2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

On the Distinguishability of Discrete Linear Time-Invariant Dynamic Systems

Paulo Rosa and Carlos Silvestre

***Abstract*— This paper introduces the notion of absolutely distinguishable discrete dynamic systems, with particular applicability to linear time-invariant (LTI) systems. The motivation for this novel type of distinguishability stems, in particular, from the stability and performance requirements of worst-case adaptive control methodologies. The main results presented herein show that, in most practical cases, a persistence of excitation type of condition and a minimum number of iterations are required to properly distinguish two dynamic systems. We also demonstrate that the former constraint can be written as a lower bound on the intensity of the exogenous disturbances. The applicability of the developed theory is illustrated with a set of examples.**

I. INTRODUCTION

The *identifiability* of dynamic systems plays an important role in certain areas, where it is fundamental to ensure that the model of the plant can be inferred from input/output data. In particular, model estimation methods require the system to be identifiable. Otherwise, the estimation problem may not be well-posed and lead to erroneous results. Such methods also require, in general, a persistence of excitation condition in the exogenous inputs, in order to avoid the issues related to the indistinguishability due to the small amplitude of the disturbances.

However, in applications such as fault detection [1], [2], [3], [4], [5], identification of hybrid systems [6], [7], and multiple-model adaptive control [8], [9], [10], it suffices to guarantee that we can identify the family of systems to which the true plant belongs, among a finite set of families of dynamic systems. In such cases, we say that the families of systems are *distinguishable*. This notion was introduced in [11], where in fact the authors relate the concept of identifiability with that of distinguishability.

Indeed, these two concepts (see, for instance, [11], [12], [13]) are naturally related to each other. If a system can be identified by using a certain input signal, then it is obvious that it can be distinguished from any other system with the same structure. A discussion on this topic can be found in [14]. The interested reader is also referred to [15], [16], [17].

However, most of the results on system identifiability are not enough for system *absolute distinguishability*. In particular, for two absolutely distinguishable systems we require, for *all* the admissible input signals, the corresponding outputs to be different from one another, unlike the most common definition of (structurally) identifiable system, which only demands that the unknown parameters of the plant can be estimated for *some* input signal.

The most common notion of (in)distinguishability [11], [14] states that two systems are indistinguishable only if there are neither initial conditions nor exogenous inputs that can generate different outputs for the systems. This definition, however, may not be enough in applications where the distinguishability of the systems must be guaranteed from a worst-case perspective.

Therefore, this paper introduces a novel definition of system distinguishability, referred to as *absolute distinguishability*, as detailed in the sequel. We also provide necessary and sufficient conditions for the absolute distinguishability of two dynamic LTI systems under different scenarios, and show how to design observers that can be used to distinguish among dynamic systems. These observers are used for model falsification, *i.e.*, to (in)validate models of the system, based upon the input/output data. Although this can be achieved by using methods such as in [18], [19], the solution adopted herein provides an iterative algorithm to solve the problem of model falsification, which is suitable for run time applications like multiple model adaptive control.

The remainder of this paper is organized as follows: in Section II, the notation and some of the basic concepts used throughout the paper are introduced; in Section III, a series of properties is derived for absolutely distinguishable LTI systems; Section IV is devoted to the introduction of the set-valued observers, used for system distinguishability; the applicability of the distinguishability theory is illustrated in a series of examples, in Section V; finally, in Section VI, some conclusions regarding the results obtained are highlighted.

II. PRELIMINARIES AND NOTATION

In this paper, we are going to start by considering the broad class of discrete time-varying dynamic systems described by

$$y(k) = F_k\left(x_0, \phi_0, \phi_1, \phi_2, \cdots, \phi_k, p\right), \tag{1}$$

where $F_k : \mathbb{R}^n \times \overbrace{\Phi \times \cdots \times \Phi}^{k+1 \text{ times}} \times \Omega \rightarrow \mathbb{R}^{n_y}$, $\phi_i \in W_d \times U =: \Phi \subseteq \mathbb{R}^{n_u+n_d}$ for $i = 0, 1, \cdots, k$, and $p \in \Omega \subseteq \mathbb{R}^{n_p}$ is a vector of parameters. The sequence $(\phi_0, \phi_1, \cdots, \phi_k)$, where $\phi_i = [d_i^{\mathrm{T}},\, u_i^{\mathrm{T}}]^{\mathrm{T}}$, denotes the exogenous disturbances, $d_i \in W_d \subseteq \mathbb{R}^{n_d}$, and control input signals, $u_i \in U \subseteq \mathbb{R}^{n_u}$, at time instant i, and $y(k)$ is the output of the system at time k. The initial state is represented by $x(0) \in X(0) \subseteq \mathbb{R}^n$.

