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Trends in Nonlinear Control

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I. Introduction

The report on an IEEE Control Systems Society Workshop held at the University of Santa Clara in 1986 reads

> *Techniques for the control of systems described by nonlinear mathematical models are difficult, but a major breakthrough occurred during the past decade with the development of techniques which solve such control problems as disturbance decoupling, input-output decoupling, and feedback linearization ....*

Since then, nonlinear control theory (and its applications) has undergone substantial developments and become one of the most active and important areas of research in the control systems community. There are several introductory and advanced textbooks devoted to nonlinear control theory, see *e.g.,* [74], [88], [38], [61], [55], [75], [39], [94], [49].

In turn, nonlinear control has been integrated into the standard graduate curricula in engineering and applied mathematics. In addition, nonlinear control theory is at the basis of the successful development and initiation of several research directions: it plays a fundamental role in the development of systems' biology, in the understanding of complex communication systems, power systems and cooperative systems, in the study of event driven and agent based systems, and in the development of an ever increasing number of industrial applications.

Nevertheless, we maintain that the conclusion of the 1986 Workshop that "techniques for the control of systems described by nonlinear mathematical models are difficult" is still accurate, although we may argue on the *meaning* of the word "difficult".

Nonlinear control theory embraces a large number of research areas, which use diverse tools and methods, each well-suited for specific problems. It is therefore extremely difficult to give a tutorial presentation which represent the joint effort of the international research community, and one has to follow personal inclinations. We have therefore decided to emphasize three research directions that (we believe) are important, both from a methodological perspective and from the applications point of view. As a consequence we have left aside several important topics, which would deserve equal attention, for example robust and adaptive control, optimal control, model predictive control, passivity- and energy-based control, variable structure control, differential geometric methods, Lyapunov design, anti-windup methods, singular perturbation methods, ...

The goal of this tutorial is therefore to illustrate three research themes that have undergone substantial developments in the past few years, and to highlight related open problems and possible avenues for future research. The paper is organized as follows. Section II discusses the role of invariant manifolds in nonlinear control design, with special attention to the problem of global observer design; Section III presents recent advances in the theory of hybrid systems, finally nonlinear digitally controlled systems are discussed in Section IV.

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II. Invariant manifolds in control and observer design – A. Astolfi

In this section we briefly (and informally) recall some prototypical, yet important, control problems, the solution of which requires the computation of invariant manifolds. For further detail on the considered problems the reader is referred to the given references.

A. Stabilization via backstepping

Backstepping [55], [61] (see also the recent results in [4]) is a constructive design method which is applicable to systems in feedback form. In its simplest formulation backstepping is applied to design global asymptotic stabilizers for systems described by equations of the form

$$\dot{x}_1 = f(x_1, x_2), \qquad \dot{x}_2 = u, \tag{1}$$

with $x = \mathrm{col}(x_1, x_2) \in (\mathbb{R}^n \times \mathbb{R})$, $u \in \mathbb{R}$ and $f(0,0) = 0$, and such that there exists a mapping $\alpha(x_1)$ such that the system

$$\dot{x}_1 = f(x_1, \alpha(x_1)) \tag{2}$$

has a globally asymptotically stable equilibrium at zero.

To construct the globally stabilizing control law it is noted that the set described by $x_2 - \alpha(x_1) = 0$ is a controlled invariant manifold for (1). The dynamics on the manifold are described by equation (2), whereas the dynamics *orthogonal*[1] to the manifold are described by

$$\frac{d}{dt}(x_2 - \alpha(x_1)) = u - \frac{\partial \alpha(x_1)}{\partial x_1} f(x_1, x_2),$$

[1] These dynamics can be used to describe attractivity properties of the invariant manifold.

and these can be assigned by a proper selection of u. It is precisely this selection that yields a control law asymptotically stabilizing the zero equilibrium. The stabilizing control law obtained using *classical* backstepping does not retain invariance of the above manifold, whereas invariance can be retained, while guaranteeing stability of the closed-loop system, exploiting the results in [8], [9].

B. Stabilization via forwarding

Forwarding [62], [41], [87], [92], [42], [25] is a constructive design procedure for the stabilization of cascaded nonlinear systems. In its simplest formulation forwarding is employed to control systems described by equations of the form

$$\dot{x}_1 = h(x_2), \qquad \dot{x}_2 = f(x_2) + g(x_2)u, \tag{3}$$

with $x = \mathrm{col}(x_1, x_2) \in (\mathbb{R} \times \mathbb{R}^n)$, $u \in \mathbb{R}$, $f(0) = 0$ and $h(0) = 0$. It is further assumed that the x_2-subsystem with $u = 0$ has a globally asymptotically stable, and locally exponentially stable, equilibrium for $x_2 = 0$. This implies that the overall system, with $u = 0$ is stable in the sense of Lyapunov [62]. To construct a Lyapunov function and hence, under certain detectability assumptions, a globally stabilizing state feedback law, it is worth noting that system (3) has a (controlled) invariant manifold described by $x_1 - \psi(x_2) = 0$, where $\psi(0) = 0$, and $\psi(x_2)$ solves the p.d.e.

$$h(x_2) - \frac{\partial \psi}{\partial x_2} f(x_2) = 0.$$

(In [62], [87] it is shown that, under the stated assumptions, such a p.d.e. has a globally defined solution.) For $u = 0$ the dynamics restricted to the manifold are described by $\dot{x}_2 = f(x_2)$, hence have a globally asymptotically stable equilibrium at $x_2 = 0$ by assumption. In addition, setting $\xi = x_1 - \psi(x_2)$ one obtains

$$\dot{\xi} = -\frac{\partial \psi}{\partial x_2} g(x_2)u,$$

which shows that, for $u = 0$, the dynamics *orthogonal* to the manifold are simply stable. Note that it is possible to assign the orthogonal dynamics with a proper selection of u and, similarly to what happens for backstepping, it is this selection that yields a stabilizing control law for the overall system.

