2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Bounded control based on saturation functions of nonlinear under-actuated mechanical systems : the cart-pendulum system case

Carlos Aguilar-Ibañez, Juan C. Martínez-García, Alberto Soria-López

***Abstract*— We are concerned in this paper by bounded control of nonlinear underactuated dynamical systems. We focus our exposition on a feedback-based stabilization bounded control action shaped by saturation functions. A simple stabilizing controller for the well-known cart-pendulum system is then designed in this paper. Our control strategy describes in lumped linear time-invariant terms the concerned underactuated nonlinear system as a cascade nonlinear dynamical system consisted of a simple chain of four integrators with a high-order smooth nonlinear perturbation, and assumes initialization of the resulting underactuated system in the upper-half plane. Our proposed feedback-based regulation design procedure involves the simultaneous combination of two control actions: one bounded linear and one bounded quasilinear. Control boundedness is provided in both involved control actions by specific saturation functions. The first bounded control action brings the non-actuated coordinate near to the upright position and keep it inside of a well-characterized small vicinity, whereas the second bounded control action asymptotically brings the whole state of the dynamical system to the origin. The necessary closed-loop stability analysis uses standard linear stability arguments as well as the traditional well-known Lyapunov method and the LaSalle's theorem. Our proposed control law ensures global stability of the closed-loop system in the upper half plane, while avoiding the necessity of solving either partial differential equations, nonlinear differential equations or fixed-point controllers. We illustrate the effectiveness of the proposed control strategy via numerical simulations.**

Keywords: Underactuated Nonlinear Mechanical Systems, Cascade Interconnected Systems, Nonlinear Feedback-Based Bounded Control, Global Stabilization, Saturation Functions, Cart-Pendulum System. .

I. INTRODUCTION

Underactuated nonlinear dynamical mechanical systems offer an interesting challenge when considering control issues. Nonlinear feedback-based stabilization of this class of dynamical systems is by no means a trivial problem. Because of its control-related challenging features, the well-known cart-pendulum system have been extensively studied in recent times by the control community (see for instance [1] and the references therein). This control benchmark consists of a free vertical rotating pendulum with a pivot point mounted on a cart horizontally moved by a horizontal force (which corresponds to the system input). The control problem ask for the pendulum to be swinged up from its stable hanging position in order to maintain it in its unstable upright position. What makes this simple mechanical system an interesting control benchmark is the fact that the pendulum angular acceleration cannot be controlled, *i.e.* the cart-pendulum system is a *two degrees-of-freedom* mechanical underactuated system. Hence, many common stabilizing control techniques developed for fully-actuated systems (mainly robot manipulators) cannot be directly applied to this particular system. It must be pointed out that the cart-pendulum system is not input-output (statically or dynamically) feedback linearizable (see for instance [2]). Moreover, the cart-pendulum system loses controllability and other control-related geometric properties when the pendulum moves through the horizontal plane (see [1]). However, since the system is locally controllable around the unstable equilibrium point, closed-loop stabilization by linear pole-placement can be used there (see for instance [3]). Being more specific, stabilizing the cart-pendulum system involves two main aspects: i) swinging up the pendulum from the stable hanging position to the unstable upright vertical position (see for instance [4]); ii) stabilizing the closed-loop system around the open-loop unstable equilibrium point. For this second aspect it is commonly assumed that the free endpoint of the pendulum is initially located above the horizontal plane, or lies inside a well-characterized open vicinity of zero (the vicinity defines the closed-loop stability domain). We focus our attention on the former more challenging problem. Let us now review some remarkable works on this aspect. In [5] a nonlinear controller, based on the backstepping procedure, is used to solve the stabilization problem in the unstable equilibrium point; the proposed controller ensures full state convergence. A controller based on nested saturation functions is proposed in [6]. A similar work is discussed in [7], where a chain of integrators is considered as a model for the cart-pendulum system. In [4] a stabilization technique using switching and saturation functions (in addition to the Lyapunov method) is introduced. A control strategy based on controlled Lagrangians is presented in [8]. A feedback control scheme based on matching conditions is described in [9], while a simple matching condition is used in [10] to solve the cart-pendulum regulation problem. A very interesting nonlinear control strategy based on energy shaping techniques combined with input-to-state stability methods is presented in [11]. A solution which exploits power-based passivity properties of the cart-pendulum system is proposed in [12]. A nonlinear controller based on both the fixed-point backstepping procedure and saturation functions is