The notion of distinguishability in [11], [14] states that two realizations of F_k, parametrized by the pair of parameter vectors (p_A, p_B), are indistinguishable in N sampling times only if there are neither initial conditions nor exogenous

This work was partially supported by Fundação para a Ciência e a Tecnologia (ISR/IST pluriannual funding) through the POS_Conhecimento Program that includes FEDER funds, and by the PTDC/EEA-ACR/72853/2006 OBSERVFLY Project. The work of P. Rosa was supported by a PhD Student Scholarship, SFRH/BD/30470/2006, from the FCT.

P. Rosa and C. Silvestre are with Institute for Systems and Robotics - Instituto Superior Tecnico, Av. Rovisco Pais, 1, 1049-001 Lisboa, Portugal
prosa@isr.ist.utl.pt, cjs@isr.ist.utl.pt

inputs that can generate different outputs for the systems parametrized by p_A and p_B. Hence, this definition is not useful in some applications, namely to provide guarantees that, regardless the input signals, the pair of parameter values is distinguishable.

Thus, a new definition of distinguishability, referred to U-input distinguishability, was presented in [7]. Consider the dynamic systems S_A and S_B, which correspond to realizations of (1), for $p = p_A$ and $p = p_B$ and with outputs at sampling time k denoted by $y_A(k)$ and $y_B(k)$, respectively. The definition in [7] states that, unless all the initial conditions and inputs are zero, if two systems are U-input distinguishable in N sampling times, then the corresponding outputs must be different at some time instant k smaller than or equal to N. This guarantees that we can distinguish two such systems in, at most, N sampling times, just by measuring the outputs. Nevertheless, exogenous disturbances, $d(k) \in W_d$, and measurement noise, $n(k) \in W_n$, are not taken into account in this definition of distinguishability.

Therefore, this motivated the introduction of the following definitions, which are going to be used extensively throughout this article:

Definition 1: Systems S_A and S_B are said to be *absolutely distinguishable in N measurements* if, for any non-zero

$$\left(x_o^A, x_o^B, d_{0,N-1}^A, d_{0,N-1}^B, n_{0,N}^A, n_{0,N}^B, u_{0,N-1}, \right) \in \mathbb{R}^{2n} \times \mathbb{R}^{2Nn_d} \times \mathbb{R}^{2(N+1)n_n} \times \mathbb{R}^{2Nn_u},$$

there exists a $k \in \{0, 1, \cdots, N\}$ such that

$$y_A(k) \neq y_B(k).$$

Moreover, two systems are said to be *absolutely distinguishable* if there exists $N \geq 0$ such that they are absolutely distinguishable in N measurements. $\diamond$

In the definition, we used the short-hand notation $v_{0,N}$ to denote a concatenation of a sequence of vectors

$$v_{0,N} := [v_o^{\mathrm{T}}, \cdots, v_N^{\mathrm{T}}]^{\mathrm{T}}.$$

Definition 2: Systems S_A and S_B are said to be *absolutely (X_o, U, W)-input distinguishable in N measurements* if, for any non-zero

$$\left(x_o^A, x_o^B, d_{0,N-1}^A, d_{0,N-1}^B, n_{0,N}^A, n_{0,N}^B, u_{0,N-1}, \right) \in X_o^2 \times W_d^{2N} \times W_n^{2(N+1)} \times U^N$$

where $W := W_n \times W_d$, there exists $k \in \{0, 1, \cdots, N\}$ such that

$$y_A(k) \neq y_B(k).$$

Moreover, two systems are said to be *absolutely (X_o, U, W)-input distinguishable* if there exists $N \geq 0$ such that they are absolutely (X_o, U, W)-input distinguishable in N measurements. $\diamond$

These two definitions are important when we want to guarantee that, regardless of the input signals, two systems can be distinguished in a given number of iterations. This fact is going to be further stressed in the sequel.

Due to space limitations, the proofs of the results presented in this paper are omitted. A 7-pages version of this paper, with the complete set of proofs, is available at *www.arxiv.org*.

III. Absolute Input-Distinguishability of LTI Systems

In this section, we are going to specialize the concept of absolute input-distinguishability for linear time-invariant (LTI) models and discuss some of the properties of this a class of dynamic systems.

Let S_i be a discrete LTI dynamic system described by

$$\begin{cases} x_i(k+1) &= A_i x_i(k) + B_i u_i(k) + L_i d_i(k), \\ y_i(k) &= C_i x_i(k) + N_i n_i(k), \end{cases} \quad (2)$$

where $x_i(0) = x_0^i$, $x_i(k) \in \mathbb{R}^n$, $u_i(k) \in U \subseteq \mathbb{R}^{n_u}$, $d_i(k) \in W_d \subseteq \mathbb{R}^{n_d}$, $y_i(k) \in \mathbb{R}^{n_y}$ and $n_i(k) \in W_n \subseteq \mathbb{R}^{n_n}$.