C. Stabilization via immersion and invariance

The immersion and invariance methodology [8], [9] is a control design method to achieve asymptotic (and adaptive) stabilization (of an equilibrium) of a general nonlinear system. The main idea is to consider a system described by equations of the form

$$\dot{x} = f(x) + g(x)u, \tag{4}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, a target system, with a globally asymptotically stable equilibrium, described by

$$\dot{\xi} = \alpha(\xi), \tag{5}$$

with $\xi \in \mathbb{R}^p$, and $p < n$, and a mapping

$$x = \pi(\xi) \tag{6}$$

such that

$$f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \frac{\partial \pi}{\partial \xi}\alpha(\xi). \tag{7}$$

for some mapping c. The condition (6), together with the stability properties of the target system, implies that the zero equilibrium of the controlled system can be asymptotically stabilized, whereas the p.d.e. (7) implies that the set $x - \pi(\xi) = 0$ is a controlled invariant manifold for the composite system (4)-(5). The dynamics on the manifold are described by equation (5), thus have a globally asymptotically stable equilibrium, whereas the dynamics *orthogonal* to the manifold are described by

$$\begin{array}{rl} \dot{z} = & f(z+\pi(\xi)) + g(z+\pi(\xi))\,c(z+\pi(\xi)) - \frac{\partial \pi}{\partial \xi}\alpha(\xi) \\ = & f(z+\pi(\xi)) + g(z+\pi(\xi))\,c(z+\pi(\xi)) - \\ & f(\pi(\xi)) + g(\pi(\xi))\,c(\pi(\xi)) \end{array}$$

and can be (partly) assigned via the selection of the mapping c, which is then used to construct a stabilizing feedback for system (4).

D. The regulator problem

The regulator problem for general nonlinear systems has been widely studied in the last decades, see *e.g.,* [17], [37]. The problem can be posed, in its simplest form, as follows. Given a nonlinear system described by equations of the form

$$\dot{x} = f(x) + g(x)u + p(x)w, \qquad y = h(x) + q(w), \tag{8}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, exogenous input $w \in \mathbb{R}^q$, $f(0) = 0$, and $h(0) = 0$. Assume that w is such that

$$\dot{w} = s(w) \tag{9}$$

with $s(0) = 0$, and that the zero equilibrium of the w system is Poisson stable. Find a control law such that, when $w = 0$ the zero equilibrium of the closed-loop system is locally asymptotically stable, and when $w \neq 0$ the closed-loop system is such that, for all initial conditions (x_0, w_0) close to $(0, 0)$,

$$\lim_{t \to \infty} y(t) = 0.$$

In [17], [38], [40] it has been shown that the problem can be solved, by a feedback of x and w, provided that there is a mapping π which solves the equations p.d.e.

$$\begin{array}{rcl} \frac{\partial \pi}{\partial w} s(w) & = & f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w, \\ 0 & = & h(\pi(w)) + q(w), \end{array} \tag{10}$$

for some mapping c, and that the linearized system around the origin is controllable. Equation (10) implies that the set $x - \pi(\xi) = 0$ is a controlled invariant manifold for the composite system (8)-(9). Moreover, the dynamics on the manifold are described by equation (9), whereas the dynamics *orthogonal* to the manifold can be made locally exponentially stable under the stated controllability assumption. Note that the controller achieving (local) asymptotic regulation is the *superposition* of a controller that renders the manifold invariant and of a controller that *stabilizes* the orthogonal dynamics.

E. Observer design

The problem of observer design for general nonlinear systems has been addressed from several points of views [51], [52], [21], [93], [29], [31], [48], [5], [30], [28], [11], [60], [10]. All these works show that invariant manifolds play a crucial role in observer design.

The classical ideas of Luenberger have been given a nonlinear counterpart, for systems without inputs, in [47]. The main idea therein is that, for a given system with output described by equations of the form

$$\dot{x} = f(x), \qquad y = h(x), \tag{11}$$

with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, a local observer is a system described by equations of the form

$$\dot{\xi} = F\xi + Gy \qquad \hat{x} = T^{-1}(\xi) \tag{12}$$

where $\xi \in \mathbb{R}^n$, F is a Hurwitz matrix, and $T(x)$ is an invertible mapping solution of the p.d.e.

$$FT(x) + Gh(x) = \frac{\partial T}{\partial x} f(x). \tag{13}$$

Existence of a solution for the p.d.e. (13) implies that the composite system (11)-(12) has an invariant manifold described by $\xi - T(x) = 0$. The dynamics on the manifold are described by $\dot{x} = f(x)$, whereas the dynamics *orthogonal* to the manifold are described by $\dot{\eta} = F\eta$, hence the *orthogonal* behaviour can be assigned selecting the matrix F, *i.e.,* the observer dynamics.

Existence of solutions for the p.d.e. (13) and invertibility of the mapping T are guaranteed by non-resonance conditions in [47]. These conditions have been relaxed in [53], and a global version of these results has been given in [3], under specific observability and completeness assumptions (see also [50] for similar ideas). Note that, in all these works the observer has linear dynamics, the (local or global) existence and invariance of the manifold is guaranteed by non-resonance conditions or completeness assumptions, the attractivity of the manifold is implied by stability of the observer dynamics, and (left) invertibility of the mapping T is guaranteed by local structural properties in [47], [53], and by a delicate dimensional argument in [3].

