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Robust stability analysis of uncertain Linear Positive Systems via Integral Linear Constraints: L_1- and L_∞-gain characterizations[†]

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***Abstract*— Copositive Lyapunov functions are used along with dissipativity theory for stability analysis of uncertain linear positive systems. At the difference of standard results, linear supply-rates are employed for robustness and performance analysis and lead to L_1- and L_∞-gain characterizations. This naturally guides to the definition of Integral Linear Constraints (ILCs) for the characterization of input-output nonnegative uncertainties. It turns out that these integral linear constraints can be linked to the Laplace domain, in order to be tuned adequately, by exploiting the L_1-norm and input/output signals properties. This dual viewpoint allows to prove that the static-gain of the uncertainties, only, is critical for stability. This fact provides a new explanation for the surprising stability properties of linear positive time-delay systems. The obtained stability and performance analysis conditions are expressed in terms of (robust) linear programming problems that are transformed into finite dimensional ones using the Handelman's Theorem. Several examples are provided for illustration.**

***Index Terms*— Positive linear systems; Integral Linear Constraints; Robustness; Robust linear programming**

I. INTRODUCTION

Linear internally positive systems are a particular class of linear systems whose state takes only nonnegative values. Such models can represent many real world processes, from biology [1], passing through ecology and epidemiology [2] to networking [3]. This is because many physical systems involve quantities that are nonnegative in nature, it seems then natural to represent them by models involving nonnegative quantities as well. By extension, positive systems can also be generalized to (internally or not) input/ouput positive systems, meaning that for any positive input, the output is also positive.

Several results focusing on the (robust) stability analysis and the (robust) stabilization of linear positive systems have already been reported in the literature, see e.g. [4], [5], [6], [7], [8], [9]. The stability of such systems can be determined using quadratic Lyapunov functions $V(x) = x^T Px$, $P = P^T \succ 0$ as any linear system. But, surprisingly, the Lyapunov matrix P can losslessly be chosen as diagonal, this is referred to as *diagonal stability* [10]. There also exists another class of Lyapunov functions leading to necessary and sufficient condition, the so-called *linear copositive Lyapunov functions* $V(x) = \lambda^T x$ for some $\mathbb{R}^n \ni \lambda > 0$ where $' >'$ is componentwise [5], [6], [11], [12], [13], [14]. In such a case, the resulting problem takes the form of a linear programming problem (convex) which can be solved efficiently using solvers implementing the simplex algorithm, active sets methods or linear interior-point algorithms [15]. The Lyapunov function being linear, there is no more relationship with the vector 2- and the $\mathcal{L}_2$-norms as for a quadratic one. The connection with the vector 1- and L_1-norms is however evident[1].

In this paper, the stability analysis of uncertain positive systems is considered in the L_1- and the L_∞-norms using linear copositive Lyapunov functions and dissipativity theory [17], [5]. Stability analysis results for unperturbed systems are first provided to set up the ideas and introduce the important tools. It is then shown that the exact computation problems of L_1- and L_∞-gains for linear internally input/output positive systems amount to solve linear programming problems. The L_1-gain is computed via a direct application of the dissipativity theory while the L_∞-gain computation relies on the concept of adjoint system.

The robust stability analysis is based on Linear Fractional Transformations (LFTs), a classical tool of robust analysis [18], which seems to have been ignored in the context of positive systems. In this framework, the uncertain positive system is rewritten as the interconnection of a nominal positive system and an uncertain diagonal matrix of input/ouput positive operators. The uncertain operators are characterized by Integral Linear Constraints (ILCs) whose name is to put in contrast with the well known Integral Quadratic Constraints (IQCs). Although the provided framework does not enjoy the availability of the Plancherel Theorem, frequency domain analysis can still be used in order to select the scalings accurately. Finally, using dissipativity theory or, equivalently, a linear counterpart of the full-block S-procedure [19], robust stability analysis tools, formulated as robust linear programming problems, are provided. The problem is made tractable by applying the Handelman's Theorem [20] which yields linear programs involving a finite number of constraints. In order to reduce the computational complexity of the approach, an elimination procedure for some of the extra variables introduced by the Handelman's Theorem is considered.