Notice that, according to Definition II, two systems are absolutely (X_0, U, W)-input distinguishable in N sampling times if, for *all* the input signals in U, the corresponding outputs are different at least at some time instant $k \leq N$. This is obviously a stronger constraint than simply saying that the two systems are distinguishable whenever the corresponding outputs are different for a particular input sequence, as in [11], [14].

Remark 1: In the remainder of this paper, we are going to use the terms *distinguishable* and *absolutely distinguishable* interchangeably. $\diamond$

For the sake of simplicity, let us consider, for the time being, that $U = \mathbb{R}^{n_u}$, $W_d = \mathbb{R}^{n_d}$ and $W_n = \mathbb{R}^{n_n}$. The following theorem can be used to test whether or not two systems are distinguishable (in the sense of Definition II).

Theorem 1: Let

$$M_N = \left[\begin{array}{cc|c|c|c} C_A & -C_B & & & \\ C_A A_A & -C_B A_B & & & \\ C_A A_A^2 & -C_B A_B^2 & \bar{Q} & \bar{R} & \bar{J} \\ \vdots & \vdots & & & \\ C_A A_A^N & -C_B A_B^N & & & \end{array}\right],$$

where

$$\bar{Q} = \mathrm{diag}(Q, Q, \cdots, Q),\ Q = \begin{bmatrix} N_A & -N_B \end{bmatrix},$$

$$\bar{R} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ R_1^1 & 0 & \cdots & 0 \\ R_1^2 & R_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_1^N & R_2^N & \cdots & R_N^N \end{bmatrix},\ \bar{J} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ J_1^1 & 0 & \cdots & 0 \\ J_1^2 & J_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J_1^N & J_2^N & \cdots & J_N^N \end{bmatrix},$$

$$R_i^k = \begin{bmatrix} C_A A_A^{k-i} L_A & -C_B A_B^{k-i} L_B \end{bmatrix},$$

$$J_i^k = \begin{bmatrix} C_A A_A^{k-i} B_A - C_B A_B^{k-i} B_B \end{bmatrix}.$$

Systems S_A and S_B are absolutely distinguishable in N sampling times if and only if there does not exist a non-zero vector $v \in \mathbb{R}^{2n+2(N+1)n_n+N(n_u+2n_d)}$ such that

$$M_N v = 0,$$

i.e., if and only if,

$$\mathrm{rank}(M_N) = 2n + 2(N+1)n_n + N(n_u + 2n_d).$$

The importance of this theorem is twofold: on the one hand, it provides necessary and sufficient conditions for the absolute distinguishability of two systems; on the other hand, it allows the use of set-valued observers [20] for system distinguishability, as shown in the sequel.

As remarked in [7] for continuous-time systems, the distinguishability of two systems implies their observability. In fact, we are going to obtain a similar conclusion for discrete-time systems, but using an alternative path.

Corollary 1: Let S_A and S_B be two absolutely distinguishable systems. Then, S_A and S_B are both observable.

Remark 2: The concept of absolute distinguishability of two systems requires the corresponding outputs to be different from each other, for *every* non-zero initial condition. However, if a system is non-observable, then there are (non-zero) initial conditions for the state which do not impact on the output of the plant. $\diamond$

Corollary 1 provides necessary conditions for the distinguishability of two discrete LTI dynamic systems. For the sake of completeness, we recall that there are also results in the literature providing sufficient conditions for absolute distinguishability. For instance, the authors of [21] show that, if $\bar{J}$ has full column rank and the sets X_0 and W are compact, then the systems can be distinguished for sufficiently large input control signals, regardless of their direction.

IV. Distinguishing Systems Using Set-Valued Observers

The results presented in the previous section give us necessary and sufficient conditions for absolute distinguishability. They do not provide, however, a systematic way of distinguishing two models, say S_A and S_B. In this section, we will show how to use set-valued observers (SVOs) to (in)validate models of systems, *i.e.*, to distinguish among them. This type of observers was applied to linear time-varying systems excited by bounded-but-unknown disturbances in [20], and were used for model falsification in [10]. At each iteration, these observers compute the set-valued estimate of the state, by taking into account the model of the plant and the most recent measurements. Since, no conservatism to the solution is added for LTI systems (see [20]) we conclude that an SVO can be used to falsify (invalidate) a given model of the plant using the following reasoning:

- if the set-valued estimate of the state, $X(k)$, at iteration k is empty, then the model of the plant is falsified, *i.e.*, the input/output data invalidates the model of the plant;
- if the set-valued estimate of the state, $X(k)$, at iteration k is not empty, then the model of the plant is not falsified, *i.e.*, the input/output data is *compatible* with the model of the plant.