Alternatively, one could consider a parameterized description of the manifold (hence there is no existence issue), and select the observer dynamics to render the manifold invariant. The crucial issue is therefore the attractivity of the manifold, which has to be achieved by a proper selection of the observer dynamics, which are in general nonlinear and partly imposed by the invariance condition. In what follows we describe this approach (see [8] for further details and illustrative examples) for a class of nonlinear systems.

Consider nonlinear, time-varying systems described by equations of the form

$$\begin{aligned} \dot{\eta} &= f_1(\eta, y, t), \\ \dot{y} &= f_2(\eta, y, t), \end{aligned} \tag{14}$$

where $\eta \in \mathbb{R}^n$ is the unmeasured part of the state and $y \in \mathbb{R}^m$ is the measurable part of the state. It is assumed that the vector fields $f_1(\cdot)$ and $f_2(\cdot)$ are forward complete, *i.e.,* trajectories starting at time t_0 are defined for all times $t \geq t_0$. (This assumption can be removed under certain conditions, see [46]).

The dynamical system

$$\dot{\xi} = \alpha(\xi, y, t), \tag{15}$$

with $\xi \in \mathbb{R}^p$, $p \geq n$, is called an observer for the system (14), if there exist mappings $\beta : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$ and $\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$ that are left-invertible (with respect to their first argument)[2] and such that the manifold

$$\mathcal{M} = \{ (\eta, y, \xi, t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \ : \ \beta(\xi, y, t) = \phi(\eta, y, t) \} \tag{16}$$

has the following properties.

(i) All trajectories of the extended system (14)-(15) that start on the manifold $\mathcal{M}$ remain there for all future times, *i.e.,* $\mathcal{M}$ is positively invariant.

(ii) All trajectories of the extended system (14)-(15) that start in a neighbourhood of $\mathcal{M}$ asymptotically converge to $\mathcal{M}$.

The above definition implies that an asymptotically converging estimate of the state η is given by

$$\hat{\eta} = \phi^{\mathrm{L}}(\beta(\xi, y, t), y, t),$$

where $\phi^{\mathrm{L}}(\cdot)$ denotes a left-inverse of $\phi(\cdot)$. Note that the state estimation error $\hat{\eta} - \eta$ is zero on the manifold $\mathcal{M}$. Moreover, if the property (ii) holds for any $(\eta(t_0), y(t_0), \xi(t_0), t_0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}$ then (15) is a *global* observer for the system (14).

We now present a general tool for constructing nonlinear (reduced-order) observers.

Theorem 1: Consider the system (14)-(15) and suppose that there exist $\mathcal{C}^1$ mappings $\beta(\xi, y, t) : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$ and $\phi(\eta, y, t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$, with a left-inverse $\phi^{\mathrm{L}} : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$, such that the following hold.

(A1) For all y, ξ and t, $\beta(\xi, y, t)$ is left-invertible with respect to ξ and $\det(\frac{\partial \beta}{\partial \xi}) \neq 0$.

(A2) The system

$$\begin{aligned} \dot{z} = &-\frac{\partial \beta}{\partial y} \left(f_2(\hat{\eta}, y, t) - f_2(\eta, y, t) \right) \\ &+ \left.\frac{\partial \phi}{\partial y}\right|_{\eta=\hat{\eta}} f_2(\hat{\eta}, y, t) - \frac{\partial \phi}{\partial y} f_2(\eta, y, t) \\ &+ \left.\frac{\partial \phi}{\partial \eta}\right|_{\eta=\hat{\eta}} f_1(\hat{\eta}, y, t) - \frac{\partial \phi}{\partial \eta} f_1(\eta, y, t) + \left.\frac{\partial \phi}{\partial t}\right|_{\eta=\hat{\eta}} - \frac{\partial \phi}{\partial t}, \end{aligned} \tag{17}$$

[2] A mapping $\psi(x, y, t) : \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$ is left-invertible (with respect to x) if there exists a mapping $\psi^{\mathrm{L}} : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^l$ such that $\psi^{\mathrm{L}}(\psi(x, y, t), y, t) = x$, for all $x \in \mathbb{R}^l$ (and for all y and t).

with $\hat{\eta} = \phi^{\mathrm{L}}(\phi(\eta, y, t)+z)$, has a (globally) asymptotically stable equilibrium at $z = 0$, uniformly in η, y and t.

Then the system (15) with

$$\alpha(\xi, y, t) = -\left(\frac{\partial \beta}{\partial \xi}\right)^{-1}\left(\frac{\partial \beta}{\partial y} f_2(\hat{\eta}, y, t) + \frac{\partial \beta}{\partial t} - \left.\frac{\partial \phi}{\partial y}\right|_{\eta=\hat{\eta}} f_2(\hat{\eta}, y, t) - \left.\frac{\partial \phi}{\partial \eta}\right|_{\eta=\hat{\eta}} f_1(\hat{\eta}, y, t) - \left.\frac{\partial \phi}{\partial t}\right|_{\eta=\hat{\eta}}\right), \tag{18}$$

where $\hat{\eta} = \phi^{\mathrm{L}}(\beta(\xi, y, t), y, t)$, is a (global) observer for the system (14).