Carlos Aguilar-Ibañez is with CIC-IPN, Av. Juan de Dios Bátiz s/n casi esq. Miguel Othón de Mendizábal, Unidad Profesional Adolfo López Mateos, Col. Nueva Industrial Vallejo, Delegación Gustavo A. Madero, C.P 07738, México D.F., México caguilar@cic.ipn.mx

Juan C. Martínez-García (Corresponding Author) and Alberto Soria-López are with the Departamento de Control Automático, CINVESTAV-IPN, A.P. 14-740, 07000 México, D.F., México juancarlos_martinez-garcia@conciliencia.org, betosoria@gmail.com

proposed in [13]. This list of published works is by no means exhaustive. Let us conclude it by mentioning the work published in [14], where the challenging nature of the cart-pendulum problem is underscored to the nonlinear control community. We proceed now to speak about our bounded control approach. As far as the description of the system is concerned, we consider the nonlinear model discussed in [13], assuming that the pendulum is initialized over the upper half plane. We propose then a very simple control strategy, based on the traditional Lyapunov method and the LaSalle's theorem. Based on a description of the concerned system as a cascaded interconnected nonlinear system, our approach splits the control scheme in two bounded control actions: the first control action consists of a bounded linear controller, whereas the second control action consists of a bounded quasilinear control law. In dynamical terms, the bounded linear control action confines both the angular position and the angular velocity in a small compact set defining the closed-loop stability domain. As far as the bounded quasilinear control action is concerned, it guarantees the full state convergence of the closed-loop system. We must point out that our solution avoids the necessity of solving partial differential equations, nonlinear differential equations or fixed point control equations. The paper is organized as follows: Section II concerns modeling issues as well as the problem statement, namely the *cart-pendulum regulation problem*. We present our proposal in Section III, which we illustrate with a simulated control scheme. We conclude with some final remarks in Section IV.

II. PROBLEM STATEMENT

In this work we consider cascade interconnected nonlinear dynamical systems expressed in state-based terms as follows:

$$\dot{z}_1 = z_2; \dot{z}_2 = z_3 + \Delta(z_3, z_4); \dot{z}_3 = z_4; \dot{z}_4 = u, \qquad (1)$$

where $\Delta(z_3, z_4)$:$\mathbb{R}^2 \to \mathbb{R}$ is a smooth nonlinear function, bounded as $|\Delta(z)| \leq \kappa_0 z_4^2$, with k_0 being a real constant parameter. A system described in this simple way can be seen as a chain of four integrators with a high-order perturbation.

Remark 1: Notice that the cascade interconnected nonlinear system (1) corresponds in fact to a nonlinearly perturbated linear system, what allows linear control actions to be included in feedback-based control schemes intended to regulate the behavior of the system.

We must point out that many underactuated nonlinear dynamical systems can be described as in (1). That is the case in [13], where this description is chosen in order to solve the stabilization problem of both the Furuta pendulum and the cart-pole system. A novel version of the backstepping technique, combined with a fixed point controller is proposed in [13] to stabilize these two nonlinear dynamical systems. The corresponding stability analysis is based on the remarkable *convergence property*, proposed in [15], satisfied by some cascaded interconnected systems. We must remark that some other works related to the study of the stabilization of this kind of feed-forward systems were introduced before [13]. For instance, a nested saturation control technique is introduced in [16]. Here we solve the regulation problem of the cart-pendulum system using the configuration (1). In what follows we show how our system can be expressed as a perturbed chain of four integrators. We finish this section introducing the following useful definitions concerning both linear saturation functions and nonlinear saturation functions:

Definition 1: Let $x \in \mathbb{R}$. The classical linear saturation function is defined as:

$$\sigma_m(x) = \left\{ x \; if \quad |x| \leq m; \; m\frac{x}{|x|}, \; if \quad |x| > m, \right\} \qquad (2)$$

for a fixed given bound $m \in \mathbb{R}$.

Definition 2: By a sigmoidal function $s_m(x)$, we mean a smooth function that is bounded, strictly increasing with the property that $s_m(0) = 0$, $xs_m(x) \geq 0$, and $|s_m(x)| \leq m$, for all $x \in \mathbb{R}$.