In order to use the SVOs for model falsification, let us pose the following assumption:

Assumption 1: The set of *admissible initial states*, X_0, *i.e.*, the minimum set containing all the possible initial states x_0^i of the systems, is convex and compact. $\square$

Under Assumption 1, the set of initial states can be represented by

$$X_0 = \text{Set}(M_{X_0}, m_{X_0}) := \{x : M_{X_0}x \leq m_{X_0}\}.$$

Analogously, suppose that:

Assumption 2: The set of *admissible exogenous disturbances*, W_d, is convex and compact, *i.e.*,

$$W_d = \text{Set}(M_d, m_d).$$

$\square$

Assumption 3: The set of *admissible measurement noise*, W_n, is convex and compact, *i.e.*,

$$W_n = \text{Set}(M_n, m_n).$$

$\square$

Then, we can state the following theorem, which is an extension of Theorem 1 for convex and compact X_0 and W:

Theorem 2: Suppose that Assumptions 1–3 are satisfied. Then, systems S_A and S_B are absolutely $(X_0, \mathbb{R}^{n_u}, W)$-input distinguishable if and only if

$$\underset{v \in \mathfrak{G}}{\forall} \quad v \neq 0 \Rightarrow \begin{bmatrix} M_N \\ -M_N \\ \tilde{M}_{X_0} \\ \tilde{M}_W \end{bmatrix} v \not\leq \begin{bmatrix} 0 \\ 0 \\ \tilde{m}_{X_0} \\ m_W \end{bmatrix}, \tag{3}$$

where

$$\tilde{M}_{X_0} = \begin{bmatrix} \text{diag}(M_{X_0}, M_{X_0}) & 0 & 0 & 0 \end{bmatrix}, \; \tilde{m}_{X_0} = \begin{bmatrix} m_{X_0} \\ m_{X_0} \end{bmatrix},$$

$$\tilde{M}_W = \begin{bmatrix} 0 & \text{diag}(M_n, \cdots, M_n) & 0 & 0 \\ 0 & 0 & \text{diag}(M_d, \cdots, M_d) & 0 \end{bmatrix},$$

$$m_W = \begin{bmatrix} m_n^{\mathrm{T}} & \cdots & m_n^{\mathrm{T}} & m_d^{\mathrm{T}} & \cdots & m_d^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

and $\mathfrak{G} = \mathbb{R}^{2n} \times \mathbb{R}^{2n_n(N+1)} \times \mathbb{R}^{2n_dN} \times \mathbb{R}^{n_uN}$.

Although Theorem 2 provides necessary and sufficient conditions for distinguishability of systems with polytopic constraints on the disturbances and initial states, the condition in (3) is seldom satisfied in practice. Therefore, we add the following constraint on the disturbances intensity, which can be interpreted as a persistence of excitation, like condition

$$\frac{1}{N}\sum_{k=1}^{N} \|d(k)\|^2 \geq \gamma. \tag{4}$$

Such a condition can be easily merged with (3), as explained in the sequel.

We start by introducing the Fourier-Motzkin elimination method, described in [22]. This method can be used to project polyhedral convex sets on to subspaces. Indeed, we are interested in projecting a polytope described by $\{x \in \mathbb{R}^{n_x} : Ax \leq b\}$ on to $\mathbb{R}^{\tilde{n}_x}$, where $\tilde{n}_x < n_x$.

Let

$$(A_{\text{LFM}}, b_{\text{LFM}}) := \text{LFM}(A, b, n),$$

and

$$(A_{\text{RFM}}, b_{\text{RFM}}) := \text{RFM}(A, b, n),$$

where LFM and RFM stand for the *left-* and *right-Fourier Motzkin* elimination methods, $n = n_x - \tilde{n}_x > 0$, and where A_{LFM}, b_{LFM}, A_{RFM} and b_{RFM}, satisfy, for all $\tilde{x} \in \mathbb{R}^{\tilde{n}_x}$,

$$A_{\text{LFM}}\tilde{x} \leq b_{\text{LFM}} \Leftrightarrow \exists_{x \in \mathbb{R}^n} : A \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \leq b,$$

and

$$A_{\text{RFM}}\tilde{x} \leq b_{\text{RFM}} \Leftrightarrow \exists_{x\in\mathbb{R}^n} : A\begin{bmatrix}\tilde{x}\\ x\end{bmatrix} \leq b.$$

At this point, we can state the following theorem:

***Theorem** 3:* Suppose that Assumptions 1–3 are satisfied and let

$$(A_N, b_N) = \text{RFM}\left\{\text{LFM}\left(\begin{bmatrix}M_N\\ -M_N\\ \tilde{M}_{X_o}\\ \tilde{M}_W\end{bmatrix}, \begin{bmatrix}0\\ 0\\ \tilde{m}_{X_o}\\ m_W\end{bmatrix}, 2n\right), n_u\right\}.$$

