2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

An internal-model principle for the synchronisation of autonomous agents with individual dynamics

Jan Lunze

Ruhr-Universität Bochum, 44780 Bochum, Germany
Lunze@atp.rub.de

***Abstract*— The task of synchronising autonomous agents is solved by a networked controller that steers the agents towards a common trajectory. This paper extends existing analysis and design methods for sets of linear agents with individual dynamics. To formulate the basic condition under which agents can be synchronised, the notion of system inclusion is introduced. It is shown that the agents can be synchronised in a leader-follower structure by an appropriate networked controller if and only if the dynamics of the extended agents include the dynamics of the synchronous trajectory. The results are illustrated by their application to vehicle platooning.**

I. Introduction

Synchronisation is an important phenomenon occurring in multi-agent systems. A networked controller should be chosen so as to give all subsystems a coherent behaviour. In this paper, synchronisation is investigated as the process of moving the subsystem outputs $\boldsymbol{y}_i(t)$, $(i = 1, 2, ..., N)$ onto a common trajectory $\boldsymbol{y}_{\rm s}(t)$.

The paper considers leader-follower synchronisation where the synchronous trajectory is prescribed by some reference system $\Sigma_{\rm s}$ (Fig. 1). The closed-loop system is said to be synchronised, if the following requirements are met:

1) **Synchronous behaviour:** For specific initial states, all outputs $\boldsymbol{y}_i(t)$ should follow a common trajectory $\boldsymbol{y}_{\rm s}(t)$:

$$\boldsymbol{y}_1(t) = ... = \boldsymbol{y}_N(t) = \boldsymbol{y}_{\rm s}(t), \quad t \geq 0. \quad (1)$$

2) **Asymptotic synchronisation:** For all other initial states, the networked controller should asymptotically synchronise the agents:

$$\lim_{t\to\infty} \|\boldsymbol{y}_i(t) - \boldsymbol{y}_{\rm s}(t)\| = 0, \quad i = 1, 2, ..., N. \quad (2)$$

An important issue of synchronisation is given by the requirement that the agents should be able to follow the synchronous trajectory $\boldsymbol{y}_{\rm s}(t)$ due to their internal dynamics. On the synchronous trajectory (1) the interactions established by the networked controller are not active and the agents generate the synchronous outputs independently from each other. This characteristic distinguishes synchronisation from many control problems, where a system is forced by its input to follow a nominal trajectory.

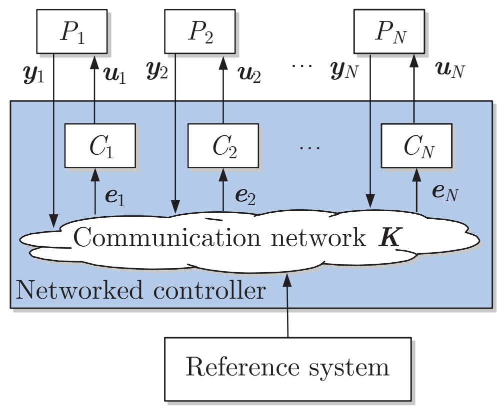

Fig. 1: Leader-follower synchronisation

Considering agents with arbitrary linear dynamics, this paper answers the question under what conditions agents are able to follow a reference trajectory $\boldsymbol{y}_{\rm s}(t)$. The result is an internal-model principle for synchronsation, which claims that each agent P_i together with its local controller C_i has to include the model $\Sigma_{\rm s}$ of the reference trajectory. This fact was proved in literature to be a necessary condition for the solution of consensus problems. The present paper shows that it is also sufficient in the sense that by using an appropriate communication topology agents with internal reference model can be synchronised.

This paper deals with the synchronisation problem in a very general set-up:

- The agents P_i have individual dynamics,
- their local controllers C_i can have any dynamical properties,
- only output information $\boldsymbol{y}_i(t)$ is exchanged,

- the communication topology $\boldsymbol{K}$ can be freely chosen.

This set-up contrasts with the literature, where generally agents with identical dynamics are controlled by some static feedback and the state information $\boldsymbol{x}_i(t)$ is communicated.

Literature survey. In the control literature on synchronisation, the focus has been laid on the design of distributed controllers for sets of identical subsystems [4], [6], [8], [9]. The leader-follower structure has been investigated e. g. in [1].

Identical agents can be synchronised, because they possess the same dynamical properties. To synchronise individual agents results in the new problem of choosing appropriate controller dynamics. It has been shown in [2], [12] that the agents have to possess an internal model of the consensus trajectory. The current paper distinguishes from these references in the control structure. It considers networked systems in which only the agent outputs $\boldsymbol{y}_i(t)$ are communicated and the agents are not extended by a common reference generator.