We proceed now to show how the cart-pendulum system can be described as a cascaded interconnected nonlinear systems consisted of a nonlinearly perturbed chain of four integrators.

A. System dynamics

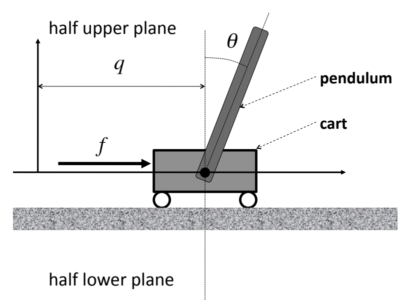

Fig. 1. Cart-pendulum system.

Consider the well-known cart-pendulum system (see Figure 1), described by the following set of normalized differential equations (see for instance [3]):

$$\cos\theta\ddot{q} + \ddot{\theta} - \sin\theta = 0, \; (1+\delta)\,\ddot{q} + \cos\theta\ddot{\theta} + \dot{\theta}^2\sin\theta = f, \qquad (3)$$

where: q is the normalized displacement of the cart; θ is the actual angle that the pendulum forms with the vertical; f is the horizontal normalized force applied to the cart (*i.e.* the system input), and $\delta > 0$ is a real constant that depends directly on both, the cart and the pendulum masses. In the non-forced case corresponding to $f = 0$ and $\theta \in (-\pi/2, \pi/2)$ the above system has only one unstable equilibrium point given by $\overline{x} = (\theta = 0, \dot{\theta} = 0, q = \overline{q}, \dot{q} = 0)$; with $\overline{q}$ being constant. Some simple algebra allows us to derive a new control variable u:

$$\ddot{q} = \frac{1}{\delta + \sin^2\theta}\left(f - \dot{\theta}^2\sin\theta - \cos\theta\sin\theta\right) \stackrel{\triangle}{=} u. \qquad (4)$$

Thus, system (3) can be written in a very simple way as:

$$\ddot{\theta} = \sin\theta - \cos\theta u, \; \ddot{q} = u. \qquad (5)$$

Now, we proceed to express system (5) as if it were a four-order chain of integrators plus an additional nonlinear perturbation. For this end we define the new coordinates:

$$z_1 = q + 2\tanh^{-1}(\tan\frac{\theta}{2}),$$
$$z_2 = \dot{q} + \dot{\theta}\sec\theta, z_3 = \tan\theta, z_4 = \dot{\theta}\sec^2\theta. \tag{6}$$

Then, system (5) can be written as:

$$\dot{z}_1 = z_2,\ \dot{z}_2 = z_3 + \alpha(z_3)z_4^2,\ \dot{z}_3 = z_4,\ \dot{z}_4 = v, \tag{7}$$

where the term $\alpha(z_3)$ is given by $\alpha(z_3) = \frac{z_3}{(1+z_3^2)^{\frac{3}{2}}}$, and v is now the new control variable defined, as:

$$v \triangleq \sec^2\theta\,(-u\cos\theta + \sin\theta) + 2\dot{\theta}^2\sec^2\theta\tan\theta. \tag{8}$$

Remark 2: Notice that the above set of transformations are well defined for all $-\pi/2 < \theta < \pi/2$. That is, the cart-pendulum model (5) works well for all the states belonging to the upper half plane. We shall then assume in what follows that the system is initialized on the upper half plane. On the other hand, we must remark that $|\alpha(z)| \leq \kappa_0 = 2/3^{1.5}$. We can say then that (7) approximates (5) in the upper half plane, *i.e.* we describe the motion of the system only in the region that concerns our control purposes. The cascaded interconnected system (7) corresponds to a *control-on-the-upper-half-plane* reduced model.

We can now formulate our control problem.

Problem formulation: *Given a cart-pendulum system described as in (7), bring the pendulum to the upright position and, simultaneously, bring the cart to the origin or any other fixed desired position.*

Remark 3: Note that this problem formulation assumes that the pendulum angle position is initialized over the horizontal plane.

We proceed now to propose our bounded control strategy.