Further define

$$P_A = \frac{1}{N}\text{diag}(0_{n_n}, \cdots, 0_{n_n}, \bar{P}, 0_{n_d}, \cdots, \bar{P}, 0_{n_d}),$$

and

$$P_B = \frac{1}{N}\text{diag}(0_{n_n}, \cdots, 0_{n_n}, 0_{n_d}, \bar{P}, \cdots, 0_{n_d}, \bar{P}),$$

and let $\gamma_{\text{min}} \geq 0$ be such that

$$\gamma_{\text{min}} \geq \max_{A_N x \leq b_N} x^{\mathrm{T}} P_A x \quad \text{and} \quad \gamma_{\text{min}} \geq \max_{A_N x\leq b_N} x^{\mathrm{T}} P_B x. \tag{5}$$

Then, the systems S_A and S_B are $(X_o, \mathbb{R}^{n_u}, W)$-input distinguishable in N measurements if

$$\frac{1}{N}\sum_{k=0}^{N}\|\bar{P}d_A(k)\|^2 > \gamma_{\text{min}} \quad \text{and} \quad \frac{1}{N}\sum_{k=0}^{N}\|\bar{P}d_B(k)\|^2 > \gamma_{\text{min}}. \tag{6}$$

Remark 3: Notice that 5 can be interpreted as a concave quadratic programming problem, which can be solved, for instance, by testing the solution at the vertices of the polytope $S = \{x : Ax \leq b\}$ (*cf.* [23]). $\diamond$

Remark 4: The Fourier-Motzkin elimination method removes the dependence of the distinguishability of systems S_A and S_B on the initial state and control inputs. Hence, it reduces the number of variables in the optimization procedure. $\diamond$

The constraint in (4) can be replaced by a similar condition on the intensity of the output, $y(.)$. To see this, we rewrite the output sequence as

$$\begin{aligned} y(k) &= Cx(k) + Nn(k), \quad k = 0, 1, \cdots, N\\ &= \begin{bmatrix}C & N\end{bmatrix}\begin{bmatrix}x(k)\\ n(k)\end{bmatrix}, \quad k = 0, 1, \cdots, N\\ &= \bar{C}\bar{x}(k), \quad k = 0, 1, \cdots, N, \end{aligned}$$

where

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) + Ld(0),\\ &\vdots\\ x(N) &= A^N x(0) + A^{N-1}Bu(0) +\\ &\quad + A^{N-1}Ld(0) + \cdots +\\ &\quad + Bu(N-1) + Ld(N-1). \end{aligned}$$

Hence,

$$\begin{aligned} \bar{x}(0) &= \bar{I}v_N,\\ \bar{x}(1) &= \bar{A}(0)v_N,\\ &\vdots\\ \bar{x}(N) &= \bar{A}(N-1)v_N, \end{aligned}$$

where

$$\begin{aligned} v_N^{\mathrm{T}} = [&x^{\mathrm{T}}(0), n^{\mathrm{T}}(0), \cdots, n^{\mathrm{T}}(N), d^{\mathrm{T}}(0), \cdots, d^{\mathrm{T}}(N-1),\\ &u^{\mathrm{T}}(0), \cdots u^{\mathrm{T}}(N-1)]^{\mathrm{T}} \end{aligned}$$

and

$$\bar{I} = \left[\begin{array}{c|cccc|cccc|cccc} I & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right],$$

$$\bar{A}(0) = \left[\begin{array}{c|cccc|cccc|cccc} A & 0 & 0 & \cdots & 0 & L & 0 & \cdots & 0 & B & 0 & \cdots & 0\\ 0 & 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right],$$

$$\bar{A}(N-1) = \left[\begin{array}{c|ccc|ccc|ccc} A^N & 0 & \cdots & 0 & A^{N-1}L & \cdots & L & A^{N-1}B & \cdots & B\\ 0 & 0 & \cdots & I & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right].$$

Then, we have that

$$\begin{aligned} \theta &:= \frac{1}{N}\sum_{k=0}^{N}\|y(k)\|^2\\ &= \frac{1}{N}\left[\bar{x}^{\mathrm{T}}(0)\bar{C}^{\mathrm{T}}\bar{C}\bar{x}(0) + \bar{x}^{\mathrm{T}}(1)\bar{C}^{\mathrm{T}}\bar{C}\bar{x}(1) + \right.\\ &\quad \left. + \cdots + \bar{x}^{\mathrm{T}}(N)\bar{C}^{\mathrm{T}}\bar{C}\bar{x}(N)\right]\\ &= \frac{1}{N}(v_N)^{\mathrm{T}}\left(\tilde{V}_N\right)^{\mathrm{T}}\mathcal{C}(N)\tilde{V}_N v_n \end{aligned}$$

where

$$\tilde{V}_N = \begin{bmatrix}\bar{I}\\ \bar{A}(0)\\ \bar{A}(1)\\ \vdots\\ \bar{A}(N-1)\end{bmatrix},$$

and

$$\mathcal{C}(N) = \text{diag}(\bar{C}^{\mathrm{T}}\bar{C}, \cdots, \bar{C}^{\mathrm{T}}\bar{C}).$$

