_{u_{[k]}}(x)$ and $\Psi_{u_{[k]}}^{-1}(z)$ are uniformly Lipschitz in Ω and $\Psi_{u_{[k]}}(\Omega)$, respectively.

B. Observer definition

The idea behind the proposed observer is to approximate the vector of the output variables from the current estimate $\hat{x}_k$. Let $\hat{x}_{k+1|k} = F_\Delta(\hat{x}_k, u_k)$ denote the one-step prediction. We have

$$z_{k+1} = \Psi_{u_{[k+1]}}(x_{k+1}) = \Psi_{u_{[k+1]}}(\hat{x}_{k+1|k}) + \sum_{j=1}^{\infty} \frac{\nabla \otimes \Psi_{u_{[k+1]}}(x)}{j!}\Big|_{x=\hat{x}_{k+1|k}} \left(x_{k+1} - \hat{x}_{k+1|k}\right)^{[j]}. \quad (16)$$

Truncating the Taylor expansion at $j = 1$ and neglecting the remainder, we have

$$z_{k+1} \simeq \Psi_{u_{[k+1]}}(\hat{x}_{k+1|k}) + Q_{u_{[k+1]}}(\hat{x}_{k+1|k})\left(x_{k+1} - \hat{x}_{k+1|k}\right). \quad (17)$$

Inverting (17) we can express the one-step estimate $\hat{x}_{k+1}$ as

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + Q_{u_{[k+1]}}^{-1}(\hat{x}_{k+1|k})\Big(z_{k+1} - \Psi_{u_{[k+1]}}(\hat{x}_{k+1|k})\Big). \quad (18)$$

Theorem 4: For a nonlinear system (11), consider the system

$$\hat{x}_{k+1} = F_\Delta(\hat{x}_k, u_k) + \left(Q_{u_{[k+1]}}(F_\Delta(\hat{x}_k, u_k))\right)^{-1} \Big(z_{k+1} - \Psi_{u_{[k+1]}}(F_\Delta(\hat{x}_k, u_k))\Big). \quad (19)$$

Then there exists a suitable $\delta > 0$ for which

$$\lim_{t \to \infty} \|\hat{x}_k - x_k\| = 0, \quad (20)$$

provided that

- $\|\hat{x}_0 - x_0\| < \delta$;
- system (11) is Lipschitz observable in $\mathbb{R}^n$;
- $h(x)$ and $F_\Delta(x, u)$ are uniformly Lipschitz in $\mathbb{R}^n$.

Proof. The proof can be derived following the guidelines of Theorem 1.2 in [8], and it is based on the following representation of system (11) using the z-coordinates

$$\begin{aligned} z_{k+1} &= A_b z_k + B_b\Big(h \circ F_\Delta(\Psi_{u_{[k]}}^{-1}(z_k))\Big) \\ y_k &= h(\Psi_{u_{[k]}}^{-1}(z_k)) \end{aligned} \quad (21)$$

where A_b, B_b are Brunowski matrices of size n.

□

Theorem 4 states that (19) is a local observer for system (11), since the convergence is guaranteed only for $\|\hat{x}_0 - x_0\| < \delta$.

The observer (19) is based on the linear approximation (17) of the inverse map $\Psi_{u_{[k]}}^{-1}(z)$. More general semi-global observers, with an arbitrary convergence region, can be designed using a polynomial approximation of $\Psi_{u_{[k]}}^{-1}(z)$, following the approach proposed in [10].