III. Regulation of the cart-pendulum system

A. Linear transformation

Inspired in what is presented in [16], we first introduce the following linear transformation:

$$x \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix},$$

which leads to when applied to (7):

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3 + x_4 + 3\alpha(z_3)q_4^2 + v, \\ \dot{x}_2 &= x_3 + x_4 + \alpha(z_3)x_4^2 + v, \\ \dot{x}_3 &= x_4 + v, \\ \dot{x}_4 &= v. \end{aligned} \tag{9}$$

In the following subsection we split the new control input v into two control actions. One part of this control, namely v_1, brings both the state x_3 and the state x_4 to a small compact set defining the closed-loop stability domain and, consequently, renders the nonlinear terms of system (9) to an arbitrarily small vicinity of zero. Acting simultaneously with the linear control action, a bounded quasilinear control action, namely v_2, stabilizes the missing states x_1 and x_2.

B. Stabilization of the states x_3 and x_4

In order to guarantee that the states x_3 and x_4 are bounded, we propose v to be splitted as:

$$v = \overbrace{-x_3 - x_4}^{v_1} + v_2, \tag{10}$$

where $|v_2| \leq \varepsilon$; with $\varepsilon > 0$. Thus, after substituting (10) into (9), we have that:

$$\begin{aligned} \dot{x}_1 &= x_2 + 3\alpha(z_3)x_4^2 + v_2, \\ \dot{x}_2 &= \alpha(z_3)x_4^2 + v_2, \\ \dot{x}_3 &= -x_3 + v_2, \\ \dot{x}_4 &= -x_3 - x_4 + v_2. \end{aligned} \tag{11}$$

We emphasize that, if $|v_2| \leq \varepsilon$ (with ε small enough), then the states x_3 and x_4 converge toward the vicinity $B(x_{34}) \leq \delta_\varepsilon$[1]; where the bound δ_ε can be made small by minimizing ε and, consequently, all the nonlinear terms in (11) can be arbitrarily approximated to zero. This results in the dominancy of the linear dynamics over their respectively nonlinear dynamics. That is, signal v_2 will be then selected as a bounding function, where ε is given latter in our discussion. In order to analyze the boundedness of both the state x_3 and the state x_4, we consider the definite positive function given by:

$$V_1(x_4, x_3) = \frac{1}{2}(x_3 - x_4)^2 + \frac{1}{2}x_4^2. \tag{12}$$

Differentiating (12) and taking into account (11) we have that:

$$\dot{V}_1(x_4, x_3) = -2x_4^2 + x_4 v_2. \tag{13}$$

Now, given the assumption $|v_2| \leq \varepsilon$ we have that $\dot{V}_1$ is in fact bounded as follows:

$$\dot{V}_1(x_4, x_3) \leq -|x_4|\left(-\varepsilon + 2|x_4|\right). \tag{14}$$

Note that if $|x_4| > \varepsilon/2$, then, from (14), we have that $\dot{V}_1 < 0$. Consequently, after a finite time T_1 we have:

$$|x_4(t)| < \varepsilon/2;\ \forall\ t > T_1. \tag{15}$$

In a similar fashion, there exists a finite time $T_2 > T_1$ such that:

$$|x_3(t)| < \varepsilon;\ \forall\ t > T_2. \tag{16}$$

It is important to remark that the right-hand side of (11) is globally Lipschitz, then the states x_1 and x_2 remain bounded in a finite time. Therefore, there does not exist a finite time of escape (see [17]).

Remark 4: From (13) we have that the following inequality holds:

$$\dot{V}_1(x_4, x_3) < \frac{3}{2}x_4^2 + \frac{v_2^2}{2}. \tag{17a}$$

[1] For simplicity $B(x_{34}) = \sqrt{x_3^2 + x_4^2}$.

C. Stabilization of both the state x_2 and the state x_1

In order to stabilize the missing state variables we propose the bounding control action v_2 as follows:

$$v_2 = -\sigma_m(x_2) - k_i\sigma_m(x_1), \tag{18}$$

with the control parameter k_i being characterized by $0 < k_i < 1$.

Remark 5: The function $\sigma_m(x_1)$ can substituted by any saturation function.

