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Adaptive consensus estimation of multi-agent systems

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***Abstract*— This paper considers the effects of a penalty term in both the state and parameter estimates for multi-agent systems. It is assumed that the plant parameters are desired to be identified on-line and N agents are available to implement adaptive observers. Using an additive term which takes the form of a penalty on the mismatch between the state and parameter estimates, the proposed adaptive consensus estimation scheme ensures that both state and parameter estimates reach consensus. While the proposed adaptive consensus identifiers assume an all-to-all connectivity, the abstract framework that the adaptive identifiers are examined under, allows for any form of agent connectivity to be examined. As a measure of agreement of the estimates that is independent of the network topology, the deviation from the mean estimate for both the state and parameter estimates is defined and shown to converge to zero. Simulation studies of a second order system provide a verification of the proposed theoretical predictions.**

I. Introduction

In this work, we consider a plant with unknown state and input matrices. It is desired to adaptively identify both matrices using a group of N agents. Each agent is generating its own adaptive identifier which is based on the identifiers presented in [1]. Despite sharing the same structure, there is no guarantee that these N adaptive identifiers will have their estimates converge to the same value, unless one imposes a persistence of excitation condition. When interagent information exchange is implemented, a consensus of both state and parameter estimates is reached.

The additional term in both the state and parameter equations, which penalizes the mismatch amongst all estimates, takes the form of a non-negative damping term which also enhances the convergence properties of the state and parameter errors. The proposed work follows from earlier work [2] on adaptive consensus control of multi-agent systems with the difference that the mismatch between the parameter estimates is also penalized, which constitutes the main contribution of this work. However, in the non-adaptive case, a linear estimator scheme was considered that penalized the mismatch of the parameter estimates [3], [4].

To examine the agreement of both the state and parameter estimates, a measure that is independent of the network topology [5] is considered, and which takes the form of the deviation from the mean estimate.

The added benefit of the proposed adaptive identifiers, which penalize both state and parameter estimates, is the abstract form that the collective error dynamics are placed. Despite the fact that the stringent all-to-all connectivity is assumed, which serves the purpose of a baseline for subsequent extensions with different inter-agent connectivity, the abstract framework allows one to decouple the connectivity (graph Laplacian) with the stability analysis by simply replacing one non-negative damping-like matrix (representing all-to-all connectivity) with another such matrix (representing more general inter-agent connectivity).

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II. Problem formulation and mathematical preliminaries

We are concerned with the adaptive estimation of plant parameters in systems with full state availability. It is assumed that the dynamical system is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the plant state and $u \in \mathbb{R}^q$ is the control input. Without resorting to any plant parametrization [1], [6], it is assumed that the matrices A and B are unknown and it is desired to identify them on-line. Following the procedure outlined in [6], one considers the following adaptive observer

$$\begin{aligned} \dot{\widehat{x}}(t) &= A_m\widehat{x}(t) + (\widehat{A}(t) - A_m)x(t) + \widehat{B}(t)u(t), \\ \widehat{x}(0) &= \widehat{x}_0 \neq x_0, \end{aligned} \tag{2}$$

where $\widehat{A}(t)$ is the adaptive estimate of A and $\widehat{B}(t)$ is the adaptive estimate of B. The matrix A_m is a design matrix that essentially defines the observer poles and satisfies an associated Lyapunov equation. Central to the extraction of the adaptation rules for these on-line estimates is the associated Lyapunov function which uses the state and parameter errors. Defining the state and parameter errors

$$e(t) \triangleq \widehat{x}(t) - x(t), \quad \widetilde{A}(t) \triangleq \widehat{A}(t) - A, \quad \widetilde{B}(t) \triangleq \widehat{B}(t) - B,$$

we have that the state error is governed by

$$\begin{aligned} \dot{e}(t) &= A_m e(t) + (\widehat{A}(t) - A)x(t) + (\widehat{B}(t) - B)u(t) \\ &= A_m e(t) + \widetilde{A}(t)x(t) + \widetilde{B}(t)u(t). \end{aligned} \tag{3}$$

The associated Lyapunov function is

$$V(t) = e^T(t)Pe(t) + \mathrm{Tr}\left\{\widetilde{A}^T(t)\Gamma_a^{-1}\widetilde{A}(t) + \widetilde{B}^T(t)\Gamma_b^{-1}\widetilde{B}(t)\right\} \tag{4}$$

where Γ_a and Γ_b are the adaptive matrix gains and the symmetric positive definite matrix P satisfies the Lyapunov equation

$$A_m^T P + PA_m = -Q, \qquad Q = Q^T > 0. \tag{5}$$

Using the above Lyapunov function, the adaptation rules are

$$\begin{aligned}\dot{\widehat{A}}(t) &= -\Gamma_a P e(t) x^T(t), \quad \widehat{A}(0) = \widehat{A}_0 \neq A, \\ \dot{\widehat{B}}(t) &= -\Gamma_b P e(t) u^T(t), \quad \widehat{B}(0) = \widehat{B}_0 \neq B,\end{aligned} \tag{6}$$

Without incorporating any robust modifications to the above adaptive laws as no unmodeled dynamics or bounded disturbances were considered in (1), one has that the signals $e(t)$, $\widehat{A}(t)$ and $\widehat{B}(t)$ are bounded and if the plant state x and control input u are bounded, then $\lim_{t\to\infty} e(t) = 0$. Convergence of the adaptive estimates to the true values can be shown when a persistence of excitation condition is imposed [6].