Theorem 1 provides an implicit description of the observer dynamics (15) in terms of the mappings $\beta(\cdot)$, $\phi(\cdot)$ and $\phi^{\mathrm{L}}(\cdot)$ which must then be selected to satisfy (A2). (Note, however, that the function $\alpha(\cdot)$ in (18) renders the manifold $\mathcal{M}$ invariant *for any* mappings $\beta(\cdot)$ and $\phi(\cdot)$.) As a result, the problem of constructing an observer for the system (14) is *reduced* to the problem of rendering the system (17) asymptotically stable by assigning the functions $\beta(\cdot)$, $\phi(\cdot)$ and $\phi^{\mathrm{L}}(\cdot)$. This non-standard stabilisation problem can be extremely difficult to solve since, in general, it relies on the solution of a set of partial differential equations (or inequalities). However, in many cases of practical interest, these equations turn out to be solvable (see [8]).

To illustrate the proposed approach, and some of its shortcomings, consider a class of nonlinear systems described by equations of the form (note that this is a particular instance of the class of systems described by equations (14))

$$\begin{array}{rcl} \dot{y} & = & f(y, u) + \Phi(y)\eta, \\ \dot{\eta} & = & h(y, u) + A(y)\eta, \end{array} \tag{19}$$

where $y \in \mathbb{R}^m$ is the measured part of the state, $\eta \in \mathbb{R}^n$ is the unmeasured part of the state (which may also include unknown parameters, *i.e.,* equations of the form $\dot{\eta}_i = 0$), and u is an external, measurable signal.

Following the approach described above (see also [45]) we define a parameterized manifold as the set described by

$$z = \hat{\eta} - \eta = \xi + \beta(y) - \eta = 0,$$

where $\xi \in \mathbb{R}^n$ is the observer state and $\beta(\cdot) : \mathbb{R}^m \to \mathbb{R}^n$ is a mapping to be determined. Defining the observer as in Theorem 1 yields

$$\dot{\xi} = h(y, u) + A(y)\hat{\eta} - \frac{\partial \beta}{\partial y}\left(f(y, u) + \Phi(y)\hat{\eta}\right)$$

and the *error system* (17) is given by

$$\dot{z} = \left[A(y) - \frac{\partial \beta}{\partial y}\Phi(y)\right] z.$$

To complete the design it is therefore necessary to assign the function $\beta(\cdot)$ so that the above system has a uniformly (asymptotically, if convergence of the estimation error is required) stable equilibrium at zero.

Note that, to be able to solve the observer design problem, we expect to be able to find an output injection matrix $B(\cdot)$ such that the system $\dot{z} = [A(y) - B(y)\Phi(y)]\, z$ has a uniformly (asymptotically) stable equilibrium at zero.

This implies that the observer design problem is solved if it is possible to find $\beta(\cdot)$ such that

$$\frac{\partial \beta}{\partial y} = B(y). \tag{20}$$

Note that, if the $y \in \mathbb{R}$ is one, equation (20) has always a solution, and this can be obtained, at least formally, with an integration. However, if $y \in \mathbb{R}^m$ with $m > 1$, it may not be possible to find a $\beta(\cdot)$ such that equation (20) holds, *i.e.,* $B(\cdot)$ may not be a Jacobian matrix.

This obstacle can be overcome introducing a filterer output and exploiting dynamic scaling, thus obtaining an observer of dimension $n + m + 1$, see [44]. Therein the idea is to employ the output filter to ensure that

$$\frac{\partial \beta}{\partial y} = \Psi(y, \hat{y}), \tag{21}$$

where $\hat{y}$ is the filtered output and $\Psi(\cdot)$ is such that $\Psi(y, y) = B(y)$, and then use dynamic scaling to compensate for the mismatch between y and $\hat{y}$. The obvious gain from this modification is that $\Psi(\cdot)$ can be chosen so that (21), in contrast with (20), is easily solvable.

We complete this section noting that the use of filtered measurements is standard in adaptive control, see for example [86], whereas dynamic scaling has been introduced in [78] (see also [79]), and further exploited in observer desing in [54], [60], [10], [43], [2]. The use of these tools in the context of observer design for general classes of nonlinear systems, as outlined above, is a current area of research.

III. HYBRID METHODS FOR NONLINEAR CONTROL – A.R. TEEL

A. Introduction

Ad-hoc hybrid control methods for nonlinear systems have been around for decades. However, only recently has the maturity of hybrid dynamical systems theory begun to approach that of classical nonlinear systems. These developments have made hybrid feedback control an emerging, multi-faceted area. In this section, we use examples to illustrate various hybrid control ideas. Space limitations preclude a full tutorial presentation on hybrid dynamical systems.

Hybrid control comprises dynamic feedback whose states can jump when certain conditions are met. The mathematical equations of a general hybrid controller for a continuous, nonlinear control system $\dot{x} = f(x, u)$ have the form

$$\begin{array}{rcll} u & = & \kappa(x, \eta) & \\ \dot{\eta} & = & \varphi(x, \eta) & (x, \eta) \in C \\ \eta^+ & \in & G(x, \eta) & (x, \eta) \in D \end{array} \tag{22}$$

where η denotes the state of the controller. When the controller state jumps, its new value, denoted η^+, must belong to the set $G(x, \eta)$. The sets C and D indicate where flowing and jumping, respectively, are allowed. The set C is called the *flow set* while D is called the *jump set*. They are taken to be closed sets. The functions φ and κ, which, together with f, govern the continuous evolution of the control system, are continuous. These functions comprise the *flow map*.

The set-valued mapping G, the *jump map*, is taken to be locally bounded with a closed graph. By closed graph, we mean that the set $\{(y,x,\eta) : y \in G(x,\eta)\}$ is closed. The closed-loop interconnection of the hybrid controller with the nonlinear control system constitutes a hybrid dynamical system. The indicated regularity conditions on κ, φ, G, C, and D ensure that, when the closed-loop hybrid system has an asymptotically stable compact set, that set is robustly, asymptotically stable. For more details on this robustness, as well as on the notions of solution and asymptotic stability for hybrid systems, see [33] and [18] for example.