The outline of the paper is as follows, Section II introduces the problem, fundamental definitions and results. Section III is devoted to the stability analysis of unperturbed systems. Section IV brings out Integral Linear Constraints. Results

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[1] This has also been noticed in the paper [16], also published in the same conference as this one.

on robust stability are detailed in Section V. Illustrative examples are treated in Section VI.

The notation are the following: $\mathbb{1}_n$ denotes a column vector with n rows containing only 1 entries. For general real matrices/vectors $A, B \in \mathbb{R}^{n\times m}$, the inequality $A < (\leq) B$ is componentwise. The vector 1-norm is denoted by $||x||_1 = \sum_{i=1}^{n} |x_i|$ while the vector ∞-norm is defined by $||x||_\infty = \max_{i\in\{1,\ldots,n\}} |x_i|$ for any vector $x \in \mathbb{R}^n$. Given $v : \mathbb{R} \to \mathbb{R}^n$, the L_1-norm $||v||_{L_1}$ and the L_∞-norm $||v||_{L_\infty}$ are defined by $||v||_{L_1} = \int_{\mathbb{R}} ||v(t)||_1 dt$ and $||v||_{L_\infty} = \text{ess sup}_{t\in\mathbb{R}} ||v(t)||_\infty$ respectively. We also define the sets $\mathbb{R}^n_{++} = \{\alpha \in \mathbb{R}^n : \alpha > 0\}$, $\mathbb{R}^n_+ := \{\alpha \in \mathbb{R}^n : \alpha \geq 0, ||\alpha|| \neq 0\}$ and $\bar{\mathbb{R}}^n_+ := \{\alpha \in \mathbb{R}^n : \alpha \geq 0\}$. A linear function $\ell(x) = c^T x$ is said to be copositive if $c^T x > 0$ for all $x \in \mathbb{R}_+$.

II. Preliminaries

Let us consider general LTI systems of the form:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Ew(t) \\ z(t) & = & Cx(t) + Fw(t) \\ x(0) & = & x_0 \end{array} \tag{1}$$

where $x, x_0 \in \mathbb{R}^n$, $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$ are respectively the system state, the initial condition, the exogenous input and the controlled output.

The following definitions and results will be useful in the sequel:

Definition 1: The autonomous version of system (1) is said to be internally positive if the matrix A is a Metzler matrix, i.e. all the off-diagonal elements are nonnegative.

Definition 2: A linear map $V(x) = \lambda^T x$ is said to be a linear copositive Lyapunov function for the internally positive system $\dot{x}(t) = Ax(t)$ if V is a linear copositive function and $\lambda^T Ax < 0$ for all $x \in \mathbb{R}^n_+$.

Definition 3: The system (1) is said to be internally input-output positive if for any $x_0 \in \bar{\mathbb{R}}^n_+$ and $w(t) \in \bar{\mathbb{R}}^p_+$, we have $x(t) \in \bar{\mathbb{R}}^n_+$ and $z(t) \in \bar{\mathbb{R}}^q_+$ for all $t \geq 0$.

Lemma 1: The system (1) is internally input-output positive if and only if the following statements hold:

1) the matrix A is Metzler,
2) the matrices E, C and F are nonnegative. △

Definition 4: The L_1-gain $||\Sigma_1||_{L_1-L_1}$ of the operator $\Sigma_1 : L_1^p \to L_1^q$ is defined as the smallest $\gamma_1 > 0$ such that $||\Sigma_1 w||_{L_1} \leq \gamma_1 ||w||_{L_1}$ holds for all $w \in L_1$. The explicit solution when the system is LTI is given by [21]:

$$||\Sigma_1||_{L_1-L_1} := \max_{j\in\{1,\ldots,q\}} \left\{ \sum_{i=1}^{p} \int_0^{+\infty} |h_{ij}(t)| dt \right\} \tag{2}$$

where $h_{ij}(t)$ is the impulse response from input j to output i.