The above error system can be placed in a compact form

$$\frac{d}{dt}\begin{bmatrix} e(t) \\ \widetilde{A}(t) \\ \widetilde{B}(t) \end{bmatrix} = \mathcal{A}(x,u) \begin{bmatrix} e(t) \\ \widetilde{A}(t) \\ \widetilde{B}(t) \end{bmatrix} \tag{7}$$

where

$$\mathcal{A}(x,u) = \left[\begin{array}{c|cc} A_m & (\cdot)x(t) & (\cdot)u(t) \\ \hline -\Gamma_a P(\cdot)x^T(t) & 0 & 0 \\ -\Gamma_b P(\cdot)u^T(t) & 0 & 0 \end{array}\right].$$

The above structure with skew-adjoint state dependent matrix is characteristic of adaptive systems [7]. The same matrix will be observed in the case of distributed adaptive consensus identifiers, in which the equivalent form will involve the same matrix as before and the contribution due to consensus enforcement. The latter can be related to the Laplacian of the graph topology.

III. ADAPTIVE CONSENSUS DISTRIBUTED OBSERVERS

We now consider a multi-agent system in which N agents are utilized to adaptively estimate the plant parameters A and B. Each agent will provide its own estimate $\widehat{A}_i(t)$ and $\widehat{B}_i(t)$, $i = 1, \ldots, N$, and it is desired to arrive at common estimates, i.e. reach consensus on the parameter adaptive estimates.

A. Overview of adaptive distributed observers

When the N agents do not interact with each other (no connectivity), then the N decoupled adaptive observers along with the adaptive estimates are simply generated by a direct application of (2) and (6), and are given by

$$\begin{aligned}\dot{\widehat{x}}_i(t) &= A_m \widehat{x}(t) + (\widehat{A}_i(t) - A_m)x(t) + \widehat{B}_i(t)u(t), \\ &\qquad \widehat{x}_i(0) = \widehat{x}_{i0} \neq x(0), \\ \dot{\widehat{A}}_i(t) &= -\Gamma_{ai} P e_i(t) x^T(t), \;\; \widehat{A}_i(0) = \widehat{A}_{i0}, \\ \dot{\widehat{B}}_i(t) &= -\Gamma_{bi} P e_i(t) u^T(t), \;\; \widehat{B}_i(0) = \widehat{B}_{i0},\end{aligned} \tag{8}$$

for $i = 1, \ldots, N$. One would like to find the extent at which the above state and parameter estimates agree with each other. A measure of agreement/disagreement, as was considered in [5] and which addresses disagreement independent of the network topology, is the deviation from the mean

$$\begin{aligned}\delta_{ie}(t) &\triangleq \widehat{x}_i(t) - \frac{1}{N}\sum_{j=1}^{N} \widehat{x}_j(t) = e_i(t) - \frac{1}{N}\sum_{j=1}^{N} e_j(t), \\ \delta_{ia}(t) &\triangleq \widehat{A}_i(t) - \frac{1}{N}\sum_{j=1}^{N} \widehat{A}_j(t) = \widetilde{A}_i(t) - \frac{1}{N}\sum_{j=1}^{N} \widetilde{A}_j(t), \\ \delta_{ib}(t) &\triangleq \widehat{B}_i(t) - \frac{1}{N}\sum_{j=1}^{N} \widehat{B}_j(t) = \widetilde{B}_i(t) - \frac{1}{N}\sum_{j=1}^{N} \widetilde{B}_j(t),\end{aligned}$$

for $i = 1, \ldots, N$, and the pairwise disagreement

$$\begin{aligned}\widehat{x}_{ij}(t) &\triangleq \widehat{x}_i(t) - \widehat{x}_j(t) = e_{ij}(t) = e_i(t) - e_j(t), \\ \widehat{A}_{ij}(t) &\triangleq \widehat{A}_i(t) - \widehat{A}_j(t) = \widetilde{A}_{ij}(t) = \widetilde{A}_i(t) - \widetilde{A}_j(t), \\ \widehat{B}_{ij}(t) &\triangleq \widehat{B}_i(t) - \widehat{B}_j(t) = \widetilde{B}_{ij}(t) = \widetilde{B}_i(t) - \widetilde{B}_j(t),\end{aligned}$$

for $i, j = 1, \ldots, N$. While the distributed adaptive observers in (8) are not expected to have any form of agreement, as no penalty of their disagreement was imposed in the adaptation rules, we will consider the dynamics for their disagreement as they will provide a baseline for the success of the adaptive consensus filters presented below.