The interest in hybrid control for nonlinear systems comes from multiple sources. For one, hybrid feedback can be used to provide efficient solutions to local and global feedback stabilization problems that cannot be solved by classical feedback control. In addition, the added flexibility of hybrid control sometimes allows the designer to achieve closed-loop responses not possible with classical feedback control. Moreover, hybrid feedback can be used to solve nonlinear control problems that are simply more difficult to solve with classical feedback control.

Other control problems where a good knowledge of hybrid systems theory is helpful is in the design and analysis of nonlinear networked control systems, in the control and analysis of dynamical systems that exhibit hybrid behavior, like mechanical systems that experience impacts, and in the design and analysis of synchronization algorithms that involve impulsive behavior.

B. *Efficient circumvention of obstacles to robust, local feedback stabilization*

1) Background results: One of the significant challenges in nonlinear control design for continuous-time systems is captured in Brockett's famous necessary condition [15] for stabilization of an equilibrium point by time-invariant, continuous feedback. Ryan [83] showed that Brockett's condition is also necessary for *robust* stabilization by time-invariant, locally bounded feedback for a large class of systems. By robust stabilization we mean that local asymptotic stability of the equilibrium point should be preserved in the presence of measurement noise whose magnitude is limited by a sufficiently small function of the state's distance to the equilibrium. Results closely related to that of [83] can be found in [26] and [59].

Obstacles to robust, local stabilization can be overcome with time-varying feedback. This approach was pioneered by Coron in the early 1990's [23], [24]. See also [77]. Another approach is to use hybrid feedback, as proposed in [36], and also used later in [81]. As is the case for periodic time-varying feedback, hybrid feedback satisfying the regularity properties listed below (22) induces asymptotic stability that is automatically robust to small perturbations, including measurement noise [33],[18]. Hybrid feedback is distinct from discontinuous feedback. Discontinuous feedback has been used for nonholonomic control systems, like in [14]. For such systems, the achieved stabilization cannot be robust to arbitrarily small measurement noise according to the results of [83], [26], and [59].

2) The nonholonomic integrator as an illustration: To illustrate the use of hybrid feedback, consider the stabilization problem for Brockett's nonholonomic integrator system

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_1u_2 - x_2u_1\ .\end{aligned}$$

The controller has a dynamic state q taking values in the set $\{1,2\}$. The feedback control law has the form $u = \kappa_q(x)$ where κ_q is defined on a closed set C_q. Jumps in the state q are permitted on the closed sets $D_q := \overline{\mathbb{R}^3 \setminus C_q}$ and satisfy the update rule $q^+ = 3 - q$, which toggles q. In particular, when $q = 1$, $3 - q = 2$ and when $q = 2$, $3 - q = 1$. Let $\rho \in (1,2)$ and consider the definitions

$$\begin{aligned}C_1 &:= \left\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \le \rho|x_3|\right\}\\ C_2 &:= \left\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \ge |x_3|\right\}\\ C &:= \bigcup_{q\in\{1,2\}} C_q \times \{q\}\\ D &:= \bigcup_{q\in\{1,2\}} D_q \times \{q\}\ ,\end{aligned}$$

$\kappa_1(x) := (1,0)^T$, and

$$\kappa_2(x) := -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{3x_3}{x_1^2 + x_2^2}\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}\ .$$

This hybrid controller is similar to the one proposed in [36]. There are various ways to see that it produces a closed-loop system having the set $\{(x,q) : x = 0, q \in \{1,2\}\}$ globally asymptotically stable. We will give an interpretation based on the notion of "patchy control Lyapunov functions", as introduced in [32]. Notice that, with $\varepsilon > 0$, the function

$$V_1(x) := -x_1 + (\sqrt{\rho} + \varepsilon)\sqrt{|x_3|}$$

is positive definite and proper on C_1. Moreover it is continuously differentiable on an open set containing $C_1 \setminus \{0\}$ and on this set satisfies

$$\langle \nabla V_1(x), f(x,\kappa_1(x))\rangle \le -1 + 0.5(\sqrt{\rho} + \varepsilon)\sqrt{\rho} < 0$$

where the last inequality comes from using $\rho \in (1,2)$ and taking $\varepsilon > 0$ sufficiently small. Thus, while there is no reason to believe that solutions to the system $\dot{x} = f(x,\kappa_1(x))$, $x \in C_1$ are complete, the solutions of this system behave as though the origin in $\mathbb{R}^3$ is globally asymptotically stable up to the point where they fail to be continuable. This notion has been called "preasymptotic stability" in [18]. The story for the solutions of $\dot{x} = f(x,\kappa_2(x))$, $x \in C_2$ is similar. The function $V_2(x) := 0.5x^Tx$ is positive definite, proper, continuously differentiable, and satisfies

$$\langle \nabla V_2(x), f(x,\kappa_2(x))\rangle = -x_1^2 - x_2^2 - 3x_3^2 \qquad \forall x \in C_2\ .$$

Thus, the solutions of $\dot{x} = f(x,\kappa_2(x))$, $x \in C_2$ behave as though the origin in $\mathbb{R}^3$ is globally asymptotically stable up to the point where they fail to be continuable, if such points exist.

The properties established above are not enough to assert global asymptotic stability in the hybrid closed-loop system, since it may be possible to cycle back and forth between the two modes without converging to the origin. However, this is ruled out by one additional key property induced by κ_2. Namely, solutions of $\dot{x} = f(x, \kappa_2(x))$, $x \in C_2$ that start in D_1 but not at the origin do not reach D_2, except perhaps at the origin. Thus, after there is a jump from mode 1 to mode 2, there are no additional jumps until the system reaches the origin. This and the other facts above establish global asymptotic stability. A general formulation of the supervisor problem for a finite family of (hybrid) controllers can be found in [85].