Definition 5: Similarly, the L_∞-gain of the operator $\Sigma_\infty : L_\infty^p \to L_\infty^q$ is defined as the smallest $\gamma_\infty > 0$ such that $||\Sigma_\infty w||_{L_\infty} \leq \gamma_\infty ||w||_{L_\infty}$ holds for all $w \in L_\infty$. Moreover, when the system is LTI we have [21]:

$$||\Sigma_\infty||_{L_\infty-L_\infty} := \max_{i\in\{1,\ldots,p\}} \left\{ \sum_{j=1}^{q} \int_0^{+\infty} |h_{ij}(t)| dt \right\}. \tag{3}$$

The L_1-gain quantifies the gain of the *most influent input* since the max is taken over the columns. In contrast, the L_∞-gain of a system is the max taken over the rows and then characterizes the *most sensitive output*. Note that in the SISO, symmetric and certain sparse cases, the two norms coincide. Another important fact, needed later, is the correspondence of the L_1-induced and L_∞-induced norms using the notion of adjoint system:

Proposition 1: The L_∞-gain of a LTI finite dimensional system H is related to the L_1-gain of the adjoint system through the equality:

$$||H||_{L_\infty-L_\infty} = ||H^*||_{L_1-L_1} \tag{4}$$

where $H^*(s) = B^T(sI - A^T)^{-1}C^T + D^T$ is the adjoint system of $H(s) = C(sI - A)^{-1}B + D$.

Proof: The proof follows from the definitions of the adjoint and the norms. ■

Coming back to internally input-output positive systems, we have the following useful facts:

Fact 1. The adjoint system of an internally input-output positive system is also internally input-output positive.

Fact 2. The L_1-norm of a nonnegative function $v : \mathbb{R}_+ \to \mathbb{R}^n_+$ is given by $||v||_{L_1} = \int_0^{+\infty} \mathbb{1}_n^T v(t) dt$.

III. Nominal Stability and Performance Analysis

In this section, stability and performance analysis criteria for unperturbed systems are derived. That is, we tacitly assume that (1) is internally input/ouput positive.

Lemma 2: The system (1) is asymptotically stable if and only if there exist $\lambda \in \mathbb{R}^n_{++}$ and a scalar $\gamma > 0$ such that the following linear program

$$\begin{bmatrix} A^T\lambda + C^T\mathbb{1}_q \\ E^T\lambda - \gamma\mathbb{1}_p + F^T\mathbb{1}_q \end{bmatrix} < 0 \tag{5}$$

is feasible. Moreover, in such a case, the L_1-gain of the transfer $w \to z$ is lower than γ. △

Proof: The proof is based on dissipativity theory for nonnegative systems [17], [5] used along with a linear copositive storage function of the form $V(x) = \lambda^T x$, $\lambda \in \mathbb{R}^n_{++}$. ■

Lemma 3: The system (1) is asymptotically stable if and only if there exist $\lambda \in \mathbb{R}^n_{++}$ and a scalar $\gamma > 0$ such that the following linear program

$$\begin{bmatrix} A\lambda + E\mathbb{1}_p \\ C\lambda - \gamma\mathbb{1}_q + F\mathbb{1}_p \end{bmatrix} < 0 \tag{6}$$

is feasible. Moreover, in such a case, the L_∞-gain of the transfer $w \to z$ is lower than γ. △

Proof: The proof is based on Proposition 1 and Lemma 2. ■

It is important to stress that the above lemmas are necessary and sufficient conditions for stability and gain-computation. Therefore, in both results, by minimizing $\gamma > 0$ the exact norm can be computed.

IV. Handling uncertainties using ILCs and Laplace domain interpretation

In robust stability analysis theories for interconnected systems, the stability condition usually consists of two separate local conditions: one for each subsystem. The first one, generally very precise, is used to characterize the nominal system usually enjoying nice properties: e.g. linearity, time-invariance, etc. The second condition, often more difficult to derive when high precision is sought, is used to characterize the uncertain part. IQCs are very powerful objects able to accurately describe uncertain operators [22] through the consideration of their input/output signals. Unfortunately, they are based on quadratic functionals and do not fall into the current 'linear functionals-based framework'. Yet, we can find inspiration in this idea.

A. Integral Linear Constraints

Our framework is indeed more adapted to Integral Linear Constraints (ILCs) taking the form

$$\int_0^{+\infty} \varphi_1^T |\alpha(t)| + \varphi_2^T |\beta(t)| dt \geq 0 \tag{7}$$

where the functions $\alpha, \beta \in L_1$ verify $\beta = \Sigma\alpha$ for some given uncertain operator Σ and where the absolute value is componentwise. For positive signals $\alpha = z_0$, $\beta = w_0$ and a positive operator $\Sigma = \Sigma_0$, the above ILC reduces to

$$\Phi := \int_0^{+\infty} \varphi_1^T z_0(t) + \varphi_2^T w_0(t) dt \geq 0. \tag{8}$$

The operator Σ_0 may consist of any nonnegative operator: nonnegative time invariant/varying parameters, delay operators, nonnegative transfer functions, etc.