The distributed Lyapunov functions for each agent are given in a similar form as in (4):

$$V_i(t) = e_i^T(t) P e_i(t) + \mathrm{Tr}\{\widetilde{A}_i^T(t)\Gamma_{ai}^{-1}\widetilde{A}_i(t) + \widetilde{B}_i^T(t)\Gamma_{bi}^{-1}\widetilde{B}_i(t)\} \tag{9}$$

for $i = 1, \ldots, N$. The stability of the collective dynamics are examined by

$$V(t) = \sum_{i=1}^{N} V_i(t). \tag{10}$$

It is straightforward to see that $\dot{V}_i(t) = -e_i^T Q e_i(t)$, $i = 1, \ldots, N$, and that

$$\dot{V}(t) = -\sum_{i=1}^{N} e_i^T Q e_i(t) = -E^T(t)\mathbb{Q}E(t)$$

where $E(t) = \begin{bmatrix} e_1^T(t) & e_2^T(t) & \ldots & e_N^T(t) \end{bmatrix}^T$, and $\mathbb{Q} = \mathrm{diag}\{Q\}$ is the block-diagonal matrix $\mathbb{Q} = \mathbf{I}_{n\times n} \otimes Q$. The above distributed adaptive observers can be placed in a form similar to (7). Define

$$\widehat{\mathbb{A}}(t) = \begin{bmatrix} \widehat{A}_1(t) \\ \widehat{A}_2(t) \\ \vdots \\ \widehat{A}_N(t) \end{bmatrix}, \quad \widehat{\mathbb{B}}(t) = \begin{bmatrix} \widehat{B}_1(t) \\ \widehat{B}_2(t) \\ \vdots \\ \widehat{B}_N(t) \end{bmatrix},$$

with $\widetilde{\mathbb{A}}(t)$ and $\widetilde{\mathbb{B}}(t)$ defined in a similar manner,

$$\mathbb{A}_m = \mathbf{I}_{n\times n} \otimes A_m, \quad \mathbb{P} = \mathbf{I}_{n\times n} \otimes P,$$

$$\Gamma_a = \begin{bmatrix} \Gamma_{a1} & \mathbf{0}_{n\times n} & \ldots \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \ldots & \Gamma_{aN} \end{bmatrix}, \Gamma_b = \begin{bmatrix} \Gamma_{b1} & \mathbf{0}_{n\times n} & \ldots \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \ldots & \Gamma_{bN} \end{bmatrix},$$

then we have

$$\begin{cases} \dot{E}(t) = \mathbb{A}_m E(t) + \widetilde{\mathbb{A}}(t)x(t) + \widetilde{\mathbb{B}}(t)u(t), \\ \dot{\widetilde{\mathbb{A}}}(t) = -\Gamma_a \mathbb{P} E(t) x^T(t), \\ \dot{\widetilde{\mathbb{B}}}(t) = -\Gamma_b \mathbb{P} E(t) u^T(t). \end{cases}$$

When the above is placed into matrix form, it yields

$$\frac{d}{dt}\begin{bmatrix} E(t) \\ \widetilde{\mathbb{A}}(t) \\ \widetilde{\mathbb{B}}(t) \end{bmatrix} = \mathbf{A}(x,u)\begin{bmatrix} E(t) \\ \widetilde{\mathbb{A}}(t) \\ \widetilde{\mathbb{B}}(t) \end{bmatrix}, \tag{11}$$

with

$$\mathbf{A}(x,u) = \left[\begin{array}{c|cc} \mathbb{A}_m & (\cdot)x(t) & (\cdot)u(t) \\ \hline -\Gamma_a \mathbb{P}(\cdot)x^T(t) & 0 & 0 \\ -\Gamma_b \mathbb{P}(\cdot)u^T(t) & 0 & 0 \end{array}\right].$$

Equation (11) is the multi-agent version of (7), and we have that each of the distributed adaptive observers in (8) enjoys the same stability and convergence properties as (2) and (6).

B. Adaptive consensus distributed observers - Main result

We now present the main results of this work. The update laws for the parameter identifiers in (8) now include a penalty of the mismatch of the parameter estimates $\widehat{A}_i(t)$ and $\widehat{B}_i(t)$. While the individual adaptive estimates will still need to have a certain condition of persistence of excitation to ensure parameter convergence, the proposed adaptive consensus modification ensures that all the parameter estimates agree with each other without enforcing persistence of excitation. This is presented in the lemma below.

Lemma 1: Consider the dynamical system (1) with x and u available and bounded, and A and B are unknown. Assume that there are N available agents capable of implementing their own distributed adaptive observers and that they have all-to-all connectivity (complete graph). Then the following distributed adaptive consensus observers

$$\begin{cases} \dot{\widehat{x}}_i(t) = A_m\widehat{x}(t) + (\widehat{A}_i(t) - A_m)x(t) + \widehat{B}_i(t)u(t) \\ \qquad -P^{-1}\sum_{j\neq i}^{N}(\widehat{x}_i(t) - \widehat{x}_j(t)), \;\; \widehat{x}_i(0) = \widehat{x}_{i0}, \\ \dot{\widehat{A}}_i(t) = -\Gamma_{ai} P e_i(t) x^T(t) \\ \qquad -\Gamma_{ai}\sum_{j\neq i}^{N}(\widehat{A}_i(t) - \widehat{A}_j(t)), \;\; \widehat{A}_i(0) = \widehat{A}_{i0}, \\ \dot{\widehat{B}}_i(t) = -\Gamma_{bi} P e_i(t) u^T(t) \\ \qquad -\Gamma_{bi}\sum_{j\neq i}^{N}(\widehat{B}_i(t) - \widehat{B}_j(t)), \;\; \widehat{B}_i(0) = \widehat{B}_{i0}, \end{cases} \tag{12}$$