The results are applied here to vehicle platooning, which is an important practical example [5].

This paper is structured as follows. Section II defines the agents with individual dynamics and the local controllers. Section III introduces the notion of system inclusion. The main result of this paper shows that synchronisation occurs if and only if the controlled agents include the reference system as an internal model and the synchronisation conditions are satisfied (Theorem 3). The results are illustrated by considering vehicle platooning as a synchronisation problem.

II. MODELS

The agents P_i have linear dynamics

$$P_i: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_i(t) &=& \boldsymbol{A}_i\boldsymbol{x}_i(t) + \boldsymbol{B}_i\boldsymbol{u}_i(t) \\ \boldsymbol{x}_i(0) &=& \boldsymbol{x}_{i0} \\ \boldsymbol{y}_i(t) &=& \boldsymbol{C}_i\boldsymbol{x}_i(t) \end{array} \right. \tag{3}$$

$(i = 1, 2, ..., N)$ with $\boldsymbol{u}_i(t) \in \mathbb{R}^{m_i}$ denoting the input, $\boldsymbol{x}_i(t) \in \mathbb{R}^{n_i}$ the state and $\boldsymbol{y}_i(t) \in \mathbb{R}^r$ the output.

The reference system Σ_s

$$\Sigma_\mathrm{s}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_\mathrm{s}(t) &=& \boldsymbol{A}_\mathrm{s}\boldsymbol{x}_\mathrm{s}(t), \ \ \boldsymbol{x}_\mathrm{s}(0) = \boldsymbol{x}_\mathrm{s0} \\ \boldsymbol{y}_\mathrm{s}(t) &=& \boldsymbol{C}_\mathrm{s}\boldsymbol{x}_\mathrm{s}(t) \end{array} \right. \tag{4}$$

has the state $\boldsymbol{x}_\mathrm{s}(t) \in \mathbb{R}^{n_\mathrm{s}}$ and the output $\boldsymbol{y}_\mathrm{s}(t) \in \mathbb{R}^r$. Note that the dimension r of the outputs of all agents and of the reference system is the same. To avoid trivial solutions, all eigenvalues $\lambda_{\mathrm{s}i}$ of $\boldsymbol{A}_\mathrm{s}$ are assumed to have a nonnegative real part.

The agents P_i are controlled by local controllers C_i, which communicate over a network (Fig. 1):

$$C_i: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{\mathrm{r}i}(t) &=& \boldsymbol{A}_{\mathrm{r}i}\boldsymbol{x}_{\mathrm{r}i}(t) + \boldsymbol{B}_{\mathrm{r}i}\boldsymbol{e}_i(t) \\ \boldsymbol{x}_{\mathrm{r}i}(0) &=& \boldsymbol{x}_{\mathrm{r}i0} \\ \boldsymbol{u}_i(t) &=& \boldsymbol{K}_{\mathrm{r}i}\boldsymbol{x}_{\mathrm{r}i}(t) + \boldsymbol{K}_{\mathrm{e}i}\boldsymbol{e}_i(t). \end{array} \right. \tag{5}$$

The communication among the agents is restricted to a transfer of the agent outputs $\boldsymbol{y}_i(t)$, $(i = 1, 2, ..., N)$ and of the reference trajectory $\boldsymbol{y}_\mathrm{s}(t)$. The generalised synchronisation error $\boldsymbol{e}_i(t)$ is given by

$$\begin{aligned} \boldsymbol{e}_i(t) &= \sum_{j=1, j\neq i}^{N} k_{ij}(\boldsymbol{y}_j(t) - \boldsymbol{y}_i(t)) + k_{i\mathrm{s}}(\boldsymbol{y}_\mathrm{s}(t) - \boldsymbol{y}_i(t)) \\ &= \sum_{j=1}^{N} k_{ij}\boldsymbol{y}_j(t) + k_{i\mathrm{s}}\boldsymbol{y}_\mathrm{s}(t), \quad i = 1, 2, .., N \end{aligned}$$

with

$$k_{ii} = -\sum_{j=1, j\neq i}^{N} k_{ij} - k_{i\mathrm{s}}. \tag{6}$$

It merely refers to the relative outputs $\boldsymbol{y}_j(t) - \boldsymbol{y}_i(t)$. The matrix with the elements k_{ij}, $(i, j = 1, 2, ..., N)$ is denoted by $\boldsymbol{K}$. With $\boldsymbol{k}_\mathrm{s} = (k_{1\mathrm{s}} \ldots k_{N\mathrm{s}})^\mathrm{T}$ the synchronisation error $\boldsymbol{e} = (\boldsymbol{e}_1^\mathrm{T}(t) \ldots \boldsymbol{e}_N^\mathrm{T}(t))^\mathrm{T}$ is given by

$$\boldsymbol{e}(t) = (\boldsymbol{K} \otimes \boldsymbol{I}_r)\boldsymbol{y}(t) + \boldsymbol{k}_\mathrm{s} \otimes \boldsymbol{y}_\mathrm{s}(t). \tag{7}$$

where $\otimes$ denotes the Kronecker product.