The efficiency of IQCs lies in the fact that, by virtue of the Plancherel Theorem, it is possible to convert the inequality to the frequency domain in which the tuning of the IQC kernel may be easier and/or more precise. Finally, by virtue of the Kalman-Yakubovich-Popov Lemma and the S-procedure, the frequency domain conditions are converted back to time-domain conditions, taking the form of tractable LMIs problems.

However, in L_1, a Plancherel Theorem does not exist and we cannot expect to switch from time to frequency domains in the same spirit as in L_2. We can, however, still link time and frequency domains together by noting that

$$\begin{array}{rcl} \Phi & = & \int_0^{+\infty} \left[\varphi_1^T z_0(t) + \varphi_2^T w_0(t)\right] e^{-st} dt\Big|_{s=0} \\ & = & \varphi_1^T \hat{z}_0(0) + \varphi_2^T \hat{w}_0(0) \end{array} \tag{9}$$

where hatted signals stand for the Laplace transform of the corresponding signals. So, we can see that ILCs characterize the static-gain of an operator, independently of its nature (i.e. time-invariant or time-varying). Hence the *static-gain only* is critical for stability in this context.

If the operator Σ_0 is linear and time invariant, the transfer function $\hat{\Sigma}_0(s)$ exists and the ILC rewrites

$$\left[\varphi_1^T + \varphi_2^T \hat{\Sigma}_0(0)\right] \hat{z}_0(0) \geq 0. \tag{10}$$

Note that since the signal $z_0(t)$ is positive, then so is $\hat{z}_0(0)$. It is hence enough to select nonzero φ_1 and φ_2 such that the left factor is always nonnegative. If $\hat{\Sigma}_0(0)$ is exactly known, then equality can be ensured by elimination, i.e. $\varphi_1^T = -\varphi_2^T \hat{\Sigma}_0(0)$. Otherwise the terms are tuned such that the inequality holds for all possible for $\hat{\Sigma}_0(0)$.

It is also important to note that in the time-invariant case, the problem always reduces to a problem with parametric uncertainty $\hat{\Sigma}_0(0)$. It turns out that this may be the case for time-varying uncertainties for which the 'static-gain' is assimilated to the worst-case (maximal) values.

B. Examples

Example 1: Let us consider the constant delay operator $\hat{\Sigma}_0(s) = e^{-sh}$, $h \geq 0$. In this case, the static-gain is $\hat{\Sigma}_0(0) = 1$ for all $h \geq 0$. Hence, it is enough to choose $\varphi_1^T = -\varphi_2^T$. This will be further developed in Section VI-B. $\diamond$

Example 2: Consider now an uncertain positive SISO transfer function $\hat{\Sigma}_0(s,\rho)$ depending on some uncertain parameters $\rho \in \mathscr{P} \subset \mathbb{R}^N$. Then, all that matters is the value of the static-gain of $\hat{\Sigma}_0(s,\rho)$ lying within $\mathcal{I} := \left\{\hat{\Sigma}_0(0,\rho) : \ \rho \in \mathscr{P}\right\}$. The problem hence essentially reduces to a problem with constant parametric uncertainties. $\diamond$

Example 3: Let us consider in this example the multiplication operator Σ_0 which multiplies the input signal by a bounded and time-varying positive parameter $\delta(t)$. Since the parameter is time-varying it is not possible to use the Laplace transform. However, we can write

$$\int_0^{+\infty} [\varphi_1^T + \varphi_2^T \delta(\theta)] z_0(\theta) d\theta \tag{11}$$

which has to be nonnegative. Since $z_0(\cdot) \geq 0$, then a sufficient condition is given by $\varphi_1^T + \varphi_2^T \delta(t) \geq 0$ for all $t \geq 0$. This condition turns out to be also necessary when no restrictions are made on the trajectories of $\delta(t) \geq 0$ and $z_0(t) \geq 0$. $\diamond$