On the other hand, we have that:

$$|v_2| \leq \varepsilon \triangleq m(k_i + 1). \tag{19}$$

Then, after substituting the above controller into the second equation of (11) we get:

$$\dot{x}_2 = -\sigma_m(x_2) - k_i s_m(x_1) + \alpha(z_3)x_4^2. \tag{20}$$

We introduce now the following positive definite function:

$$V_2 = \int_0^{x_2} \sigma_m(s)ds.$$

in order to verify the boundedness of the state x_2. Differentiating V_2 and using (20) we have that:

$$\dot{V}_2 = \sigma_m(x_2)\left(-\sigma_m(x_2) - k_i s_m(x_1) - \alpha(z_3)x_4^2\right). \tag{21}$$

Selecting $m > k_i m + \kappa_0\varepsilon^2/4$ we can assure that $\dot{V}_2 < 0$, if $|x_2| > k_i + \kappa_0\varepsilon^2/4m$. Therefore, there is a finite time $T_3 > T_2 > 0$ such that:

$$|x_2(t)| \leq k_{m_2} \triangleq k_i + \frac{\kappa_0\varepsilon^2}{4m};\ \forall\ t > T_3.$$

We emphasize that the restriction $m > k_i m + \kappa_0\varepsilon^2/4$ can be always satisfied. Indeed, from the definition of ε given in (19) we evidently have:

$$1 > k_i + \frac{\kappa_0 m}{4}(1 + k_i)^2 \tag{22}$$

(just to illustrate how this inequality holds take for instance $k_i = 2/3$ and $m = 1$, for a given $\kappa_0 = 0.39$). Finally, once the state x_2 is confined to move inside the region defined by k_{m_2}, the linear saturation function no longer acts over this state; that is, $\sigma_m(x_2) = x_2$. Therefore, v_2 turns out to be $v_2 = -x_2 - k_i\sigma_m(x_1)$. In the same way, after $t > T_3$, we can claim that the model in (11) leads to:

$$\begin{aligned}
\dot{x}_1 &= -k_i\sigma_m(x_1) + 3\alpha(z_3)x_4^2, \\
\dot{x}_2 &= -x_2 - k_i\sigma_m(x_1) + \alpha(z_3)x_4^2, \\
\dot{x}_3 &= -x_3 - \sigma_m(x_2) - k_i\sigma_m(x_1) \\
\dot{x}_4 &= -x_4 - x_3 - \sigma_m(x_2) - k_i\sigma_m(x_1).
\end{aligned} \tag{23}$$

Now, in instead of showing that the state x_1 is bounded, we show in what follows that, after a finite period of time $t > T_3$, all the states asymptotically converge to zero. Let us first introduce the following useful Lemma.

Lemma 1: Consider the first two equation in (23) and the following positive definite function:

$$V_m(x_2, x_1) = \int_0^{x_2} \sigma_m(s)ds + k_i \int_0^{x_1} \sigma_m(s)ds. \tag{24}$$

After a finite period of time $t > T_3$, the following inequity holds $\dot{V}_m(x_2, x_1) \leq K_m x_4^2 - \frac{1}{2}\left(x_2^2 + k_i^2\sigma_m^2(x_1)\right) - \frac{1}{2}v_2^2$, where $K_m \triangleq m\kappa_0(3k_i + 1)$.

The Proof of this Lemma is given in the Appendix.

D. Asymptotic convergence to the origin of the whole state

From the above discussion we conclude that after the finite time $t > T_3 > 0$, the states x_1, x_2 and x_3 are bounded in some compact set, which defines the closed-loop stability domain. To guarantee that all the states asymptotically converge to zero we propose the Lyapunov function $V_T(x) = V_1(x_4, x_3) + V_m(x_4, x_3)$, where V_1 and V_m were previously defined in (12) and (24), respectively. Since functions $V_1(*)$ and $V_m(*)$ are strictly positive definite function, with their respective arguments, we can claim that $V_T(x)$ qualifies as a candidate Lyapunov function. So, in case that $t > T_3$ we have that the time derivative of V_T satisfies the following inequality (see **Lemma 1** and **Remark 4**):

$$\dot{V}_T(x) \leq -(\frac{3}{2} - K_m)x_4^2 - \frac{1}{2}\left(x_2^2 + k_i^2 s_m^2(x_1)\right). \tag{25}$$