C. Implementation issues

In practice, a gain vector $K \in \mathbb{R}^n$ may be introduced on the correction term to speed up the convergence of the observer. It helps to keep into account the sensitivity of

the system dynamics to initial errors on the different state variables. This yields the observer

$$\hat{x}_{k+1} = F_\Delta(\hat{x}_k, u_k) + \big(Q_{u_{[k+1]}}(F_\Delta(\hat{x}_k, u_k))\big)^{-1} K\Big(z_{k+1} - \Psi_{u_{[k+1]}}(F_\Delta(\hat{x}_k, u_k))\Big). \tag{22}$$

For many high-gain observers based on a coordinate transformation, the computation of the Jacobian of the coordinate map is a complex task. Frequently, the computation of derivatives of order up to the size of the system is mandatory. A nice feature of the observer (19) is that its implementation requires only the first derivative of $F_\Delta(x,u)$ and $h(x)$. Consider again the Jacobian (15). The computation of the i-th row of $Q_{u_{[k]}}(x_k)$, $i>0$, can be carried out as

$$\begin{aligned}\frac{d}{dx}H^i(x_k) &= \frac{d}{dx}\Big(h\circ F^{(i)}_{-\Delta}(x_k)\Big)\\ &= \frac{d}{dx}\Big(h\circ F_{-\Delta}\circ\cdots\circ F_{-\Delta}(x_k)\Big)\\ = \left(\frac{dh}{dx}\Big|_{F^{(i)}_{-\Delta}(x_k)}\right)&\left(\frac{dF_{-\Delta}}{dx}\Big|_{F^{(i-1)}_{-\Delta}(x_k)}\right)\cdots\left(\frac{dF_{-\Delta}}{dx}\Big|_{x_k}\right),\end{aligned} \tag{23}$$

that is, the requested row is obtained as the product of dh/dx by the matrix $dF_{-\Delta}/dx$ evaluated i times along the discretized trajectory that ends at x_k.

The evaluation of $dF_{-\Delta}/dx$ (or, equivalently, dF_Δ/dx), is straightforward, due to the polynomial nature of $F_\Delta(x)$,

$$\frac{dF_\Delta(x,u)}{dx} = \nabla_x \otimes F_\Delta(x,u) = I_n + \sum_{j=1}^{\nu}\Big(\nabla_x \otimes C_j(x)\Big)\frac{\Delta^j}{j!}. \tag{24}$$

From the structure of the coefficients C_j reported in Table I, it is easy to compute $\nabla_x \otimes C_j(x)$, reminding that

$$\nabla_x \otimes A^1_j = (j+1)A^1_{j+1}, \tag{25}$$

and using the following derivation rule of matrix product

$$\nabla_x \otimes \big(A\cdot B\big) = \big(\nabla_x \otimes A\big)(I_n \otimes B) + A\big(\nabla_x \otimes B\big), \tag{26}$$

where n is the size of x, and $A \in I\!R^{l\times m}$, $B \in I\!R^{m\times q}$ are generic compatible matrices.

IV. Simulation results

The example presented in this section in not casual, in fact it motivated the study of the discretization/observation approach presented previously. The system that we consider is a simple model for the avascular tumor growth that can be used to estimate the kinetic parameter of the growth as well as the effects of chemotherapy treatment. In the last decades, many mathematical models have been proposed to describe the untreated and treated tumors growth at different level of complexity, from macroscopic to the microscopic molecular scale of analysis. However they have been only marginally used in clinical applications, due to the intrinsic complexity of the phenomena related to tumor growth and to the difficulties of arranging these models on the basis of the limited set of available measurements. This problem is dramatically true for *in vivo* applications, since the available information is provided just by means of biomedical image techniques like TAC and RNM that, among other things, are bound to provide measurements with very large discretization intervals.

Detailed mathematical models for tumor growth must take into account of several factors and, specifically, the nutrient concentration profiles within the tumor mass. To accomplish this task, partial differential equations must be solved, to include the space and time dependence for all variables of interest. On the other hand, some interesting results can be achieved considering models based on ODE, whose aim is to describe how the number of proliferant, quiescent and dead cells evolves over time; even if they do not describe the nutrients concentration profiles, are able to reproduce with reasonable accuracy the typical observed tumor growth over time. These models generally rely on an empirical definition of the dynamics of the tumor cell growth, based on parameters that characterize some kinetic features. Among them, a relevant place is occupied by the Gompertz model [6], [18], that has been seen chosen as a benchmark for the description of the tumor growth along time.