Therefore, we are now in conditions of stating the following theorem:

***Theorem** 4:* Define $\tilde{P}_B = \frac{1}{N}\tilde{C}_B^N$, and let $\theta_{\text{min}} \geq 0$ be such that

$$\theta_{\text{min}} \geq \max_{\tilde{A}_N x\leq \tilde{b}_N} x^{\mathrm{T}}\tilde{P}_B x, \tag{7}$$

where

$$(\tilde{A}_N, \tilde{b}_N) = \text{RFM}\left\{\text{LFM}\left(\begin{bmatrix}\begin{bmatrix}M_A^N & -M_B^N\end{bmatrix}\\ \begin{bmatrix}-M_A^N & M_B^N\end{bmatrix}\\ M_{X_0}^*\\ M_W^*\end{bmatrix}, \begin{bmatrix}0\\ 0\\ \tilde{m}_{X_0}\\ m_W\end{bmatrix}, n + (N+1)n_n + N(n_d+n_u)\right), N(n_d+n_u)\right\},$$

$$\tilde{C}_B^N = \left(\tilde{V}_B^N\right)^{\mathrm{T}}\begin{bmatrix}\bar{C}_B^{\mathrm{T}}\bar{C}_B & 0 & \cdots & 0\\ 0 & \bar{C}_B^{\mathrm{T}}\bar{C}_B & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{C}_B^{\mathrm{T}}\bar{C}_B\end{bmatrix}\tilde{V}_B^N,$$

$$\tilde{V}_B^N = \begin{bmatrix} \bar{I} \\ \bar{A}_B^0 \\ \bar{A}_B^1 \\ \vdots \\ \bar{A}_B^{N-1} \end{bmatrix},$$

$$M_{X_o}^* = \left[\begin{array}{c|c|c|c|c|c|c|c} M_{X_o} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{X_o} & 0 & 0 & 0 \end{array} \right],$$

$$M_W^* = \left[\begin{array}{c|c|c|c|c|c|c|c} 0 & D_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_n & 0 & 0 \\ 0 & 0 & D_d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_d & 0 \end{array} \right],$$

$$D_n = \operatorname{diag}(M_n, \cdots, M_n), \quad D_d = \operatorname{diag}(M_d, \cdots, M_d),$$

and, for $i \in \{A, B\}$,

$$M_i^N = \left[\begin{array}{c|c|c|c} \begin{matrix} C_i \\ C_i A_i \\ \vdots \\ C_i A_i^N \end{matrix} & \operatorname{diag}(N_i) & \tilde{L}_i^N & \tilde{B}_i^N \end{array} \right],$$

$$\tilde{L}_i^N = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C_i L_i & 0 & \cdots & 0 \\ C_i A_i L_i & C_i L_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_i A_i^{N-1} L_i & C_i A_i^{N-2} L_i & \cdots & C_i L_i \end{bmatrix},$$

$$\tilde{B}_i^N = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C_i B_i & 0 & \cdots & 0 \\ C_i A_i B_i & C_i B_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_i A_i^{N-1} B_i & C_i A_i^{N-2} B_i & \cdots & C_i B_i \end{bmatrix}.$$

If an SVO is designed for system S_A (denoted SVO$_A$) and the true plant is S_B, then SVO$_A$ is able to invalidate model S_A in less than N iterations if

$$\frac{1}{N} \sum_{k=1}^{N} \|y(k)\|^2 > \theta_{\min}. \tag{8}$$

V. Examples

In this section, we illustrate the concepts introduced in this paper with a couple of examples.

A. Example I

Consider the LTI discrete-time systems described by

$$S_A \begin{cases} x(k+1) = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(k), \\ y(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) + n(k), \end{cases}$$

$$S_B \begin{cases} x(k+1) = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(k), \\ y(k) = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix} x(k) + n(k), \end{cases} \tag{9}$$

where $|d| \leq 1$ and $|n| \leq 0.001$. Notice that system S_B can be seen as system S_A with a sensor failure.

Suppose we design an SVO for system S_A (denoted SVO$_A$), and that $|x(0)| \leq 1$. Further suppose that we simulate SVO$_A$ with the outputs coming from S_B. Then, we might want to ask: *What is the minimum intensity of the disturbances such that the SVO falsifies model S_A, with probability 1?* The answer to this question can be obtained using Theorem 3 and is illustrated in Fig. 1. Notice that in this case we only considered the disturbances intensity, while no assumptions have been posed regarding the intensity of the measurement noise.