Selecting $K_m < 3/2$ we have that $\dot{V}_T(x) \leq 0$.[2] From Lyapunov's direct method we ensure the stability of the whole state in the Lyapunov sense. In order to prove now asymptotic stability, we use the well-known LaSalle's theorem [17]. In the region defined as:

$$S = \{x \in \mathbb{R}^4 : \dot{V}_T(x) = 0\}$$

we have that $x_4(t) = 0$, $x_2(t) = 0$ and $x_1(t) = 0$. Thus, in the set S, we also have $v_2 = 0$. Now, from the four chained integrators model (11) we have $x_3(t) = 0$, in the set S. Therefore, the largest invariant set $M \subset S$ is given by $x = 0$. Thus, according to LaSalle's theorem all the trajectories of system (23) asymptotically converge towards the largest invariant set defined by $M = \{x = 0\}$.

We summarize our previous discussion with the next proposition, which corresponds to our main result:

Proposition 1: Consider the closed-loop cart-pendulum system as described by model (7) with:

$$v = -z_3 - 2z_4 - \sigma_m(z_2 + 2z_3 + z_4) - k_i\sigma_m(z_1 + 3z_2 + 3z_3 + z_4). \tag{26}$$

Then the closed-loop system is globally asymptotically stable and locally exponentially stable, provided that the parameters m and k_i satisfy the inequalities $1 > k_i + \frac{\kappa_0 m}{4}(k_i + 1)^2$ and $m\kappa_0(3k_i + 1) < \frac{3}{2}$. ■

Remark 6: Note that in order to simplify as much as possible the previous stability analysis we use the proposed $v_2 = -\sigma_m(x_2) - k_i\sigma_m(x_1)$, which is formed using a linear saturation function. However, the control action is not unique since nonlinear saturation functions can also be used.

Taking into account this remark we introduce the following result:

Proposition 2: Consider the cart-pendulum model given by (7) and apply the feedback control law $v = -z_3 -$

[2] For example, $k_i = 2/3$ and $m = 1$, for a given $\kappa_0 = 0.39$.

$2z_4 - s_m(z_2 + 2z_3 + z_4) - k_i s_m(z_1 + 3z_2 + 3z_3 + z_4)$. Then, the closed-loop system can be globally asymptotically stable and locally exponentially stable, for a suitable set of the parameters m and k_i. ■

The Proof of this Proposition can be found in the Appendix.

In what follows we illustrate our results through the numerical simulation of a closed-loop control scheme.

E. An illustrative example

In order to show the effectiveness of our proposed nonlinear control strategy we chose the controller parameter values to be $m = 1$ and $k_i = 0.666$. As far as the initial conditions are concerned we take $(\theta, \dot{\theta}, q, \dot{q}) = (1.15[rad], 0, 1, 0.25)$. We consider two cases, the first one taking into account a linear saturation function (**LSF**), while the second case takes into account a nonlinear saturation function (**NSF**). Notice that these cases correspond to the results given by Proposition 1 and Proposition 2, respectively. Figure 2 shows the results coming out from the numerical simulations. Notice that the selected nonlinear saturation function is fixed as $s_m(x) = \tanh(x)$. As can be seen we have, as expected, a quite effective performance for both controllers. Also, we can observe that the closed-loop behavior for both control schemes is in fact very similar, nevertheless the NSF strategy displays a more smoother response. We conclude now our discussion with some final comments.

IV. Concluding Remarks

In this paper a new simple control strategy is proposed in order to solve the well-known cart-pendulum regulation problem, assuming that the pendulum is initialized in the upper half plane. Our control strategy used a control-oriented model of the considered system (a model consisted of a nonlinearly perturbed linear system consisted of a chain of four integrators), previously introduced in [13]. The model choice let us to design a simple composite stabilizer consisted of two control actions. The first control action characterizes a bounded linear controller, devoted to bring the nonactuated coordinate (that is, both the angular position and the angular velocity) near to the unstable vertical position and keep it inside of a small vicinity which defines the closed-loop stability domain. The second control action is a bounded nonlinear controller which, in conjunction with the linear bounded control action, ensures that the closed-loop whole state of the system asymptotically converges to the origin. The combined control law ensures then the regulation of the system. As discussed, our proposed control strategy can be displayed in two different versions. The first version concerns a bounded controller which uses a linear saturation function, while the second version uses a nonlinear saturation function. Our stability analysis was carried out using standard arguments from linear systems theory in conjunction with the traditional Lyapunov method and the famous LaSalle's theorem. We strongly believe that many other nonlinear underactuated dynamical systems can be stabilized using our simple control approach. We must point out that a main advantage of this work is that we did not need to solve **PDE**, nonlinear differential equations and nested saturation functions. Finally, the numerical experiments carried out with an academic example illustrated the effectivity of our control strategy.