C. Efficient circumvention of obstacles to robust, global feedback stabilization

Even when robust, local stabilization by classical feedback is possible, robust, global stabilization by classical feedback may be impossible. This phenomenon occurs for the problem of asymptotically stabilizing a point on a circle.

Consider the constrained control system

$$\dot{x} = u \cdot (x_2, -x_1)^T \qquad x \in S^1 := \left\{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \right\}$$

and the objective of globally asymptotically stabilizing the point $x = (1,0)^T$. Notice that the choice $u = x_2$ almost globally asymptotically stabilizes this point. The point $(-1,0)^T$ is excluded from the basin of attraction. Similarly, the choice $u = x_1$ almost globally asymptotically stabilizes the point $(0,-1)^T$. In this case, the point $(0,1)^T$ is excluded from the basin of attraction.

We consider the effect of using a hybrid controller that switches between the two controllers mentioned above. The controller will have a state q taking values in $\{1,2\}$ and the update rule for q will be $q^+ = 3 - q$, which toggles q. The control will be $u = \kappa_q(x) := x_q$. Let $\widetilde{C}_2$ be a closed subset of the unit circle covering most of the unit circle but not a small interval surrounding the point $(-1,0)^T$ and let C_2 be similar to but a little bigger than $\widetilde{C}_2$. For example, take $\widetilde{C}_2 := \left\{ x \in S^1 : |x_2| \geq -0.5 x_1 \right\}$ and $C_2 := \left\{ x \in S^1 : |x_2| \geq -0.25 x_1 \right\}$. Then the system

$$\dot{x} = \kappa_2(x) \cdot (x_2, -x_1)^T \ , \ x \in C_2 \tag{23}$$

has the point $(1,0)^T$ asymptotically stable and all complete solutions converge to this point. Next, take $\widetilde{C}_1$ to be a closed subset of the unit circle covering most of the unit circle but not a small interval surrounding $(0,1)^T$. For example, take $\widetilde{C}_1 := \left\{ x \in S^1 : |x_1| \geq 0.25 x_2 \right\}$. Then take $C_1 := \overline{\widetilde{C}_1 \setminus \widetilde{C}_2}$. Now, we note that C_1 has no points near $(1,0)^T$ and the system $\dot{x} = \kappa_1(x) \cdot (x_2, -x_1)^T$, $x \in C_1$ has no complete solutions. While it may seem counterintuitive, this can be viewed as a stabilizer for the point $(1,0)^T$ up to the point where solutions cannot be continued. Finally, define $D_2 := \overline{S^1 \setminus C_2}$ and $D_1 := \overline{S^1 \setminus C_1}$. It remains to show that a finite number of switches can occur. Suppose there is a switch from mode 1 to mode 2. In other words, $x \in D_1$. Now, the system (23) is such that solutions starting in D_1 do not ever reach D_2. So it is not possible for the mode to switch back to mode 1 from mode 2. This establishes global asymptotic stability of the set $\{(x_1, x_2, q) : (x_1, x_2) = (1,0) \ , \ q \in \{1,2\}\}$.

D. Systematic hybrid design tools for robust, global feedback stabilization

The development of hybrid control algorithms has followed the natural progression from ad-hoc algorithms for specific systems toward systematic design tools. For example, the ideas in [36], which applied to the nonholonomic integrator in particular, gave rise to algorithms for a more general class of nonholonomic systems and, eventually, to a result on global hybrid stabilization that applies to all asymptotically controllable nonlinear systems. Various versions of such general-purpose hybrid control algorithms have been developed, including one [82] based on the patchy vector fields of Ancona and Bressan [1], and another captured in the notion of a smooth patchy control Lyapunov function [32], which extends the classical notion of a control Lyapunov function [7], [89] to hybrid control. Both of these general approaches to hybrid stabilization work by guaranteeing eventual monotonicity in the switches among logical modes of the controller, as was the case for the nonholonomic integrator and global stabilization of a point on the circle discussed above.

E. Non-classical responses

Researchers have used reset control systems over the years to achieve closed-loop responses that are not possible to achieve with classical feedback control. A reset control system is a type of hybrid control system. Recent examples of reset control systems include [12] for linear systems and [16] and [34] for nonlinear systems. Perhaps the first reset control system was designed by Clegg via op amps and diodes, as documented in [22]. The Clegg integrator acts like a normal integrating circuit until its input changes sign, at which point the state of the Clegg integrator jumps to zero. In this way, the Clegg integrator forces its state to have the same sign as its input. A simple generalization of the Clegg integrator is captured in the notion of a first-order reset element (FORE) that behaves like a first-order linear system as long as the input to the system and the state of the system have the same sign. As an illustration, consider the closed-loop interconnection of a first order plant $\dot{x}_p = \lambda_p x_p + b_p u_p$ with a first order reset element $\dot{x}_r = \lambda_r x_r + b_r u_r$ via the interconnection conditions $u_r = -x_p$, $u_p = x_r$. We use the jump rule $x_p^+ = x_p$, $x_r^+ = 0$ so that the state of the first-order system is reset to zero when it is deemed appropriate. We assume that $b_r > 0$ so that when $x_r = 0$ and $x_p \neq 0$, i.e., after a jump that is not to the origin,

$$\frac{d}{dt}(x_r x_p) = \dot{x}_r x_p = -b_r x_p^2 < 0 \ . \tag{24}$$

Then we let $\varepsilon > 0$ and define the flow and jump sets

$$\begin{array}{rcl} C & := & \left\{ (x_p, x_r) : x_p x_r \leq \varepsilon x_p^2 \right\} \\ D & := & \left\{ (x_p, x_r) : x_p x_r \geq \varepsilon x_p^2 \right\} \ . \end{array}$$