The practical applicability of this model is limited by two factors, namely

1) Parameter identification: there is no straightforward method to infer the value of the parameters contained in the model from biological evidence.
2) Discretization: as many other models based on nonlinear ODEs, the Gompertz model cannot be immediately applied when only discrete-time measurements are available. It must be noticed that the time interval between consecutive measurements is usually much larger than the threshold allowed for simple linearization techniques.

We present here a simplified version of the model that we studied in [5], in order to provide a case study for the proposed discretization/observation approach. [5] includes the case of chemotherapic infusion but a different observer is used. The Gompertz model that we use, in the case of avascular untreated growth, is defined by the equation:

$$\frac{dN(t)}{dt} = \gamma \cdot \log\left(\frac{N_\infty}{N(t)}\right), \tag{27}$$

where $N(t)$ is the number of cells of the tumor in the avascular phase, γ is a kinetic parameter and N_∞ is the theoretical saturation value, that depends on the kind of the cells.

The problem that we analyze here is to infer the value of N_∞ and γ from the measurements of $N(t)$. To this end we set $x_1 = \log(N)$, $x_2 = \log(N_\infty)$, $x_3 = \gamma$, thus we can recast equation (27) in the following dynamical system

$$\begin{aligned}\dot{x}_1(t) &= x_3(t)\big(x_2(t) - x_1(t)\big)\\ \dot{x}_2(t) &= 0\\ \dot{x}_3(t) &= 0\\ y(t) &= e^{x_1(t)}\end{aligned} \tag{28}$$

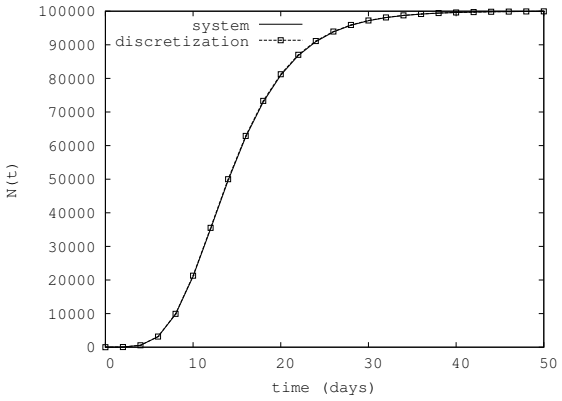

Fig. 1. Gompertz continuous and discrete-time models, a free growth case with $\nu = 3$ and $\Delta = 2$ days.

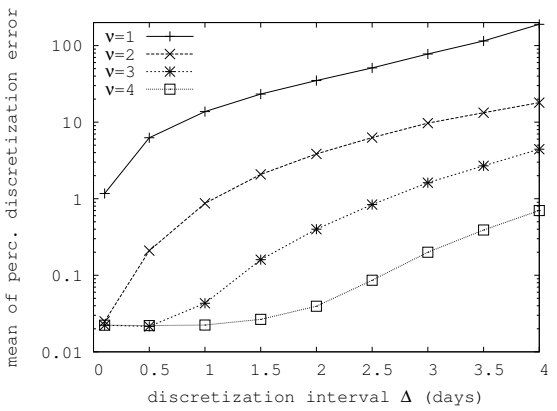

Fig. 2. Mean of the percentual discretization error $10^2 \cdot |N_k - N(t)|/N(t)$ for ν=1 – 4.