Appendix

Proof of Lemma 1

We must remark that the time derivative of V_m (24) around the trajectories defined by the first two equation of (23) is given by:

$$\dot{V}_m = \overbrace{\alpha(z_3)x_4^2\left(3k_i\sigma_m(x_1) + \sigma_m(x_2)\right)}^{\varpi_0(x)} + \overbrace{k_i\sigma_m(x_1)x_2 - v_2^2}^{\varpi_1(x,v_2)}. \quad (27)$$

Then after $t > T_3$ we must have $v_2 = -x_2 - k_i\sigma_m(x_1)$. Therefore, $\varpi_1(x, v_2)$ can be expressed as:

$$\varpi_1(x, v_2) = -\frac{1}{2}\left(x_2^2 + k_i^2\sigma_m^2(x_1)\right) - \frac{1}{2}v_2^2 \quad (28)$$

and evidently $\varpi_0(x)$ can be bounded by:

$$|\varpi_0(x)| \leq K_m x_4^2 \triangleq m\kappa_0(3k_i + 1)x_4^2. \quad (29)$$

Substituting (28) and (29) into (27), we get the inequity shown in Lema 1, which concludes this proof ■

Proof of Proposition 2

For the sake of simplicity we use the sigmoidal function introduced by **Definition 2**, which is to say $s_m(x) = m\tanh(x)$. That is, v is formed as:

$$v = \overbrace{(-x_3 - x_4)}^{v_1} + \overbrace{(-m\tanh(x_2) - k_i m\tanh(x_1))}^{v_2}.$$

Selecting v_1 and the bound for v_2 as discussed in Section III, and taking into account the expressions (10) and (19), we guarantee that there exists a time $t > T_2$, such that $|x_4(t)| < \frac{\varepsilon}{2} = \frac{m(k_i+1)}{2}$; $\forall\ t > T_2 > T_1$. Therefore, the first and the second equations of (23) become:

$$\begin{aligned} \dot{x}_1 &= x_2 - m\tanh(x_2) - k_i m\tanh(x_1) + 3\alpha(z_3)x_4^2, \\ \dot{x}_2 &= -m\tanh(x_2) - k_i m\tanh(x_1) + \alpha(z_3)x_4^2. \end{aligned} \quad (30)$$

To analyze the boundedness of x_2, we use the positive definite function $E_2 = x_2^2/2$, whose time derivative can be bounded as:

$$\begin{aligned} \dot{E}_2 &= -mx_2\tanh(x_2) - k_i m x_2\tanh(x_1) + x_2\alpha(z_3)x_4^2 \\ &\leq -m\,|x_2|\left(|\tanh(x_2)| - k_i - \tfrac{\kappa_0 m(k_i+1)^2}{4}\right). \end{aligned} \quad (31)$$

Hence, selecting m and k_i, such that $k_i + \frac{\kappa_0 m(k_i+1)^2}{4} \triangleq \eta_{mk_i} < 1$. Therefore, there is a time $t > T_3$ such that $|x_2| < \tanh^{-1}(\eta_{mk_i}) \triangleq \overline{x}_{mk_i}$; for all $t > T_3$. Indeed, it follows because, if $\tanh(x_2) > \eta_{mk_i}$, then $\dot{E}_2 < 0$. Notice that the state x_2 can be confined to move inside of a compact set relying on the bound $\overline{x}_{mk_i}$. Note that this bound can be manipulated, almost, as desired. Then we can select for instance $\overline{x}_{mk_i} < 1.9$ to make $|x - \tanh(x)| < |\tanh(x)|$; for all $|x| < 1.9$ to hold. Simple geometric arguments can be applied to prove this inequality. Until now we have only provided sufficient conditions to guarantee that