C. Parameter Dependent Scalings

When the problem can be formulated as a problem with constant parametric uncertainties (Examples 1 and 2) or as a time-varying parametric uncertainty (Example 3), it may be interesting to make the scalings φ_1 and φ_2 parameter dependent. Such a choice can be useful in order to saturate the inequality constraint over the entire parameter domain. A simple constraint saturation can be obtained by choosing $\varphi_1(\delta) = -\Delta(\delta)^T \varphi_2(\delta)$, where $\Delta(\delta)$ is the uncertain parametric operator with parametric uncertainties δ. Since the matrix $\Delta(\delta)$ can be considered w.l.o.g. as linear in the parameters, then it turns out that $\varphi_1(\delta)$ and $\varphi_2(\delta)$ can be chosen as polynomials. If the polynomials are adequately chosen, the obtained results will be non conservative. This approach leads however to robust linear optimization problems. An approach based on the Handelman's Theorem [20], parameterizing positive polynomials over polyhedra, will be used to convert the optimization problem into a tractable finite-dimensional one. One of the feature of the

Handelman theorem is the preservation of the linearity of the optimization program. Sum-of-Squares [23] could have also been used but would have resulted in semidefinite programs.

V. Robust stability and performance analysis

In this section, robust stability analysis results will be derived for the following uncertain linear positive system subject to $N \in \mathbb{N}$ distinct parametric uncertainties $\delta \in \boldsymbol{\delta} := [0,1]^N$:

$$\begin{array}{rcl} \dot{x}(t) & = & A_\delta(\delta)x(t) + E_\delta(\delta)w_1(t) \\ z_1(t) & = & C_\delta(\delta)x(t) + F_\delta(\delta)w_1(t) \\ x(0) & = & x_0 \end{array} \qquad (12)$$

where $x, x_0 \in \bar{\mathbb{R}}_+^n$, $w_1 \in \bar{\mathbb{R}}_+^p$ and $z_1 \in \bar{\mathbb{R}}_+^q$ are the system state, the initial condition, the exogenous input and the performance output respectively.

We assume in this section that for all $\delta \in \boldsymbol{\delta}$, the matrix $A_\delta(\delta)$ is Metzler and that $E_\delta(\delta), C_\delta(\delta)$ and $F_\delta(\delta)$ are nonnegative. For exposure, we will derive in this section results on uncertain systems with constant parametric uncertainties. The methodology straightforwardly generalizes to any type of positive operators: delays, uncertain/neglected dynamics, nonlinearities, etc. The reason for this stems from the fact that it is possible in many cases to bring back the stability analysis problem to a problem with parametric uncertainties, as illustrated in Section IV-B.

A. Linear Fractional Transformations (LFTs)

Using LFT, the system (12) is rewritten as

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + E_0 w_0(t) + E_1 w_1(t) \\ z_0(t) & = & C_0 x(t) + F_{00} w_0(t) + F_{01} w_1(t) \\ z_1(t) & = & C_1 x(t) + F_{10} w_0(t) + F_{11} w_1(t) \\ w_0(t) & = & \Delta(\delta) z_0(t) \end{array} \qquad (13)$$

where the virtual robustness channel $z_0, w_0 \in \bar{\mathbb{R}}_+^{n_0}$ has been added. Since the original system is positive, it is always possible to choose positive matrices for the output z_0 in order to make z_0 positive (e.g. identity matrices). The matrices with negative entries can be placed at the input, acting on w_0. The uncertainty matrix $\Delta(\delta)$ can be w.l.o.g. chosen to belong to the set $\boldsymbol{\Delta}$ defined by:

$$\boldsymbol{\Delta} := \left\{ \operatorname*{diag}_{i=1}^{N}[\delta_i I_{\ell_i}] : \ \delta \in \boldsymbol{\delta} \right\} \qquad (14)$$

where $\ell_i \in \mathbb{N}$ is the number of occurrence of the i^{th} parameter in the matrix $\Delta(\delta)$. Since $\Delta(\delta) \geq 0$ for all $\delta \in \boldsymbol{\delta}$, the uncertainty matrix is also an input/output positive operator.

The LFR of the adjoint system of (12) is given by

$$\begin{array}{rcl} \dot{x}(t) & = & A^T x(t) + \tilde{E}_0 w_0(t) + C_1^T w_1(t) \\ z_0(t) & = & \bar{C}_0 x(t) + \bar{F}_{00} w_0(t) + \bar{F}_{01} w_1(t) \\ z_1(t) & = & E_1^T x(t) + \tilde{F}_{10} w_0(t) + F_{11}^T w_1(t) \\ w_0(t) & = & \Delta(\delta) z_0(t) \end{array} \qquad (15)$$

where the matrices $\tilde{F}_{10}$ and $\tilde{E}_0$ are specific matrices of the adjoint system. All the other matrices are those of systems (12) and (13).