Extended agents. If the agent model (3) is combined with the controller (5), the *extended agent* is obtained

$$\begin{aligned} \begin{pmatrix} \dot{\boldsymbol{x}}_i(t) \\ \dot{\boldsymbol{x}}_{\mathrm{r}i}(t) \end{pmatrix} &= \underbrace{\begin{pmatrix} \boldsymbol{A}_i & \boldsymbol{B}_i\boldsymbol{K}_{\mathrm{r}i} \\ \boldsymbol{O} & \boldsymbol{A}_{\mathrm{r}i} \end{pmatrix}}_{\boldsymbol{A}_{0i}} \underbrace{\begin{pmatrix} \boldsymbol{x}_i(t) \\ \boldsymbol{x}_{\mathrm{r}i}(t) \end{pmatrix}}_{\bar{\boldsymbol{x}}_i(t)} \\ &\quad + \underbrace{\begin{pmatrix} \boldsymbol{B}_i\boldsymbol{K}_{\mathrm{e}i} \\ \boldsymbol{B}_{\mathrm{r}i} \end{pmatrix}}_{\boldsymbol{B}_{0i}} \boldsymbol{e}_i(t) \\ \boldsymbol{y}_i(t) &= \underbrace{(\boldsymbol{C}_i \ \ \boldsymbol{O})}_{\boldsymbol{C}_{0i}} \begin{pmatrix} \boldsymbol{x}_i(t) \\ \boldsymbol{x}_{\mathrm{r}i}(t) \end{pmatrix} \end{aligned} \tag{8}$$

and abbreviated as

$$\Sigma_{0i}: \left\{ \begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}\bar{\boldsymbol{x}}_i(t) &=& \boldsymbol{A}_{0i}\bar{\boldsymbol{x}}_i(t) + \boldsymbol{B}_{0i}\boldsymbol{e}_i(t) \\ \boldsymbol{y}_i(t) &=& \boldsymbol{C}_{0i}\bar{\boldsymbol{x}}_i(t). \end{array} \right. \tag{9}$$

It is assumed that all extended agents are completely controllable and completely observable.

Model of the overall system. The overall system consists of N extended agents (9) that are coupled by eqn. (7). The model $\Sigma_{\rm F}$ of the interacting followers is

$$\Sigma_{\rm F}: \left\{ \begin{array}{rcl} \frac{\rm d}{{\rm d}t}\bar{\boldsymbol{x}}(t) & = & \bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{B}}\boldsymbol{y}_{\rm s}(t) \\ \boldsymbol{y}(t) & = & \bar{\boldsymbol{C}}\bar{\boldsymbol{x}}(t) \end{array} \right. \quad (10)$$

with

$$\bar{\boldsymbol{x}}(t) = (\bar{\boldsymbol{x}}_1^{\rm T}(t) ... \bar{\boldsymbol{x}}_N^{\rm T}(t))^{\rm T},\ \boldsymbol{y}(t) = (\boldsymbol{y}_1^{\rm T}(t) ... \boldsymbol{y}_N^{\rm T}(t))^{\rm T}$$

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A}_{01} & & \\ & \ddots & \\ & & \boldsymbol{A}_{0N} \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}_{01} & & \\ & \ddots & \\ & & \boldsymbol{B}_{0N} \end{pmatrix} (\boldsymbol{K} \otimes \boldsymbol{I}_r) \begin{pmatrix} \boldsymbol{C}_{01} & & \\ & \ddots & \\ & & \boldsymbol{C}_{0N} \end{pmatrix} \quad (11)$$

$$\bar{\boldsymbol{B}} = \begin{pmatrix} k_{1{\rm s}}\boldsymbol{B}_{01} \\ k_{2{\rm s}}\boldsymbol{B}_{02} \\ \vdots \\ k_{N{\rm s}}\boldsymbol{B}_{0N} \end{pmatrix}, \quad \bar{\boldsymbol{C}} = \begin{pmatrix} \boldsymbol{C}_{01} & & \\ & \ddots & \\ & & \boldsymbol{C}_{0N} \end{pmatrix}. \quad (12)$$