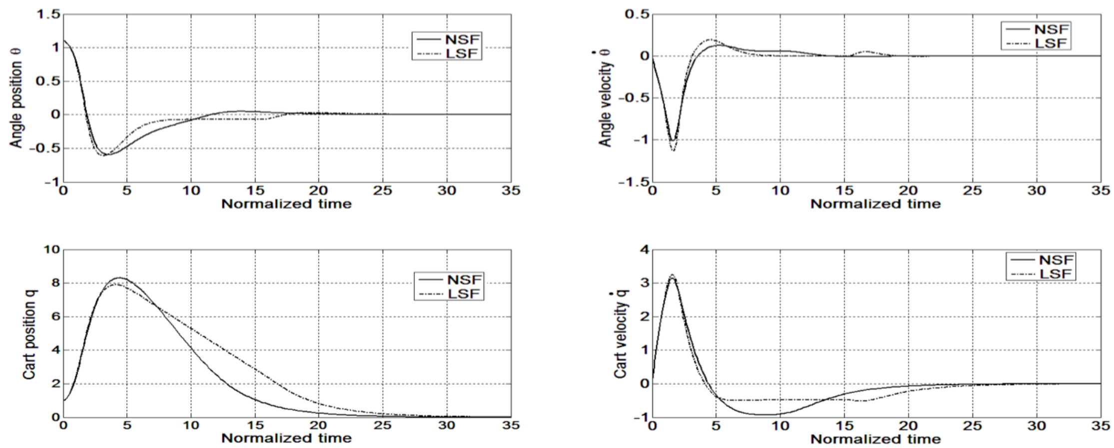

Fig. 2. Comparison between the closed-loop responses of both control strategies: **LSF** and **NLS**.

x_2, x_3 and x_4 are bounded (with the corresponding bounds being freely fixed). Now we are in conditions to proof that the whole state asymptotically converges to the origin. We first choose a positive function (similar to the one used in **Lemma 1**) defined as $E_m(x_2, x_1) = \int_0^{x_2} s_m(s)ds + k_i \int_0^{x_1} s_m(s)ds$. Differentiating the above equation with respect to (30), we have, after using simple algebra as in **Lemma 1**, the following inequity:

$$\begin{aligned}&\dot{E}_m(x_2, x_1) \le K_m x_4^2 \\ &\overbrace{+k_i \tanh(x_1)x_2 - \frac{1}{2}\left(\tanh(x_2) + k_i \tanh(x_1)\right)^2}^{\varpi(x)} \\ &-\tfrac{1}{2}v_2^2.\end{aligned} \tag{32}$$

Notice that $\varpi(x)$ can be expressed as $\varpi(w) = -\frac{1}{2}\tanh^2(x_2) - \frac{k_i^2}{2}\tanh^2(x_1) + k_i \tanh(x_1)(x_2 - \tanh(x_2))$. Now, under the assumption $t > T_3$, selecting $\overline{x}_{mk_i} < 1.9$, we have $\varpi(w) \le -\frac{1}{2}\tanh^2(x_2) - \frac{k_i^2}{2}\tanh^2(x_1) + k_i |\tanh(x_1)| |\tanh(x_2)| \le -\frac{1}{2}\left(|\tanh(x_2)| + k_i |\tanh(x_1)|\right)^2$. Thus, $\dot{E}_m$ can be bounded as:

$$\begin{aligned}&\dot{E}_m(x_2, x_1) \le K_m x_4^2 \\ &-\tfrac{1}{2}\left(|\tanh(x_2)| + k_i |\tanh(x_1)|\right)^2 - \tfrac{1}{2}v_2^2.\end{aligned} \tag{33}$$

We built now the candidate Lyapunov function $E_T = E_m + V_1$, with V_1 defined as in (12). Then, using some simple algebra and **Remark 4** it is easy to show that $\dot{E}_T$ can be bounded as $\dot{E}_T(x) \le -(\frac{3}{2} - K_m)x_4^2 - \frac{1}{2}\left(|\tanh(x_2)| + k_i |\tanh(x_1)|\right)^2$. Selecting $K_m < 3/2$ (as in Proposition 1), we have that $\dot{E}$ is semi-definite negative. Hence all the states are bounded. Finally, invoking the LaSalle's thorem, and following standard standard arguments, we can show that the whole state of the closed-loop system asymptotically converges to the origin. This concludes the proof. ■

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