Remark 1: It must be stressed here that the Linear Fractional Transformation operation does not commute with the adjoint transformation. In other words, the adjoint of the LFR does not coincide in general with the LFR of the adjoint system. Indeed, some matrices may remain unchanged (non transposed) while some can be really different. This has motivated the use of the 'bar' and 'tilde' notation to differ possibly unchanged matrices from completely different matrices respectively. For instance, when the system depends polynomially on the parameters, it is then possible to have $\bar{F}_{00} = F_{00}, \bar{F}_{01} = F_{01}$ and $\bar{C}_0 = C_0$: they are not transposed.

B. Main Results

Theorem 1: The uncertain linear positive system (12) is asymptotically stable if there exist $\lambda \in \mathbb{R}_{++}^n$, $\varphi_1(\delta), \varphi_2(\delta) \in \mathbb{R}^{n_0}$ and $\gamma > 0$ such that the robust linear program

$$\begin{array}{rcl} \lambda^T A + \varphi_1(\delta)^T C_0 + \mathbb{1}_q^T C_1 & < & 0 \\ \lambda^T E_0 + \varphi_2(\delta)^T + \varphi_1(\delta)^T F_{00} + \mathbb{1}_q^T F_{10} & < & 0 \\ \lambda^T E_1 - \gamma \mathbb{1}_p^T + \varphi_1(\delta)^T F_{01} + \mathbb{1}_q^T F_{11} & < & 0 \end{array} \qquad (16)$$

$$\varphi_1(\delta)^T + \varphi_2(\delta)^T \Delta(\delta) \geq 0 \qquad (17)$$

is feasible for all $\delta \in \boldsymbol{\delta}$. Moreover, in such a case, the L_1-gain of the transfer from $w_1 \to z_1$ is bounded from above by γ. △

Proof: The proof relies on dissipativity theory. Define the storage function to be $V(x) = \lambda^T x$ and the supply-rate $s(w,z)$ as

$$\varphi_1(\delta)^T z_0(t) + \varphi_2(\delta)^T w_0(t) - \gamma \mathbb{1}_p^T w_1(t) + \mathbb{1}_q^T z_1(t) \qquad (18)$$

where $z = \mathrm{col}(z_0, z_1)$ and $w = \mathrm{col}(w_0, w_1)$. Consider now the functional $\mathcal{H}(x,w,z) = V(x(t)) + \int_0^t s(w(\theta), z(\theta))d\theta$ whose derivative along the trajectories of the system (13) is given by:

$$\begin{bmatrix} (\lambda^T A + \varphi_1(\delta)^T C_0 + \mathbb{1}_q^T C_1)^T \\ (\lambda^T E_0 + \varphi_2(\delta)^T + \varphi_1(\delta)^T F_{00} + \mathbb{1}_q^T F_{10})^T \\ (\lambda^T E_1 - \gamma \mathbb{1}_p^T + \varphi_1(\delta)^T F_{01} + \mathbb{1}_q^T F_{11})^T \end{bmatrix}^T \begin{bmatrix} x(t) \\ w_0(t) \\ w_1(t) \end{bmatrix}$$

and is negative on $\mathbb{R}_+^{n+n_0+p}$ if and only if (16) holds. The nonnegativity of the ILC for robustness is guaranteed by (17). ■

Theorem 2: The uncertain linear positive system (12) is asymptotically stable if there exist $\lambda \in \mathbb{R}_{++}^n$, $\varphi_1(\delta), \varphi_2(\delta) \in \mathbb{R}^{n_0}$ and a scalar $\gamma > 0$ such that the componentwise inequalities:

$$\begin{array}{rcl} A\lambda + \bar{C}_0 \varphi_1(\delta) + E_1 \mathbb{1}_p & < & 0 \\ \tilde{E}_0^T \lambda + \varphi_2(\delta) + \bar{F}_{00}^T \varphi_1(\delta) + \tilde{F}_{10}^T \mathbb{1}_p & < & 0 \\ C_1 \lambda - \gamma \mathbb{1}_q + \bar{F}_{01} \varphi_1(\delta) + F_{11} \mathbb{1}_p & < & 0 \end{array} \qquad (19)$$

$$\varphi_1(\delta)^T + \varphi_2(\delta)^T \Delta(\delta) \geq 0, \ \delta \in \boldsymbol{\delta} \qquad (20)$$

is feasible for all $\delta \in \boldsymbol{\delta}$. Moreover, the L_∞-gain of the transfer from $w_1 \to z_1$ is bounded from above by γ. △