The discretization of (28) is easily accomplished. The first three terms of the resulting Taylor expansion are

$$C_1 = \begin{bmatrix} x_3(x_2 - x_1) \\ 0 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -x_3^2(x_2 - x_1) \\ 0 \\ 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} x_3^3(x_2 - x_1) \\ 0 \\ 0 \end{bmatrix}, \tag{29}$$

thus, with $\nu = 3$ the discrete-time system is

$$\begin{aligned} x_{k+1} &= x_k + \begin{bmatrix} C_1(x_k)\Delta + C_2(x_k)\frac{\Delta^2}{2} + C_3(X_k)\frac{\Delta^3}{6} \\ 0 \\ 0 \end{bmatrix} \\ y_k &= e^{(x_k)_1}. \end{aligned} \tag{30}$$

It may be noticed that $f(x)$ is polynomial, consequently the blocks A_j^1 of the matrix M_k in (7) are identically null for $j > 2$. This makes straightforward the derivation of higher order terms C_j for $j > 3$.

We test the discretization algorithm in free growth scenario with $N(0) = 1$, $N_\infty = 10^5$, and $\gamma = 0.2$ day^{-1}. The steady state is reached after approximately 50 days. Fig. 1 compares the continuous-time and discrete-time system output $N(t)$ with a discretization interval $\Delta = 2$ days. Notice that the exponential dynamics of the population growth is a challenging test for the discretization scheme.

In Fig. 2 the mean of the absolute difference between the continuos and discrete-time system, $||N(k\Delta) - N_k||$, is plotted as a function of the discretization interval Δ for a choice of ν in the interval $[1,\ 4]$. Clearly, $\nu = 1$ corresponds to a linearization of the system.

The performance of the discrete-time observer for system (28) with $\nu = 3$ are reported in Fig. 3. Since $n = 3$ the

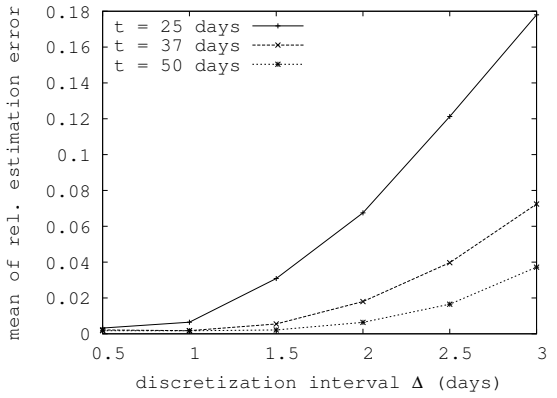

Fig. 3. Mean of the relative estimate error of the observer at 25, 37, 50 days as a function of the discretization interval Δ with $\nu = 3$.

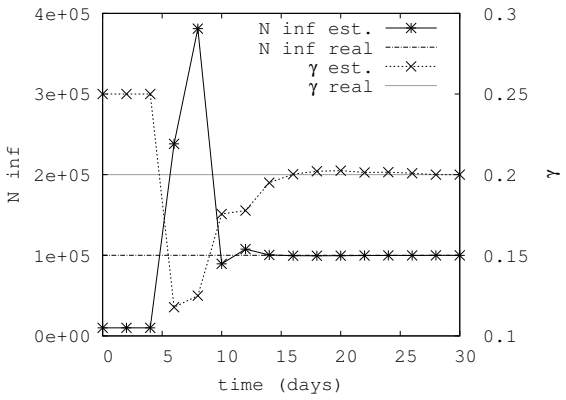

Fig. 4. An example of parameter estimation with $\Delta = 2$ and $\nu = 3$.