for $i = 1,\ldots,N$ and with P satisfying (5), ensure that both state and parameter estimates reach consensus via

$$\lim_{t\to\infty}\widehat{x}_{ij}(t) = 0, \; \lim_{t\to\infty}\widehat{A}_{ij}(t) = 0, \; \lim_{t\to\infty}\widehat{B}_{ij}(t) = 0, \;\; i,j = 1,\ldots,N,$$

and the distributed state errors converge to zero,

$$\lim_{t\to\infty} e_i(t) = 0, \;\; i = 1,\ldots,N.$$

Proof: We consider the dynamics of the state and parameter errors associated with the proposed laws in (12)

$$\begin{aligned} \dot{e}_i(t) &= A_m e_i(t) + \widetilde{A}_i(t)x(t) + \widetilde{B}_i(t)u(t) - P^{-1}\sum_{j\neq i}^{N} e_{ij}(t) \\ \dot{\widetilde{A}}_i(t) &= -\Gamma_{ai} P e_i(t) x^T(t) - \Gamma_{ai}\sum_{j\neq i}^{N}\widetilde{A}_{ij}(t) \\ \dot{\widetilde{B}}_i(t) &= -\Gamma_{bi} P e_i(t) u^T(t) - \Gamma_{bi}\sum_{j\neq i}^{N}\widetilde{B}_{ij}(t), \end{aligned} \tag{13}$$

for $i = 1,\ldots,N$. We use the same distributed Lyapunov functions (9) to assess the stability of (12). The derivative of the individual V_i's along (13) are given by

$$\begin{aligned} \dot{V}_i(t) &= e_i^T(t)\left(A_m^T P + P A_m\right)e_i(t) \\ &\quad + 2e_i^T(t)\sum_{j\neq i}^{N} e_{ij}(t) + 2e_i^T(t)P\widetilde{A}_i(t)x(t) \\ &\quad + 2e_i^T(t)P\widetilde{B}_i(t)u(t) + 2\mathrm{Tr}\{\dot{\widetilde{A}}_i^T(t)\Gamma_{ai}^{-1}\widetilde{A}_i(t)\} \\ &\quad + 2\mathrm{Tr}\{\dot{\widetilde{B}}_i^T(t)\Gamma_{bi}^{-1}\widetilde{B}_i(t)\} \\ &= -e_i^T(t)Qe_i(t) - 2e_i^T(t)\sum_{j\neq i}^{N}(e_i(t) - e_j(t)) \\ &\quad -2\widetilde{A}_i^T(t)\sum_{j\neq i}^{N}\widetilde{A}_{ij}(t) - 2\widetilde{B}_i^T(t)\sum_{j\neq i}^{N}\widetilde{B}_{ij}(t), \end{aligned} \tag{14}$$

where we used the fact that

$$\sum_{j\neq i}^{N}(\widehat{x}_i(t) - \widehat{x}_j(t)) = \sum_{j\neq i}^{N}(e_i(t) - e_j(t)) = \sum_{j\neq i}^{N} e_{ij}(t),$$

$$\sum_{j\neq i}^{N}(\widehat{A}_i(t) - \widehat{A}_j(t)) = \sum_{j\neq i}^{N}(\widetilde{A}_i(t) - \widetilde{A}_j(t)) = \sum_{j\neq i}^{N}\widetilde{A}_{ij}(t),$$

and

$$\sum_{j\neq i}^{N}(\widehat{B}_i(t) - \widehat{B}_j(t)) = \sum_{j\neq i}^{N}(\widetilde{B}_i(t) - \widetilde{B}_j(t)) = \sum_{j\neq i}^{N}\widetilde{B}_{ij}(t).$$

Now we consider the collective dynamics via (10). Using (14), the derivative is given by

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{N}\dot{V}_i = -\sum_{i=1}^{N} e_i^T(t)Qe_i(t) - 2\sum_{i=1}^{N} e_i^T(t)\sum_{j\neq i}^{N}(e_i(t) - e_j(t)) \\ &\quad -2\sum_{i=1}^{N}\widetilde{A}_i^T(t)\sum_{j\neq i}^{N}(\widetilde{A}_i(t) - \widetilde{A}_j(t)) - 2\sum_{i=1}^{N}\widetilde{B}_i^T(t)\sum_{j\neq i}^{N}(\widetilde{B}_i(t) - \widetilde{B}_j(t)) \\ &= -\sum_{i=1}^{N} e_i^T(t)Qe_i(t) - \sum_{i=1}^{N}\sum_{j\neq i}^{N}\|e_{ij}(t)\|^2 \\ &\quad -\sum_{i=1}^{N}\sum_{j\neq i}^{N}\|\widetilde{A}_{ij}(t)\|^2 - \sum_{i=1}^{N}\sum_{j\neq i}^{N}\|\widetilde{B}_{ij}(t)\|^2 \leq 0, \end{aligned}$$

where we used the fact that

$$2\sum_{i=1}^{N} a_i \sum_{j\neq i}^{N}(a_i - a_j) = \sum_{i=1}^{N}\sum_{j\neq i}^{N}|a_i - a_j|^2.$$