This model is combined with the reference system (4) to get the overall leader-follower system

$$\Sigma_{\rm LF}: \left\{ \begin{array}{rcl} \frac{\rm d}{{\rm d}t}\begin{pmatrix} \boldsymbol{x}_{\rm s}(t) \\ \bar{\boldsymbol{x}}(t) \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{A}_{\rm s} & \boldsymbol{O} \\ \bar{\boldsymbol{B}}\boldsymbol{C}_{\rm s} & \bar{\boldsymbol{A}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{\rm s}(t) \\ \bar{\boldsymbol{x}}(t) \end{pmatrix} \\ \boldsymbol{y}(t) & = & (\boldsymbol{O}\ \ \bar{\boldsymbol{C}}) \begin{pmatrix} \boldsymbol{x}_{\rm s}(t) \\ \bar{\boldsymbol{x}}(t) \end{pmatrix}. \end{array} \right.$$

The design problem is to find a networked controller for which all agents follow the trajectory $\boldsymbol{y}_{\rm s}(t)$.

III. System inclusion

This section defines the notion of system inclusion that describes systems having some dynamics in common. Consider the two systems

$$\Sigma_{\rm s}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{\rm s}(t) & = & \boldsymbol{A}_{\rm s}\boldsymbol{x}_{\rm s}(t), \quad \boldsymbol{x}_{\rm s}(0) = \boldsymbol{x}_{\rm s0} \\ \boldsymbol{y}_{\rm s}(t) & = & \boldsymbol{C}_{\rm s}\boldsymbol{x}_{\rm s}(t) \end{array} \right. \quad (13)$$

$$\Sigma: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) & = & \boldsymbol{C}\boldsymbol{x}(t) \end{array} \right. \quad (14)$$

with $\dim \boldsymbol{x}_{\rm s} = n_{\rm s}$, $\dim \boldsymbol{x} = n$ and $\dim \boldsymbol{y} = \dim \boldsymbol{y}_{\rm s} = r$.

Definition 3.1: **(System inclusion)** The system Σ is said to include the system $\Sigma_{\rm s}$ (in symbols: $\Sigma_{\rm s} \subseteq \Sigma$) if for every initial state $\boldsymbol{x}_{\rm s0} \in {\rm I\!R}^{n_{\rm s}}$ there exists an initial state $\boldsymbol{x}_0 \in {\rm I\!R}^n$ such the outputs are identical:

$$\boldsymbol{y}(t) = \boldsymbol{y}_{\rm s}(t), \quad t \geq 0. \quad (15)$$

System inclusion can be tested as follows:

Theorem 1: **(System inclusion)** The system Σ includes the system $\Sigma_{\rm s}$ if and only if there exists an $(n \times n_{\rm s})$-matrix $\boldsymbol{P}$ such that the following relations hold:

$$\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{A}_{\rm s} \quad (16)$$

$$\boldsymbol{C}\boldsymbol{P} = \boldsymbol{C}_{\rm s}. \quad (17)$$

The sufficiency of these relations follow from the initial state $\boldsymbol{x}_0 = \boldsymbol{P}\boldsymbol{x}_{\rm s0}$ for which eqn. (15) holds. The necessity of (17) results from the requirement $\boldsymbol{y}(0) = \boldsymbol{y}_{\rm s}(0)$, which claims the existence of $\boldsymbol{P}$. Then the consideration of the derivatives of $\boldsymbol{y}(t)$ and $\boldsymbol{y}_{\rm s}(t)$ leads to the condition (16).

Remark. The notion of system inclusion is less restrictive than the notion of *system equivalence* introduced in [13]. The latter is symmetric requiring for any state of one system the existence of a state of the other system for which both systems have the same output. The notions of *simulation* and *bisimulation* investigated in [7], [10] extensively use the freedom given by a disturbance input to both systems, which is not available here.

IV. Leader-follower synchronisation

A. Basic leader-follower structure

The main idea of synchronisation analysis is first demonstrated by considering the basic structure shown in Fig. 2 consisting of the agent P and the local controller C, which are described in the sequel by eqns. (3) and (5) without index i:

$$P: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) & = & \boldsymbol{C}\boldsymbol{x}(t) \end{array} \right. \quad (18)$$

$$C: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{\rm r}(t) & = & \boldsymbol{A}_{\rm r}\boldsymbol{x}_{\rm r}(t) + \boldsymbol{B}_{\rm r}\boldsymbol{e}(t) \\ \boldsymbol{u}(t) & = & \boldsymbol{K}_{\rm r}\boldsymbol{x}_{\rm r}(t) + \boldsymbol{K}_{\rm e}\boldsymbol{e}(t). \end{array} \right. \quad (19)$$

The synchronisation error is given by

$$\boldsymbol{e}(t) = \boldsymbol{y}_{\rm s}(t) - \boldsymbol{y}(t). \quad (20)$$