Due to (24) and the definition of the jump map, after one jump the flow is restricted to the second and fourth quadrant and subsequent jumps come about due to x_p reaching zero. We have taken the flow set to be a little larger than just the second and fourth quadrants to prevent a jump from D back to D. Indeed, with our definitions of C and D, jumps from D send the state to points in $C \setminus D$ or else to the origin. Suppose the parameters of the system are such that the closed-loop system matrix

$$A := \left[\begin{array}{cc} \lambda_p & b_p \\ -b_r & \lambda_r \end{array} \right]$$

either has complex eigenvalues, eigenvalues with negative real part, or no eigenvectors in C. These conditions guarantee that the origin of $\dot{x} = Ax$, $x \in C$ is "preasymptotically stable" in the sense that if solutions start near the origin they remain near the origin and complete solutions, if there are any, converge to the origin. So, taking jumps and flows into account together, a jump either moves the state to the origin, where it stays forever, or else it moves the state to C from which the next time the state reaches D we have $x_p = 0$ and then the state jumps to the origin. This establishes that the origin is globally asymptotically stable. In fact, as long as A has no eigenvectors in C, the state reaches the origin in finite time.

F. Other problems

There are several other nonlinear control problems for which hybrid control provides novel solutions. For example, hybrid control is used to overcome certain limitations in classical adaptive control in [35]. It is used to switch robustly from a globally stabilizing controller to a locally stabilizing controller when the closed-loop system reaches a neighborhood of the equilibrium point in [80]. Hybrid systems ideas also appear in the problem of synchronizing multiple clocks via impulsive control, which is related to the study of firefly synchronization in [63]. We discuss the latter two problems here as illustrations.

1) Uniting local and global controllers: Consider the nonlinear control system $\dot{x} = f(x,u)$ and suppose two controllers have been found: $u = \kappa_2(x)$, defined on a closed neighborhood of the origin C_2, which locally asymptotically stabilizes the origin and for which all complete solutions of $\dot{x} = f(x,\kappa_2(x))$, $x \in C_2$ converge to the origin; and $u = \kappa_1(x)$, defined on $\mathbb{R}^n$, which globally asymptotically stabilizes the origin. Let $D_1 \subset C_2$ be such that solutions of $\dot{x} = f(x,\kappa_2(x))$ starting in D_1 do not reach the boundary of C_2. Then define $C_1 := \overline{\mathbb{R}^n \setminus D_1}$ and $D_2 := \overline{\mathbb{R}^n \setminus C_2}$. Define the jump rule for the hybrid system to be $q^+ = 3 - q$, which toggles q. Like for the nonholonomic integrator, we note that for each mode q, the solutions of $\dot{x} = f(x,\kappa_q(x))$, $x \in C_q$ behave as though the origin is asymptotically stable as long as solutions exist. Still, we must rule out the possibility of cycling back and forth between the two modes. We note that for a switch from mode 1 to mode 2 the state x must reach D_1. But then from D_1 solutions of $\dot{x} = f(x,\kappa_2(x))$, $x \in C_2$ cannot reach the boundary of C_2 and thus cannot reach D_2. So, after a switch from mode 1 to mode 2, no additional switches are possible. This establishes that the set $\{(x,q) : x = 0\ ,\ q \in \{1,2\}\}$ is globally asymptotically stable.

2) Synchronizing two clocks: Consider the problem of designing small, nonnegative, impulsive controls to synchronize two clocks. The states of the clocks are denoted x_1 and x_2, they take values in the interval $[0,1]$, and their continuous behavior is governed by the equations $\dot{x}_i = 1$. Thus, the flow set is $C := [0,1] \times [0,1]$ and the jump set is $D := (1 \times [0,1]) \cup ([0,1] \times \{1\})$. The impulsive controls u, which can take values in the interval $[0,\varepsilon]$ where $\varepsilon > 0$, appear in the jump equations for the clock states:

$$x_i^+ \in \begin{cases} x_i + u_i & \text{when } x_i + u_i < 1 \\ 0 & \text{when } x_i + u_i > 1 \\ \{0,1\} & \text{when } x_i + u_i = 1\ . \end{cases}$$

Consider the function

$$V(x) := \min\{|x_1 - x_2|\ ,\ 1 + k - |x_1 - x_2|\}$$

where $k > 0$. Observe that V remains constant during flows. At jumps, at least one of the clock variables is equal to one. Let $x \in D$ and, without loss of generality, assume that $x_1 = 1$. Then

$$V(x) = \min\{1 - x_2\ ,\ k + x_2\}$$

and

$$V(x^+) = \min\left\{x_2^+\ ,\ 1 + k - x_2^+\right\}\ .$$

If $x_2^+ = 0$ then the clocks have synchronized and will remain that way for all future times as long as $u_i > 0$ when $x_i = 1$ for $i = 1,2$. If $x_2^+ \neq 0$ then $x_2^+ = x_2 + u_2$. In this case, in order for V to be decreasing at jumps, it is enough to have that $x_2 < (1-k)/2$ implies $x_2 + u_2 < k + x_2$ and that $x_2 > (1-k)/2$ implies $x_2 + u_2 > k + x_2$. For example, we can take $u_i = 0$ for $x_i < (1-k)/2$ and $u_i = 2k$ for $x_i > (1-k)/2$. For $x_i = (1-k)/2$ we can take $u_i \in \{0, 2k\}$. Using the invariance principle-based stability results for hybrid systems given in [84], it follows that the compact set $\{x \in C : x_1 = x_2\}$ is globally asymptotically stable, i.e., global synchronization is achieved. Picking u_i to be a continuous function of x_i would result in almost global synchronization. For example, picking $u_i = \varepsilon x_i$ results in almost global synchronization via continuous, impulsive control, as can be established using $k = \varepsilon/(2+\varepsilon)$ in the function V defined above.