From the above, we have that

$$V(t)+\lambda_{min}(Q)\int_0^t\sum_{i=1}^N\|e_i(\tau)\|^2d\tau+\int_0^t\sum_{i=1}^N\sum_{j\neq i}^N\|e_{ij}(\tau)\|^2d\tau$$
$$+\int_0^t\sum_{i=1}^N\sum_{j\neq i}^N\|\widetilde{A}_{ij}(\tau)\|^2d\tau+\int_0^t\sum_{i=1}^N\sum_{j\neq i}^N\|\widetilde{B}_{ij}(\tau)\|^2d\tau\leq V(0),$$

implying $E\in L_2(0,\infty;\mathbb{R}^{nN})\cap L_\infty(0,\infty;\mathbb{R}^{nN})$ or equivalently

$$e_i\in L_2(0,\infty;\mathbb{R}^n)\cap L_\infty(0,\infty;\mathbb{R}^n),\quad i=1,\dots,N,$$

with $\widehat{A}_i\in L_\infty(0,\infty;\mathbb{R}^{n\times n})$, $\widehat{B}_i\in L_\infty(0,\infty;\mathbb{R}^{n\times q})$, and $e_{ij}\in L_2(0,\infty;\mathbb{R}^n)$, $i,j=1,\dots,N$,

$$\widehat{A}_{ij}\in L_2(0,\infty;\mathbb{R}^{n\times n}),\ \widehat{B}_{ij}\in L_2(0,\infty;\mathbb{R}^{n\times q}),\quad i,j=1,\dots,N.$$

By virtue of the boundedness of $\widehat{A}_i$ and $\widehat{B}_i$, one then has

$$\widehat{A}_{ij}\in L_\infty(0,\infty;\mathbb{R}^{n\times n}),\ \widehat{B}_{ij}\in L_\infty(0,\infty;\mathbb{R}^{n\times q}),\quad i,j=1,\dots,N.$$

Examination of the first equation of (13) reveals that $\dot{e}_i\in L_\infty(0,\infty;\mathbb{R}^n)$ provided that x and u are bounded. That immediately implies via the use of Barbălat's Lemma ($e_i\in L_2\cap L_\infty$ and $\dot{e}_i\in L_\infty$) that

$$\lim_{t\to\infty}\|e_i(t)\|=0,\quad i=1,\dots,N.$$

Once again, using the boundedness and square integrability of e_i with a bounded derivative, we have that e_{ij} is bounded with a bounded derivative. The latter along with the square integrability of e_{ij} implies that

$$\lim_{t\to\infty}\|e_{ij}(t)\|=\lim_{t\to\infty}\|\widehat{x}_{ij}(t)\|=0,\ \ i,j=1,\dots,N.$$

Examining the second and third components of (14), we have that $\dot{\widehat{A}}_{ij}$ and $\dot{\widehat{B}}_{ij}$ are bounded provided that x and u are bounded. Once again, application of Barbălat's Lemma [8] yields

$$\lim_{t\to\infty}\|\widehat{A}_{ij}(t)\|=0,\quad\text{and}\quad\lim_{t\to\infty}\|\widehat{B}_{ij}(t)\|=0.$$

■

A consequence of the proposed adaptive consensus distributed filters is that there is guaranteed convergence of the pairwise difference of the adaptive estimates. This was a consequence of the L_2 boundnedness of the pairwise disagreement of the parameter estimates.

To better examine the dynamic agreement of the parameter and state estimates, we consider the error system (13) in a compact form. Towards this end, we define

$$\mathbb{J}=\begin{bmatrix}(N-1)I_{n\times n} & -I_{n\times n} & \dots & -I_{n\times n}\\ -I_{n\times n} & (N-1)I_{n\times n} & \dots & -I_{n\times n}\\ \vdots & \dots & \ddots & \vdots\\ -I_{n\times n} & \dots & -I_{n\times n} & (N-1)I_{n\times n}\end{bmatrix},$$

then we have

$$\begin{cases}\dot{E}(t)=\mathbb{A}_mE(t)+\widetilde{\mathbb{A}}(t)x(t)+\widetilde{\mathbb{B}}(t)u(t)-\mathbb{J}E(t)\\ \dot{\widetilde{\mathbb{A}}}(t)=-\Gamma_a\mathbb{P}E(t)x^T(t)-\Gamma_a\mathbb{J}\widetilde{\mathbb{A}}(t)\\ \dot{\widetilde{\mathbb{B}}}(t)=-\Gamma_b\mathbb{P}E(t)u^T(t)-\Gamma_b\mathbb{J}\widetilde{\mathbb{B}}(t)\end{cases}$$

The above placed in a matrix form yields

$$\frac{d}{dt}\begin{bmatrix}E(t)\\ \widetilde{\mathbb{A}}(t)\\ \widetilde{\mathbb{B}}(t)\end{bmatrix}=\Big(\mathbf{A}(x,u)-\mathbf{GJ}\Big)\begin{bmatrix}E(t)\\ \widetilde{\mathbb{A}}(t)\\ \widetilde{\mathbb{B}}(t)\end{bmatrix},$$

with

$$\mathbf{G}=\begin{bmatrix}I & 0 & 0\\ 0 & \Gamma_a & 0\\ 0 & 0 & \Gamma_b\end{bmatrix},\quad\mathbf{J}=\begin{bmatrix}\mathbb{J} & 0 & 0\\ 0 & \mathbb{J} & 0\\ 0 & 0 & \mathbb{J}\end{bmatrix},$$

which has similar structure to (11), differing only in the additional term $\mathbf{J}$ due to consensus.