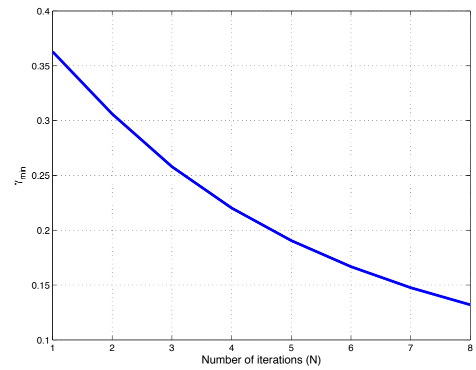

Fig. 1. Minimum disturbances intensity that guarantee the distinguishability between systems S_A and S_B in Example I.

Indeed, let us consider that we simulate system S_B with SVO$_A$. The results for 1000 Monte-Carlo runs are summarized in Table I. The green cells represent the combinations of γ_{min} and N such that the falsification of system S_A are guaranteed. Despite the fact that, in most of the cases analyzed in Table I, system S_A is falsified in more than 90% of the simulation runs, there are, however, worst-case disturbances that can help concealing the differences between the two systems, and thus it may happen that SVO$_A$ does not falsify S_A. Hence, although such disturbances may have a small probability of occurrence, they cannot be disregarded in worst-case approaches, such as the one presented herein.

TABLE I

Percentage of simulation runs that falsified system S_A in Example I.

	Number of iterations (N)					
γ_{min}	1	2	3	4	5	6
0.0001	3.7	30	51.4	66.9	76.8	82.9
0.0005	78.2	95.2	99	99.6	99.9	100
0.001	97.9	99.9	99.9	100	100	100
0.01	100	100	100	100	100	100
0.1	100	100	100	100	100	100
0.2	100	100	100	100	100	100
0.5	100	100	100	100	100	100

B. Example II

In this second example we consider a harder distinguishability problem. Indeed, let us define S_A and S_B as in Example I, but with the second element of the diagonal of the C matrix of system S_B being 0.9, rather than 0.6. In comparison with Example I, the C matrix of systems S_A and S_B are much similar. In fact, while Example I can illustrate a sensor loss-of-effectiveness of 40%, Example II can represent a sensor loss-of-effectiveness of only 10%. Repeating the design procedures as in Example I, we obtain the minimum intensity of the disturbances, depicted in Fig. 2, such that the SVO falsifies model S_A, with probability 1.

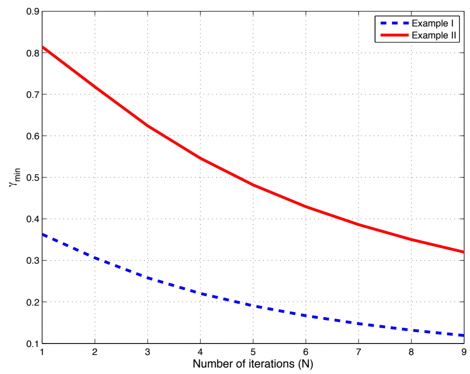

Fig. 2. Minimum disturbances intensity that guarantee the distinguishability between systems S_A and S_B in Examples I and II. The results from Example I are also shown for comparison purposes.

As expected, for the same intensity of the disturbances, we require a higher number of iterations to guarantee the distinguishability between the two systems, when compared to Example I. Alternatively, for the same number of iterations, we require a higher intensity of the disturbances to guarantee the distinguishability.

These results are further sustained by the Monte-Carlo simulations, summarized in Table II.

TABLE II

PERCENTAGE OF SIMULATION RUNS THAT FALSIFIED SYSTEM S_A IN EXAMPLE II.

	Number of iterations (N)					
$\gamma_{\min}$	1	2	3	4	5	6
0.0001	0	0	0	0	0	0
0.0005	0	0	0.9	2.3	4.4	6.2
0.001	0	6.5	19.8	32.3	42.9	50.8
0.01	84.1	96.8	99	100	100	100
0.1	100	100	100	100	100	100
0.2	100	100	100	100	100	100
0.5	100	100	100	100	100	100

VI. CONCLUSIONS

In this paper, we introduced the concept of absolutely distinguishable discrete dynamic systems, highlighting its applicability to linear time-invariant (LTI) models. We further demonstrated that, in general, a persistence of excitation condition is required on the exogenous disturbances. It turned out that this condition can be written as a lower bound on the intensity of the perturbations. Necessary and sufficient conditions for the distinguishability of two systems were derived, showing that, under mild assumptions, set-valued observers are adequate for model falsification of LTI systems. The theory was illustrated with a set of examples that demonstrate the applicability of the results presented.

REFERENCES

[1] R. Isermann, "Supervision, fault-detection and fault-diagnosis methods - an introduction," *Control Engineering Practice*, vol. 5, no. 5, pp. 639–652, 1997.