The *extended agent* Σ_0 has the model (9)

$$\Sigma_0: \left\{ \begin{array}{rcl} \frac{\rm d}{{\rm d}t}\bar{\boldsymbol{x}}(t) & = & \boldsymbol{A}_0\bar{\boldsymbol{x}}(t) + \boldsymbol{B}_0\boldsymbol{e}(t) \\ \boldsymbol{y}(t) & = & \boldsymbol{C}_0\bar{\boldsymbol{x}}(t) \end{array} \right. \quad (21)$$

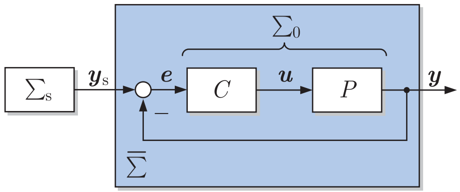

Fig. 2: Basic leader-follower structure

with $\bar{\boldsymbol{x}}(t) = (\boldsymbol{x}^{\mathrm{T}}(t),\ \boldsymbol{x}_{\mathrm{r}}^{\mathrm{T}}(t))^{\mathrm{T}}$ and

$$\begin{aligned} \boldsymbol{A}_0 &= \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}\boldsymbol{K}_{\mathrm{r}} \\ \boldsymbol{O} & \boldsymbol{A}_{\mathrm{r}} \end{pmatrix},\ \boldsymbol{B}_0 = \begin{pmatrix} \boldsymbol{B}\boldsymbol{K}_{\mathrm{e}} \\ \boldsymbol{B}_{\mathrm{r}} \end{pmatrix}, \\ \boldsymbol{C}_0 &= (\boldsymbol{C}\ \boldsymbol{O}). \end{aligned}$$

The *controlled agent* $\bar{\Sigma}$ results from combining eqns. (20) and (21):

$$\bar{\Sigma}: \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{\boldsymbol{x}}(t) &= (\boldsymbol{A}_0 - \boldsymbol{B}_0\boldsymbol{C}_0)\bar{\boldsymbol{x}}(t) + \boldsymbol{B}_0\boldsymbol{y}_{\mathrm{s}}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_0\bar{\boldsymbol{x}}(t). \end{cases} \quad (22)$$

The controller C should be chosen so that the controlled agent (22) together with the leader (4) satisfies the requirements (1) and (2) (without index i).

B. Internal-model principle for synchronisation

If the condition (1) is satisfied, the synchronisation error $\boldsymbol{e}(t)$ vanishes and the systems Σ_{s} and Σ_0 do not interact. Hence, the (open-loop) follower Σ_0 has to include the leader Σ_{s}.

Theorem 2: **(Internal-model principle for the basic leader-follower structure)** If the closed-loop agent $\bar{\Sigma}$ is synchronised with the leader $\Sigma_{\mathrm{s}} = (\boldsymbol{A}_{\mathrm{s}}, \boldsymbol{C}_{\mathrm{s}})$, then the extended follower $\Sigma_0 = (\boldsymbol{A}_0, \boldsymbol{C}_0)$ includes the leader:

$$\Sigma_{\mathrm{s}} \subseteq \Sigma_0. \quad (23)$$

The part of the extended agent Σ_0, which after some state transformation $\boldsymbol{P}$ coincides with the pair $(\boldsymbol{A}_{\mathrm{s}}, \boldsymbol{C}_{\mathrm{s}})$, is called an *internal model of the leader* in analogy to multivariable control, where the open-loop system has to include an internal model of the command signal generator. For the initial state $\boldsymbol{x}_0 = \boldsymbol{P}\boldsymbol{x}_{\mathrm{s}0}$ the follower generates the state trajectory of the leader and, hence, satisfies the requirement (1).

C. Control structures satisfying the internal-model principle

If the agent (18) includes the leader ($\Sigma_{\mathrm{s}} \subseteq P$), the internal-model principle is satisfied. Otherwise, the controller C has to introduce the internal model into the extended agent Σ_0. If the agent (18) is combined with the controller

$$C_{\mathrm{IM}}: \begin{cases} \dot{\boldsymbol{x}}_{\mathrm{r}}(t) &= \boldsymbol{A}_{\mathrm{s}}\boldsymbol{x}_{\mathrm{r}}(t) + \boldsymbol{B}_{\mathrm{r}}\boldsymbol{e}(t) \\ \boldsymbol{u}(t) &= \boldsymbol{K}_{\mathrm{r}}\boldsymbol{x}_{\mathrm{r}}(t) + \boldsymbol{K}_{\mathrm{e}}\boldsymbol{e}(t), \end{cases} \quad (24)$$

the extended agent satisfies eqn. (23) if the relation

$$\mathrm{Rank}(\boldsymbol{C}(\lambda_{\mathrm{s}i}\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}) = r \quad (25)$$

holds, which means that the eigenvalues $\lambda_{\mathrm{s}i}$ must not be transmission zeros of the agent (18) [3].