Remark 1: In the not so conservative requirement of an all-to-all connectivity assumed here, one may have an undirected graph which when considered in the abstract form proposed here, will result in a matrix $\mathbb{J}$ that is still non-negative but may have a lower maximum eigenvalue.

To assess the convergence properties of the deviation from the mean, we can express the deviation as follows

$$\delta_{ie}(t)=e_i(t)-\frac{1}{N}\sum_{j=1}^Ne_j(t)=\frac{1}{N}\sum_{j\neq i}^Ne_{ij}(t).$$

Similar expressions can be written for $\delta_{ia}(t)$ and $\delta_{ib}(t)$. The convergence of the deviation from the mean can easily be established when one uses the fact that the pairwise disagreement of the state and parameter errors converge to zero and therefore we have

$$\lim_{t\to\infty}\delta_{ie}(t)=0,\ \lim_{t\to\infty}\delta_{ia}(t)=0,\ \lim_{t\to\infty}\delta_{ib}(t)=0,\ i=1,\dots,N.$$

The interpretation of the above is that the individual deviations of the adaptive state and parameter estimates from their mean (static average) converge to zero. This is summarized in the following lemma.

Lemma 2: Consider the deviations of the state and parameter estimates from their mean (static average) given by

$$\begin{aligned}\delta_{ie}(t)&=\widehat{x}_i(t)-\frac{1}{N}\sum_{j=1}^N\widehat{x}_j(t)=e_i(t)-\frac{1}{N}\sum_{j=1}^Ne_j(t)\\&=\frac{1}{N}\sum_{j\neq i}^N\widehat{x}_{ij}(t)=\frac{1}{N}\sum_{j\neq i}^Ne_{ij}(t),\\ \delta_{ia}(t)&=\widehat{A}_i(t)-\frac{1}{N}\sum_{j=1}^N\widehat{A}_j(t)=\widetilde{A}_i(t)-\frac{1}{N}\sum_{j=1}^N\widetilde{A}_j(t)\\&=\frac{1}{N}\sum_{j\neq i}^N\widehat{A}_{ij}(t)=\frac{1}{N}\sum_{j\neq i}^N\widetilde{A}_{ij}(t),\\ \delta_{ib}(t)&=\widehat{B}_i(t)-\frac{1}{N}\sum_{j=1}^N\widehat{B}_j(t)=\widetilde{B}_i(t)-\frac{1}{N}\sum_{j=1}^N\widetilde{B}_j(t)\\&=\frac{1}{N}\sum_{j\neq i}^N\widehat{B}_{ij}(t)=\frac{1}{N}\sum_{j\neq i}^N\widetilde{B}_{ij}(t).\end{aligned}$$

Then we have convergence of these deviations to zero

$$\lim_{t\to\infty}\|\delta_{ie}(t)\|=0,\ \lim_{t\to\infty}\|\delta_{ia}(t)\|=0,\ \lim_{t\to\infty}\|\delta_{ib}(t)\|=0,$$

for $i=1,\ldots,N$.

Proof: The proof easily follows from Lemma 1 which provides the pairwise convergence $\lim_{t\to\infty}\widehat{x}_{ij}=\lim_{t\to\infty}\widehat{A}_{ij}=\lim_{t\to\infty}\widehat{B}_{ij}=0$ and the fact that $N<\infty$. ■

One can also consider the deviations in matrix form and relate them to the graph Laplacian

$$\delta_e(t)=\begin{bmatrix}\delta_{1e}(t)\\ \delta_{2e}(t)\\ \vdots\\ \delta_{Ne}(t)\end{bmatrix},\ \delta_a(t)=\begin{bmatrix}\delta_{1a}(t)\\ \delta_{2a}(t)\\ \vdots\\ \delta_{Na}(t)\end{bmatrix},\ \delta_b(t)=\begin{bmatrix}\delta_{1b}(t)\\ \delta_{2b}(t)\\ \vdots\\ \delta_{Nb}(t)\end{bmatrix}$$

via $\delta_e(t)=\left(\frac{1}{N}\right)\mathbb{J}E(t)$, $\delta_a(t)=\left(\frac{1}{N}\right)\mathbb{J}\widetilde{\mathbb{A}}(t)$, $\delta_b(t)=\left(\frac{1}{N}\right)\mathbb{J}\widetilde{\mathbb{B}}(t)$, since $\sum_{j\neq i}\Box_{ij}$ is the ith row of $\mathbb{J}\Box$. From Lemma 2, we have the following convergence

$$\lim_{t\to\infty}\|\mathbb{J}E(t)\|=\lim_{t\to\infty}\|\mathbb{J}\widetilde{\mathbb{A}}(t)\|=\lim_{t\to\infty}\|\mathbb{J}\widetilde{\mathbb{B}}(t)\|=0.$$

However, the convergence

$$\lim_{t\to\infty}\|\widetilde{\mathbb{A}}(t)\|=0\quad\text{and}\quad\lim_{t\to\infty}\|\widetilde{\mathbb{B}}(t)\|=0$$

cannot be established unless one imposes the additional condition of persistence of excitation. The above demonstrates the beneficial effects of information sharing (connectivity) where the absence of $\mathbb{J}$ removes the convergence results on consensus unless persistence of excitation is imposed.