Proof: The proof is based on the use the Linear Fractional Representation of the adjoint system (15). ■

C. Solving Robust Linear Programs with Complexity Reduction

We propose in this section, a resolution scheme based on Handelman's Theorem [20] recalled below for completeness:

Theorem 3 (Handelman's Theorem): If $\mathcal{S}$ is a compact polytope in the Euclidean N-space, defined by linear inequalities $g_i(\cdot) \geq 0$, and if P is a polynomial in N variables that is (negative) positive on $\mathcal{S}$, then P can be expressed as a linear combination with nonnegative (nonpositive) coefficients of products of members of $\{g_i\}$. $\triangle$

When dealing with univariate polynomials, finding such a representation is somehow easy. However, when multivariate polynomials are considered, we face the problem of determining the number and order of products of linear basis functions. A bound on the necessary order has been provided in [24] and generalizes immediately to matrix polynomials.

The Handelman's Theorem introduces additional variables and supplementary linear equality constraints increasing then the computational complexity but still preserving a tractable structure to the problem. When the considered problem is large, it is interesting to reduce the computational burden by a suitable preprocessing of the problem. Solving equality constraints first, indeed allows to reduce the number of decision variables.

VI. Examples

A. Computation of Norms

In this example, many linear positive systems have been randomly generated and their induced-norms computed on a laptop equipped with an Intel U7300 processor of 1.3GHz with 4GB of RAM. The Yalmip interface [25] has been used with the solver LINPROG. The mean computation time and the standard deviation for different systems are gathered in Table I. The number of variables is $n+1$ for both problems while the number of constraints is $2n+p+1$ and $2n+1+q$ for the L_1-gain and the L_∞-gain respectively. We can see that, roughly speaking, both norms take the same computation time. Exactness of the computed values can be checked using the explicit formulas of Definitions 4 and 5. It is also important to mention that the approach based on linear programming is faster than a direct norm computation from the definitions.

B. Theoretical Robustness Analysis

This example aims at illustrating that the provided robustness analysis tool based on LFT and ILC may be intrinsically nonconservative. To this aim, let us consider the positive system with constant time-delay:

$$\dot{x}(t) = Ax(t) + A_h x(t-h) \tag{21}$$

for some $h \geq 0$. It is well known that such a system is internally positive if and only if the matrix A is Metzler and the matrix A_h is nonnegative. The corresponding LFR is

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + A_h w_0(t) \\ z_0(t) &=& x(t) \\ w_0(t) &=& \nabla_c(z_0)(t) \end{array} \tag{22}$$

where ∇_c is the constant delay operator with transfer function $\widehat{\nabla}_c(s) = e^{-sh}$. According to the discussion of Section IV-B and Theorem 1, the system is asymptotically stable if there exist $\lambda \in \mathbb{R}^n_{++}$ and $\varphi_1, \varphi_2 \in \mathbb{R}^n$ such that the following conditions are satisfied

$$\begin{array}{rcl} \lambda^T A + \varphi_1^T &<& 0 \\ \lambda^T A_h + \varphi_2^T &<& 0 \\ \varphi_1^T + \varphi_2^T &=& 0 \end{array} \tag{23}$$

where the two first inequalities are obtained from Theorem 1 and the last one is deduced from the saturation of (10). This yields the conditions $\lambda^T A + \varphi_1^T < 0$ and $\lambda^T A_h - \varphi_1^T < 0$ which are in turn equivalent to $\lambda^T(A+A_h) < 0$, known to be a necessary and sufficient condition for asymptotic stability of internally positive systems with constant time-delays [26].