D. Leader-follower synchronisation of agents with individual dynamics

The conditions for the synchronism in the basic leader-follower structure are extended now for a set of N followers (3) with individual dynamics. Theorem 2 applies to all followers (9) separately:

$$\Sigma_{\mathrm{s}} \subseteq \Sigma_{0i}, \quad i = 1, 2, ..., N. \quad (26)$$

The following investigations show which additional requirements the overall system has to possess in order to be synchronised.

Synchronisation error. If eqn. (1) is not satisfied, synchronisation errors

$$\boldsymbol{e}_{i\mathrm{s}}(t) = \boldsymbol{y}_{\mathrm{s}}(t) - \boldsymbol{y}_i(t), \quad i = 1, 2, ..., N \quad (27)$$

occur and the followers interact over the communication network with each other and with the leader. The following lemma gives a representation of the synchronisation error $\boldsymbol{e}_{\mathrm{s}}(t) = (\boldsymbol{e}_{1\mathrm{s}}(t)^{\mathrm{T}} \ldots \boldsymbol{e}_{N\mathrm{s}}(t))^{\mathrm{T}}$.

Lemma 1: The synchronisation error $\boldsymbol{e}_{\mathrm{s}}(t)$ appearing in the leader-follower system Σ_{LF} is described by

$$\Sigma_{\mathrm{E}}: \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\check{\boldsymbol{x}}(t) &= \bar{\boldsymbol{A}}\check{\boldsymbol{x}}(t), \quad \check{\boldsymbol{x}}(0) = \check{\boldsymbol{x}}_0 \\ \boldsymbol{e}_{\mathrm{s}}(t) &= -\bar{\boldsymbol{C}}\check{\boldsymbol{x}}(t) \end{cases} \quad (28)$$

with the matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{C}}$ defined in eqns. (11) and (12) for some initial state $\check{\boldsymbol{x}}_0$.

The proof is given in [3].

The important aspect of this model is the fact that the matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{C}}$ are the same as in the model (10) of the interacting followers. The pair $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{C}})$ is completely observable and the model may have any initial state $\check{\boldsymbol{x}}_0 \in \mathbb{R}^{\bar{n}}$ in the state space of the model (28). Hence, the synchronisation error $\boldsymbol{e}_{\rm s}(t)$ vanishes if and only if $\bar{\boldsymbol{A}}$ is asymptotically stable.

Theorem 3: **(Leader-follower synchronisation)**
The set of extended followers Σ_{0i}, $(i = 1, 2, ..., N)$ described by eqn. (9) is synchronised with the leader $\Sigma_{\rm s}$ by the interconnection (7) if and only if the following conditions are satisfied:

1) All extended agents $\Sigma_{0i} = (\boldsymbol{A}_{0i}, \boldsymbol{B}_{0i}, \boldsymbol{C}_{0i})$ include the leader (cf. eqn. (26)).
2) The interacting followers $\Sigma_{\rm F} = (\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}}, \bar{\boldsymbol{C}})$ are asymptotically stable.

Note that this condition applies to arbitrary communication topologies. The next section, however, shows that cycle-free topologies are sufficient for synchronisation and lead to simpler synchronisation conditions.

E. Cycle-free communication topologies

Equation (11) shows that the structure of the matrix $\bar{\boldsymbol{A}}$ depends upon the communication topology described by the matrix $\boldsymbol{K}$. If the communication graph does not have any cycle, then there exists a permutation matrix $\boldsymbol{P}$ such that the matrix $\hat{\boldsymbol{K}} = \boldsymbol{P}\boldsymbol{K}\boldsymbol{P}^{\rm T}$ is a triangular matrix. After the enumeration of the agents has been changed accordingly, the matrix $\bar{\boldsymbol{A}}$ has block-triangular form with the diagonal blocks

$$\bar{\boldsymbol{A}}_{ii} = \boldsymbol{A}_{0i} + \hat{k}_{ii}\boldsymbol{B}_{0i}\boldsymbol{C}_{0i}, \quad i = 1, 2, ..., N \tag{29}$$

and off-diagonal blocks $\bar{\boldsymbol{A}}_{ij} = \boldsymbol{O}$ for $i \leq j$. Hence, the interacting followers are asymptotically stable if and only if all the controlled agents $\bar{\Sigma}_i$ are asymptotically stable.