IV. Special case

We now consider a special case that can use the adaptive consensus structure presented in the previous section.

A. Adaptive consensus using partial state measurements

We consider SISO systems given by the following parametrization

$$\dot{x}(t)=Ax(t)+B\theta\varphi(y,u),\qquad y(t)=Cx(t),\tag{15}$$

where $\varphi(y,u)$ is a known function of the input and output signals and θ represents the unknown parameter. Further it is assumed that the triple (A,B,C) satisfies Lur'e equation

$$A^TP+PA=-Q,\quad PB=C^T.\tag{16}$$

The proposed adaptive consensus distributed filters are given along the lines of (12)

$$\begin{aligned}\dot{\widehat{x}}_i(t)&=A\widehat{x}_i(t)+B\widehat{\theta}_i(t)\varphi(y,u)-P^{-1}\sum_{j\neq i}\widehat{x}_{ij}(t),\\ \dot{\widehat{\theta}}_i(t)&=-\gamma\varepsilon_i(t)\varphi(y,u)-\gamma_i\sum_{j\neq i}\widehat{\theta}_{ij}(t).\end{aligned}\tag{17}$$

The resulting ith error system is given by

$$\begin{aligned}\dot{e}_i(t)&=Ae_i(t)+B\widetilde{\theta}_i(t)\varphi(y,u)-P^{-1}\sum_{j\neq i}e_{ij}(t)\\ \varepsilon_i(t)&=C\widehat{x}_i(t)-y(t),\\ \dot{\widetilde{\theta}}_i(t)&=-\gamma\varepsilon_i(t)\varphi(y,u)-\gamma_i\sum_{j\neq i}\widetilde{\theta}_{ij}(t),\end{aligned}\tag{18}$$

To assess the stability of (18), the following Lyapunov candidate is used

$$V_i(t)=e_i^T(t)Pe_i(t)+\frac{1}{\gamma_i}\widetilde{\theta}_i^2(t),\ \ i=1,\ldots,N,\tag{19}$$

which produces

$$\dot{V}_i(t)=-e_i^T(t)Qe_i(t)-2e_i^T\sum_{j\neq i}e_{ij}(t)-2\widetilde{\theta}_i(t)\sum_{j\neq i}\widetilde{\theta}_{ij}(t),\tag{20}$$

via the use of (16) and (18). The collective dynamics are once again examined via

$$\dot{V}(t)=-\sum_{i=1}^{N}e_i^T(t)Qe_i(t)-\sum_{i=1}^{N}\sum_{j\neq i}^{N}\|e_{ij}(t)\|^2-\sum_{i=1}^{N}\sum_{j\neq i}^{N}|\widetilde{\theta}_{ij}(t)|^2$$

The convergence results are stated in the following lemma without proof, as most of the arguments are similar to the full-state case.

Lemma 3: Consider the SISO system whose nominal transfer function is strictly positive real, i.e. satisfies (16). Assume that the state x and the known function $\varphi(y,u)$ are bounded. Then the distributed adaptive consensus observers (17) ensure that all signals are bounded, the state estimation errors asymptotically converge to zero $\lim_{t\to\infty}\|e_i(t)\|=0$, $i=1,\ldots,N$, and the pairwise parameter disagreement errors converge to zero $\lim_{t\to\infty}\widetilde{\theta}_{ij}(t)=0$, $i,j=1\ldots,N$, with the deviation from their mean converging to zero

$$\lim_{t\to\infty}\left(\widehat{\theta}_i(t)-\frac{1}{n}\sum_{i=1}^{N}\widehat{\theta}_i(t)\right)=0.$$

V. Numerical Results

We consider the following second order system as the plant

$$\ddot{z}+7\dot{z}+4z=\theta\dot{z}+u,\quad y=\dot{z},$$

which can be placed in state space form

$$\dot{x}=Ax+B\theta y+Bu,\quad y=Cx,$$

with

$$A=\begin{bmatrix}0&1\\-4&-7\end{bmatrix},\ B=\begin{bmatrix}0\\1\end{bmatrix},\ x(t)=\begin{bmatrix}z(t)\\ \dot{z}(t)\end{bmatrix},\ C=B^T.$$

The plant parameters are assumed known and it is desired to adaptively estimate θ using available signals (i.e. u,y). The above parametrization assumes that there is an uncertainty in the damping parameter $7-\theta$, where only the nominal value of 7 is known. In this case, the condition of SPR is simplified since we have collocated input and output matrices. The nonlinear function $\varphi(y,u)=y$ and $P=I$ in (16) with $-Q=A+A^T$. In fact, due to the collocated input and output, one can have $-Q=A_k+A_k^T$ where $A_k=A-kBB^T$ and k is a static output feedback gain to be designed. We take $N=5$ and these distributed consensus observers are given by

$$\left.\begin{aligned}\dot{\widehat{x}}_i&=A_k\widehat{x}_i+kBy+Bu+B\widehat{\theta}_iy-\sum_{i=1}^{5}(\widehat{x}_i-\widehat{x}_j)\\ \dot{\widehat{\theta}}_i&=-\gamma_i(C\widehat{x}_i-y)y-\gamma_i\sum_{j\neq i}^{5}(\widehat{\theta}_i-\widehat{\theta}_j)\end{aligned}\right.\quad i=1,\ldots,5.$$