observer equations are

$$\hat{x}_{k+1} = F_\Delta(\hat{x}_k) + Q^{-1}(F_\Delta(\hat{x}_k))K\left(\begin{bmatrix} y_{k+1} \\ y_k \\ y_{k-1} \end{bmatrix} - \begin{bmatrix} h(F_\Delta(\hat{x}_k)) \\ h(\hat{x}_k) \\ h(F_{-\Delta}(\hat{x}_k)) \end{bmatrix}\right), \tag{31}$$

where $h(x) = e^x$, $Q(x)$ is defined as in (15) using (23)

$$Q(x) = \begin{bmatrix} \left(\frac{dh}{dx}\right)_x \\ \left(\frac{dh}{dx}\right)_{F_{-\Delta}(x)} \cdot \left(\frac{dF_{-\Delta}}{dx}\right)_x \\ \left(\frac{dh}{dx}\right)_{F^2_{-\Delta}(x)} \cdot \left(\frac{dF_{-\Delta}}{dx}\right)_{F_{-\Delta}(x)} \cdot \left(\frac{dF_{-\Delta}}{dx}\right)_x \end{bmatrix} \tag{32}$$

We have performed 1000 simulations for each discretization interval Δ in the range $\{0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3\}$ with initial condition $N(0) = 2$, $N_\infty = 10^5$, $\gamma = 0.2$, and setting the initial estimate $\hat{x}(0)$ at a random variable with normal distribution, mean $[\log(2),\ \log(10^4),\ 0.25]$ and standard deviation $[0,\ \log(10^3),\ 0.02]$. We have then computed the mean across simulations of the relative estimate error for the two non-measured state variables $x_2 = \log(N_\infty)$ and $x_3 = \gamma$ at 3 points, namely $t = 25$, $t = 37$ and $t = 50$ days. As it can be readily noticed in the plots, the final mean error on the parameter estimate is under 4% at any value of the discretization interval. Moreover with $\Delta < 2$ the mean estimate error is below 5% at $t = 25$ days, that corresponds to half of the time needed by the avascular tumor to reach its steady state. Thus, the discrete-time observer allows to estimate the growth parameters efficiently even for large values of the discretization interval. One simulation for $\Delta = 2$ is shown in Fig. 4.

V. Conclusions

The scheme proposed in this paper highlights a theoretical connection between the Carleman linearization method and the Taylor series approach and provides a fast recursive procedure for the computation of the the coefficients of the Taylor series, a useful tool to design a precise time-discrete approximation of a time-continuous system. Moreover, we have shown that the resulting discretization system has a structure that allows to design a simple yet efficient observer. We have shown that the resulting observer is robust with respect to the discretization interval. This approach is very general and can be extended further in two directions, namely the design of semi-global observers in the spirit of [10] and to the design of filters for nonlinear systems with time-discrete measurements.

References

[1] G. Besançon, H. Hammouri, S. Benamor, State equivalence of discrete-time nonlinear control systems to state affine form up to input/output injection, *System & Control Letters*, 33, 1 (1998), pp. 1–10.

[2] E. Biyik, and M. Arcak, A hybrid redesign of Newton observers in the absence of an exact discrete-time model, *System & Control Letters*, 55, 6 (2006), pp. 429–436.

[3] M. Boutayeb, H. Rafaralahy, and M. Darouach, Convergence analysis of the Extended Kalman Filter used as an observer for nonlinear deterministic discrete-time Systems, *IEEE Transactions on Automatic Control*, 42, 4 (1997), pp. 581–586.

[4] M. Boutayeb, D. Aubry, A strong tracking extended Kalman observer for nonlinear discrete-time systems, *IEEE Transactions on Automatic Control*, 44, 8 (1999), pp. 1550–1556.

[5] F. Cacace, V. Cusimano, L. Di Paola, and A. Germani, Observer-based techniques for the identification and analysis of a avascular tumor growth, *submitted for pubblication*, 2010.

[6] P. Castorina, and D. Zappalà, Tumor Gompertzian growth by cellular energetic balance, *Phys Stat Mech Appl*, 365, 2 (2006), pp. 473–480.

[7] G. Ciccarella, M. D. Mora, and A. Germani, Observers for discrete-time nonlinear systems, *System & Control Letters*, 20, 5 (1993), pp. 373–382.

[8] G. Ciccarella, M. D. Mora, and A. Germani, A robust observer for discrete-time nonlinear systems, systems, *System & Control Letters*, 24, 4 (1995), pp. 291–300.