C. Numerical Robustness Analysis

Let us consider the uncertain system with constant parametric uncertainty $\delta \in [0,1]$:

$$\begin{array}{rcl} \dot{x}(t) &=& \left(\sum_{i=0}^{2} A_i \delta^i\right) x(t) + \left(\sum_{i=0}^{2} E_i \delta^i\right) w_1(t) \\ z_1(t) &=& \left(\sum_{i=0}^{2} C_i \delta^i\right) x(t) + \left(\sum_{i=0}^{2} F_i \delta^i\right) w_1(t) \end{array} \tag{24}$$

with the matrices

$$\begin{array}{rclrcl} A_0 &=& \begin{bmatrix} -10 & 2 & 4 \\ 3 & -8 & 1 \\ 2 & 1 & -5 \end{bmatrix}, & A_1 &=& \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ -1 & 2 & -1 \end{bmatrix}, \\ A_2 &=& \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, & E_0 &=& \begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}, \\ E_1 &=& \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}^T, & E_2 &=& \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}^T, \\ C_0 &=& \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}, & C_1 &=& \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \\ C_2 &=& \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 1 \end{bmatrix}, & F_0 &=& \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \\ F_1 &=& \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, & F_2 &=& \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}. \end{array} \tag{25}$$

Rewriting the system into the forms (13) and (15) allows us to apply the results of the paper. Using Theorems 1 and 2 (with equality constraints elimination) and different structures for the scalings φ_1 and φ_2, we obtain the results gathered in Tables II and III. We can see that the L_1-gain is not very well estimated compared to the L_∞-gain for parameter independent scalings. Using scalings of degree two considerably reduces the conservatism of the approach. In such a case, we are indeed able to estimate exactly the L_∞-gain while few conservatism persists for the L_1-gain. The computation of the L_∞-gain is also much faster. Finally, it seems important to mention that, by using the same reasoning as in Example 3, the above numerical results also hold when the uncertainty is time-varying.

VII. Conclusion

A framework based on LFR and ILCs has been developed to deal with robust stability analysis of positive systems. The

nb. of systems	(n, p, q)	L_1-gain	L_∞-gain
20	(300,100,150)	$\mu = 12.282$, $\sigma = 1.1406$	$\mu = 14.186$, $\sigma = 1.4151$
100	$(50, 20, 30)$	$\mu = 0.53973$, $\sigma = 0.27486$	$\mu = 0.50735$, $\sigma = 0.080446$

TABLE I

MEAN COMPUTATION TIME μ AND STANDARD DEVIATION σ FOR GAIN COMPUTATION

$\varphi_1(\delta)$	$\varphi_2(\delta)$	constraints	computed L_1-gain	time
φ_1^0	φ_2^0	$\varphi_1^0 \geq 0$, $\varphi_1^0 + \varphi_2^0 \geq 0$	133.95	2.7844s
$\varphi_1^1\delta$	φ_2^0	$\varphi_1^1 = -\varphi_2^0$	133.95	3.829s
$\varphi_1^1\delta + \varphi_1^2\delta^2$	$\varphi_2^0 + \varphi_2^1\delta$	$\varphi_1^1 = -\varphi_2^0$, $\varphi_1^2 = -\varphi_2^1$	94.167	4.2758s

TABLE II

L_1-GAIN COMPUTATION OF THE TRANSFER $w_1 \to z_1$ OF SYSTEM (24) USING THEOREM 1 – EXACT L_1-GAIN: 92.8358

$\varphi_1(\delta)$	$\varphi_2(\delta)$	constraints	computed L_∞-gain	time
φ_1^0	φ_2^0	$\varphi_1^0 \geq 0$, $\varphi_1^0 + \varphi_2^0 \geq 0$	86.195	0.68989s
$\varphi_1^1\delta$	φ_2^0	$\varphi_1^1 = -\varphi_2^0$	86.195	1.4629s
$\varphi_1^1\delta + \varphi_1^2\delta^2$	$\varphi_2^0 + \varphi_2^1\delta$	$\varphi_1^1 = -\varphi_2^0$, $\varphi_1^2 = -\varphi_2^1$	82.025	1.7509s

TABLE III

L_∞-GAIN COMPUTATION OF THE TRANSFER $w_1 \to z_1$ OF SYSTEM (24) USING THEOREM 2 – EXACT L_∞-GAIN: 82.0249

considered techniques allow for the exact characterization of L_1- and L_∞-induced norms. It has been proved that only the static gain of uncertainty operators is critical for stability in this context. Examples have emphasized the potential of the approach.

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