TABLE I

INITIAL CONDITIONS

variable	I.C.	variable	I.C.
$\widehat{z}_1(0)$	2.5	$z(0)$	2
$\dot{\widehat{z}}_1(0)$	4	$\dot{z}(0)$	4
$\widehat{z}_2(0)$	1	θ	1.5
$\dot{\widehat{z}}_2(0)$	5	$\dot{\widehat{z}}_5(0)$	0.5
$\widehat{z}_3(0)$	0.5	$\widehat{\theta}_1(0)$	0.5
$\dot{\widehat{z}}_3(0)$	2.5	$\widehat{\theta}_2(0)$	0.6
$\widehat{z}_4(0)$	0	$\widehat{\theta}_3(0)$	0.9
$\dot{\widehat{z}}_4(0)$	1.5	$\widehat{\theta}_4(0)$	3.2
$\widehat{z}_5(0)$	3	$\widehat{\theta}_5(0)$	2.3

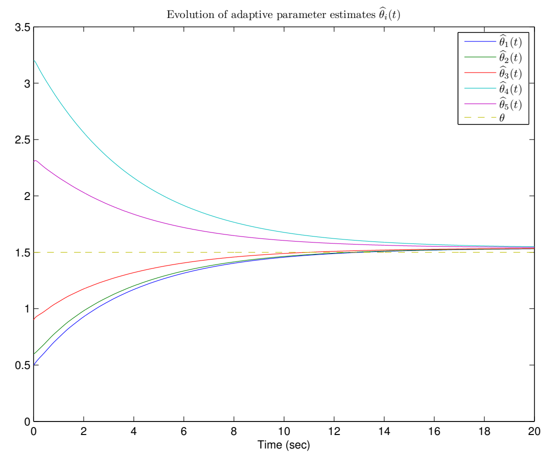

Fig. 1. Adaptive parameter estimates with consensus.

The plant along with the 5 adaptive consensus observers were simulated in the time interval $[0,20]$ with $\gamma_i = 0.05$, $k = 1$ and $u(t) = 5\sin(10t)$. The initial conditions for the state and parameter estimates are given in Table I.

Figures 1 and 2 depict the evolution of the 5 adaptive estimates $\widehat{\theta}_i(t)$ with and without consensus. It is observed that when penalty disagreement is enforced on the parameter estimates, they all converge to the same value, which in this case is the true value of the parameter θ. The same conclusion can be drawn from Figure 3, which depicts the state and parameter deviations from the mean estimates.

VI. CONCLUSIONS

This paper considered the adaptive estimation of a class of multi-agent systems with full connectivity. A modification in the standard adaptive law of the distributed adaptive observers was proposed which penalized the pairwise disagreement of the parameter adaptive estimates. This constituted the main contribution of this work which ensured that both state and parameter estimates reach consensus. Enforcing parameter consensus was shown to provide better parameter convergence properties, thus artificially yielding persistence of excitation. The numerical studies of a second order system in which the unknown parameter was related to the damping ratio, complemented the theoretical predictions of the agreement of the parameters generated by the multiple adaptive consensus estimators.

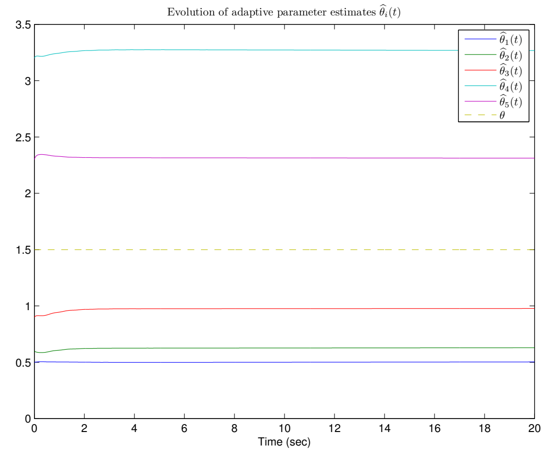

Fig. 2. Adaptive parameter estimates without consensus.

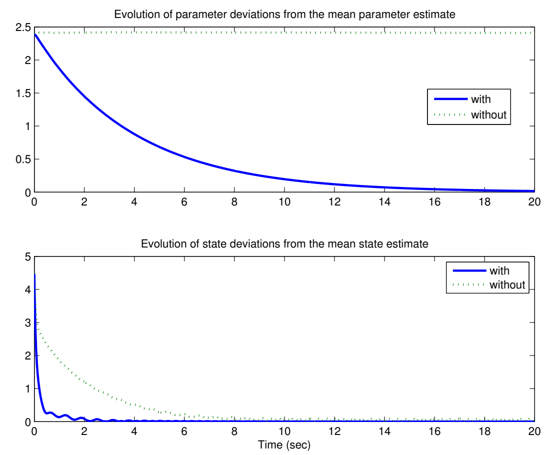

Fig. 3. State and parameter deviations from the mean estimates.

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