[9] A. Germani, and C. Manes, The Polynomial Extended Kalman Filter as an exponential observer for nonlinear discrete-time systems, *Proc. of the 47th IEEE Conf. on Decision and Control (CDC '08)*, Cancun, Mexico, December 2008.

[10] A. Germani, and C. Manes, A Discrete-time Observer Based on the Polynomial Approximation of the Inverse Observability Map, *European Journal of Control*, 15, 2 (2009), pp. 143–156.

[11] W. Gröbner, *Die lie-reihen und ihre anwendungen*. Berlin, VEB Deutscher Verlag der Wissenschaften (1967).

[12] S. Hohmann, A. Konrad, V. Krebs, Exact Sampled Data Representation of Continuous Time Nonlinear Systems by Finite Polynomials with Exactly Determined Coefficients, *Proc. of the American Contro Conference 2001*, Arlington, VA, USA, pp. 1628–1633.

[13] I. Karafyllis, and C. Kravaris, From Continuous-Time Design to Sampled-Data Design of Observers, *IEEE Transactions on Automatic Control*, 54, 9 (2009), pp. 2169–2174.

[14] N. Kazantzis,and C. Kravaris, System-theoretic properties of sampled-data representation of nonlinear systems obtained via Taylor-Lie series, *International Journal of Control*, 67, 6 (1997), pp. 997–1020.

[15] N. Kazantzis,and C. Kravaris, Time-discretization of nonlinear control systems via Taylor methods, *Computers & Chemical Engineering*, 23 (1999), pp. 763–784.

[16] N. Kazantzis,and C. Kravaris, Discrete-time nonlinear observer design using functional equations, *System & Control Letters*, 42, 2 (2001), pp. 81–94.

[17] K. Kowalski, and W. H. Steeb, *Nonlinear dynamical systems and Carleman linearization*, World Scientific, 1991.

[18] F. Kozusko, Combining Gompertzian growth and cell population dynamics, *Math Biosci*, 185, 2 (2003), pp. 153–167.

[19] S. Monaco, and D. Normand-Cyrot, On the sampling of a linear analytic control system, *Proc. 24th IEEE Conference on Control and Decision (CDC '85)*, Fort Lauderdale, FL, USA, (1985), pp. 1457–1462.

[20] S. Monaco, D. Normand-Cyrot, and S. Stornelli, On the linearizing feedback in nonlinear sampled data control systems. *Proc. 25th IEEE Conference on Control and Decision (CDC '86)*, Athens, Greece, (1986), pp. 2056–2060.

[21] P.E. Moraal, and J. E. Grizzle, Observer design for nonlinear systems with discrete-time measurements, *IEEE Transactions on Automatic Control*, 40, 3 (1995), pp. 395–404.

[22] K. Reif, and R. Unbehauen, The extended Kalman filter as an exponential observer for nonlinear systems, *IEEE Transactions on Signal Processing*, 47, 8 (1999), pp. 2324–2328.

[23] Y. Song, J. W. Grizzle, The extended Kalman filter as a local asymptotic observer for discrete-time nonlinear systems, *J. Math. Systems Estimate and Control*, 5 (1995), pp. 59–78.

[24] S. A. Svoronos, D. Papageorgiu, and C. Tsiligiannis, Discretization of nonlinear control systems via the Carleman linearization, *chemical Engineering Science*, 49 (1994), pp 3263–3267.

[25] M. Xiao, N. Kazantzis, C. Kravaris, and A. J. Krener, Nonlinear discrete-time observer design with linearizable error dynamics, *IEEE Transactions on Automatic Control*, 48, 4 (2003), pp. 622–626.

[26] M. Xiao, A direct method for the construction of nonlinear discrete-time observer with linearizable error dynamics, *IEEE Transactions on Automatic Control*, 51, 1 (2006), pp. 128–135.