[2] P. Frank and X. Ding, "Survey of robust residual generation and evaluation methods in observer-based fault detection systems," *Journal of Process Control*, vol. 7, no. 6, pp. 403–424, 1997.

[3] R. Patton and J. Chen, "Observer-based fault detection and isolation: Robustness and applications," *Control Engineering Practice*, vol. 5, no. 5, pp. 671–682, 1997.

[4] M. Blanke, R. Izadi-Zamanabadi, S. Bogh, and C. Lunau, "Fault tolerant control systems – a holistic view," *Control Engineering Practice*, vol. 5, no. 5, pp. 693–702, 1997.

[5] A. Marcos, S. Ganguli, and G. Balas, "An application of $\mathcal{H}_\infty$ fault detection and isolation to a transport aircraft," *Control Engineering Practice*, vol. 13, pp. 105–119, 2005.

[6] R. Vidal, A. Chiuso, and S. Soatto, "Observability and identifiability of jump linear systems," in *Proceedings of the 41st IEEE Conference on Decision and Control*, 2002, pp. 3614–3619.

[7] H. Lou and P. Si, "The distinguishability of linear control systems," *Nonlinear Analysis: Hybrid Systems*, vol. 3, pp. 21–38, 2009.

[8] S. Fekri, M. Athans, and A. Pascoal, "Issues, progress and new results in robust adaptive control," *Int. Journal of Adaptive Control and Signal Processing*, vol. 20, pp. 519–579, 2006.

[9] M. Kuipers and P. Ioannou, "Multiple model adaptive control with mixing," *IEEE Transactions on Automatic Control*, vol. 55, no. 8, pp. 1822–1836, February 2010.

[10] P. Rosa, C. Silvestre, J. Shamma, and M. Athans, "Multiple-model adaptive control with set-valued observers," in *Proceedings of the 48th IEEE Conference on Decision and Control*, December 2009.

[11] M. Grewal and K. Glover, "Identifiability of linear and nonlinear dynamical systems," *IEEE Trans. on Automatic Control*, vol. 21, no. 6, pp. 833–837, 1976.

[12] J. DiStefano III, "On the relationships between structural identifiability and the controllability, observability properties," *IEEE Trans. on Automatic Control*, vol. 22, p. 652, 1977.

[13] J. DiStefano III and C. Cobelli, "On parameter and structural identifiability: Nonunique observability/reconstructibility for identifiable systems, other ambiguities and new definitions," *IEEE Trans. on Automatic Control*, vol. 25, no. 4, pp. 830–833, 1980.

[14] E. Walter, Y. Lecourtier, and J. Happel, "On the structural output distinguishability of parametric models, and its relations with structural identifiability," *IEEE Trans. on Automatic Control*, vol. 29, no. 1, pp. 56–57, 1984.

[15] T. Soderstrom, L. Ljung, and I. Gustavsson, "Identifiability conditions for linear multivariable systems operating under feedback," *Automatic Control, IEEE Transactions on*, vol. 21, no. 6, pp. 837 – 840, Dec. 1976.

[16] C. R. Rojas, J. S. Welsh, G. C. Goodwin, and A. Feuer, "Robust optimal experiment design for system identification," *Automatica*, vol. 43, no. 6, pp. 993 – 1008, 2007.

[17] A. Bazanella, M. Gevers, and L. Miskovic, "Closed-loop identification of mimo systems: a new look," *European Journal of Control*, vol. 16, no. 3, pp. 228 – 239, 2010.

[18] K. Poolla, P. Khargonekar, A. Tikku, J. Krause, and K. Nagpal, "A time-domain approach to model validation," *IEEE Trans. on Automatic Control*, vol. 39, no. 5, pp. 951–959, 1994.

[19] A. Johansson, "A sufficient condition for invalidation of linear state-space systems with uncertain time-varying parameters," *IEEE Trans. on Automatic Control*, vol. 54, no. 12, pp. 2915–2920, 2009.

[20] J. Shamma and K.-Y. Tu, "Set-valued observers and optimal disturbance rejection," *IEEE Transactions on Automatic Control*, vol. 44, no. 2, pp. 253–264, 1999.

[21] M. Baglietto, G. Battistelli, and L. Scardovi, "Active mode observation of switching systems based on set-valued estimation of the continuous state," *Int. J. of Robust and Nonlinear Control*, vol. 19, pp. 1521–1540, 2009.

[22] S. Keerthi and E. Gilbert, "Computation of minimum-time feedback control laws for discrete-time systems with state-control constraints," *IEEE Transactions on Automatic Control*, vol. 32, no. 5, pp. 432–435, 1987.

[23] R. Rockefeller, *Convex Analysis*. Princeton: Princeton University Press, 1970.