Corollary 1: **(Synchronisation with cycle-free communication)** Consider leader-follower systems with cycle-free communication structures. Such systems are synchronised if and only if the following conditions are satisfied:

- All the extended agents Σ_{0i}, $(i = 1, 2, ..., N)$ include the leader.
- All the controlled followers $\bar{\Sigma}_i$, $(i = 1, 2, ..., N)$ are asymptotically stable.
- The communication graph has a spanning tree with the leader as root node.

The synchronisation condition is simplified considerably, because now it refers only to the matrices $\boldsymbol{A}_{ii}$, $(i = 1, 2, ..., N)$ given in eqn. (29) rather than to the overall system matrix $\bar{\boldsymbol{A}}$ defined in eqn. (11).

V. Example: Control of a vehicle platoon

In vehicle platooning, the vehicles should be synchronised with respect the (scalar) trajectory

$$y_{\rm s}(t) = s_0 + \bar{v}t, \tag{30}$$

which is given by the leading vehicle. The initial state of the reference system

$$\Sigma_{\rm s}: \left\{ \begin{array}{rcl} \begin{pmatrix} \dot{x}_{\rm s1}(t) \\ \dot{x}_{\rm s2}(t) \end{pmatrix} &=& \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\boldsymbol{A}_{\rm s}} \begin{pmatrix} x_{\rm s1}(t) \\ x_{\rm s2}(t) \end{pmatrix} \\ \begin{pmatrix} x_{\rm s1}(0) \\ x_{\rm s2}(0) \end{pmatrix} &=& \begin{pmatrix} s_0 \\ \bar{v} \end{pmatrix} \\ y_{\rm s}(t) &=& \underbrace{(1 \;\; 0)}_{\boldsymbol{C}_{\rm s}} \begin{pmatrix} x_{\rm s1}(t) \\ x_{\rm s2}(t) \end{pmatrix} \end{array} \right. \tag{31}$$

fixes the initial position s_0 and the velocity $\bar{v}$. The i-th vehicle should follow this trajectory with the inter-vehicle distance $\bar{s}_i$.

The vehicles P_i have a velocity controller that adapts the velocity $v_i(t)$ to the reference velocity $u_i(t)$, which is the input used by the networked controller. The position $s_i(t)$ is obtained by integrating the velocity $v_i(t)$. The i-th vehicle has the model

$$\begin{pmatrix} \dot{y}_i \\ \dot{\boldsymbol{x}}_{{\rm a}i} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \boldsymbol{c}_{{\rm a}i}^{\rm T} \\ \boldsymbol{0} & \boldsymbol{A}_{{\rm a}i} \end{pmatrix}}_{\boldsymbol{A}_i} \underbrace{\begin{pmatrix} y_i(t) \\ \boldsymbol{x}_{{\rm a}i}(t) \end{pmatrix}}_{\boldsymbol{x}_i(t)} + \underbrace{\begin{pmatrix} 0 \\ \boldsymbol{b}_{{\rm a}i} \end{pmatrix}}_{\boldsymbol{B}_i} u_i(t)$$

$$y_i(t) = \underbrace{(1 \;\; \boldsymbol{0}^{\rm T})}_{\boldsymbol{C}_i} \begin{pmatrix} y_i(t) \\ \boldsymbol{x}_{{\rm a}i}(t) \end{pmatrix}.$$

The vehicles distinguish with respect to the matrix $\boldsymbol{A}_{{\rm a}i}$ and the vectors $\boldsymbol{b}_{{\rm a}i}$ and $\boldsymbol{c}_{{\rm a}i}^{\rm T}$.

Application of the internal-model principle. The vehicles do not include the reference system $\Sigma_{\rm s}$ because they have only one vanishing eigenvalue whereas the reference system has two of them. Hence, the local controllers C_i have to include one vanishing eigenvalue. For the control law

$$C_i: \left\{ \begin{array}{rcl} \dot{x}_{{\rm r}i}(t) &=& b_{{\rm r}i}e_i(t), \quad x_{{\rm r}i}(0) = x_{{\rm r}i0} \\ u_i(t) &=& k_{{\rm r}i}x_{{\rm r}i}(t) + k_{{\rm e}i}e_i(t) \end{array} \right. \tag{32}$$

the internal-model principle is satisfied. The controller structure does not depend upon the vehicle parameters.

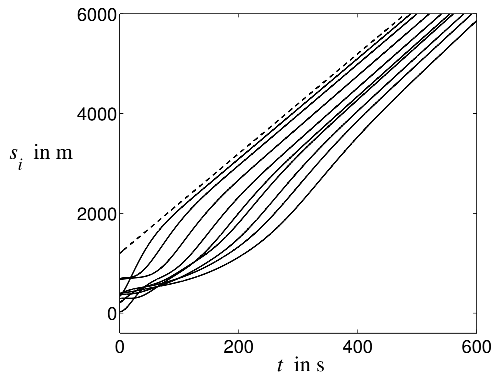

Fig. 3: Synchronisation with communication of $s_i(t)$

Leader-follower synchronisation. Figure 3 shows the synchronisation of ten vehicles with random initial position and initial velocity and with information exchange among the neighbouring vehicles. The vehicles synchronise after about $400\,\mathrm{s}$.

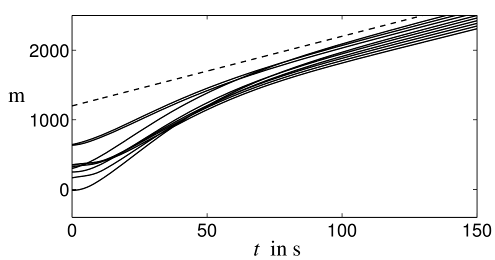

Fig. 4: Synchronisation with communication of $(s_i(t),\ v_i(t))$

The performance of the synchronisation can be considerably improved if not only the position $s_i(t)$ but also the velocity $v_i(t)$ is communicated among the vehicles (which means that $\boldsymbol{y}_i(t) = (s_i(t),\ v_i(t))^{\mathrm{T}}$ holds). Figure 4 shows the system behaviour with the same communication topology and initial states as in Fig. 3. The vehicles are already synchronised after $120\,\mathrm{s}$.

VI. Conclusions

This paper has solved the synchronisation problem for sets of agents with arbitrary linear dynamics. The main results are the internal-model principle for synchronisation and a necessary and sufficient synchronisation condition. For uni-directional communication structures, the synchronisation conditions can be stated as separate conditions on the controlled agents.

These results show that synchronisation of agents with individual dynamics is possible for very simple communication topologies like the uni-directional communication from the leader to all followers. The least requirement that the communication structure has to satisfy is the existence of a spanning tree in the communication graph with the leader as the root node.

The result of this paper can be extended for leaderless synchronisation, where the synchronous trajectory is a weighted average of the agent trajectories [3].

References

[1] Guo, W.; Chen, S.; Lü, J.; Yu, X.: Consensus of multi-agent systems with an active leader and asymmetric adjacency matrix, *IEEE Conf. on Decision and Contr. and Chinese Contr. Conf.*, Shanghai 2009, 3697-3702.

[2] Kim, H.; Shim, H.; Seo, J. H.: Output consensus of heterogeneous uncertain linear multi-agent systems, *IEEE Trans.* **AC-56** (2011), 200-206.

[3] Lunze, J.: *Synchronisation of Autonomous Agents with Individual Dynamics by Network Controllers*, Internal Report, Institute of Automation and Computer Control, Ruhr-University Bochum, 2010.

[4] Lunze, J.: Synchronisable nodes in networked systems, *J. Phys. A: Math. Theor.* **44** (2011) 045103.

[5] Naus, G. J. L.; Vugts, R. P. A.; Ploeg, J.; van de Molengraft, M. J. G.; Steinbuch, M.: String-stable CACC design and experimental validation: a frequency-domain approach, *IEEE Trans. on Vehicular Technology* **59** (2010), 4268-4279.

[6] Olfati-Saber, R.; Fax, J. A.; Murray, R. M.: Consensus and cooperation in networked multi-agent systems, *Proc. of the IEEE* **95** (2007), 215-233.

[7] Pappas, G. J.: Bisimilar linear systems, *Automatica* **39** (2003), 2035-2047.

[8] Scardovi, L.; Sepulchre, R.: Synchronization in networks of identical linear systems, *Automatica* **45** (2009), 2557-2562.

[9] Tuna, S. E.: Synchronizing linear systems via partial-state coupling, *Automatica* **44** (2008), 2179-2184.

[10] van der Schaft, A. J.: Equivalence of dynamical systems by bisimulation, *IEEE Trans.* **AC-49** (2004), 2160-2172.

[11] Wieland, P.; Allgöwer, F.: An internal model principle for consensus in heterogeneous linear multi-agent systems, *IFAC Workshop on Estimation and Control of Networked Systems*, Venice 2009, 7-12.

[12] Wieland, P.; Sepulchre, R.; Allgöwer, F.: An Internal Model Principle is necessary and sufficient for linear output synchronization, *Automatica* **47** (2011), 1068-1074.

[13] Zadeh, L. A.; Desoer, C. A.: *Linear System Theory: The State Space Approach*, McGraw-Hill Book Co., New York 1963.