IV. DIGITALLY CONTROLLED SYSTEMS – D. NEŠIĆ

At the end of the 20^{th} century we have witnessed a rapid development of digital technology that now permeates manufacturing and process industries, mining industry, defence, agriculture, transportation, as well as domestic appliances. In the context of control engineering, digital technology offers cheaper and more flexible controller realization platforms, standardized equipment that is easier to install and maintain and it often leads to reduced weight and volume of the overall system that is essential in certain applications, such as transportation. Moreover, digital technology is often more reliable for controller implementation than its analog counterparts and it typically requires less energy for its operation.

The numerous advantages of digital technology over its analogue counterparts have lead to the current domination of *computer controlled systems (CCS)* in majority of control engineering applications. Moreover, emerging digital technologies have lead to novel control architectures and paradigms, such as the *networked control systems (NCS)*, in which the control loop is closed via a local area network (LAN) that may service a range of other users besides the sensors and actuators in the closed loop. Drive-by-wire and fly-by-wire technologies that are respectively used in the automotive and aerospace industries are prime examples of NCS.

While CCS and NCS offer numerous advantages as outlined above, their design is harder than that of the classical continuous-time control systems, and especially so in the context of nonlinear systems that this paper concentrates on. The reason for this is that sampling and quantization that are intrinsic in digital technology often can not be ignored when designing a CCS or NCS. Moreover, as NCS exhibit communication bottlenecks within the control loop, effects of delays and dropouts must also be taken into account to ensure satisfactory performance of the closed loop. These issues complicate controller design for nonlinear plants and we will need years of coordinated research for this area to mature.

We overview some of the recent developments and trends in the area of nonlinear CCS and NCS. As it is impossible to present a comprehensive survey of all work done in this area, we concentrate on results that either follow from or are closely related to our own work. While research on linear CCS and NCS is related to this overview and will be cited when appropriate, we focus only on results that apply to nonlinear plants and systems.

A. Computer control systems

A wealth of design approaches and methodologies are available for linear CCS [20], such as the emulation, discrete-time design and sampled-data design method. Recently, good progress has been made on emulation design and discrete-time design for nonlinear CCS. We summarize below some of these developments.

1) Emulation: The emulation consists of first designing a controller for the continuous-time plant model (we ignore sampling during the design) and then discretizing the controller and implementing it digitally with fast sampling. This method is simpler to use but it may not be always feasible if the required sampling rate is faster than the fastest achievable sampling with the available hardware. Moreover, for slow sampling rates the performance of the system may not be satisfactory and the system may even become unstable. Consider a general nonlinear plant:

$$\dot{x}_P = f_P(x_P, u, w), \quad y = g_P(x_P) , \tag{25}$$

where x_P, u, y and w are respectively the plant state, control input, measured output and disturbance[3]. The first step in the emulation procedure is to design a continuous-time controller that stabilizes plant (25) in some sense (e.g. UGES, UGAS, ISS or L_2):

$$\dot{x}_C = f_C(x_C, y), \quad u = g_C(x_C). \tag{26}$$

Note that (26) is designed ignoring the sampling process and it can be obtained by using any continuous-time design method, including any of the techniques outlined in Section II and references cited therein. The second step of the emulation process is to "discretize" the controller and implement it with sufficiently fast sampling. First, we explain the sampling process. We assume that there is a sequence of sampling times

$$t_{s_i} = i\tau, \qquad \tau > 0 , \tag{27}$$

where τ is the sampling period at which the plant and controller are allowed to communicate. Moreover, we assume that the control signal is constant at each sampling interval, i.e. we use a zero order hold. We model the zero order hold by introducing an auxiliary variable $\hat{u}$ whose derivative is zero on each sampling interval, that is:

$$\left.\begin{array}{rcl} \dot{x}_P &=& f_P(x_P, \hat{u}, w), \\ y &=& g_P(x_P) \\ \dot{\hat{u}} &=& 0 \end{array}\right\} t \in [t_{s_i}, t_{s_{i+1}}] . \tag{28}$$

We use a particular discretization of the designed controller (26) that is sometimes referred to as the zero order hold equivalent [20]:

$$\begin{array}{l} \left.\begin{array}{rcl} \dot{x}_C &=& f_C(x_C, \hat{y}) \\ u &=& g_C(x_C) \\ \dot{\hat{y}} &=& 0 \end{array}\right\} t \in [t_{s_i}, t_{s_{i+1}}] \\ \hat{y}(t_{s_i}^+) = y(t_{s_i}), \quad \hat{u}(t_{s_i}^+) = u(t_{s_i}) . \end{array} \tag{29}$$

Hence, the closed-loop CCS with the zero order hold equivalent discretization of the controller is described by (28), (29). Denote the discrete time model of (28)-(29) as:

$$\begin{array}{rcl} x_P(i+1) &=& F_\tau(x_P(i), x_C(i), w_f[i]) \\ x_C(i+1) &=& G_\tau(x_P(i), x_C(i), w_f[i]) \end{array} \tag{30}$$

where $x_P(i) := x_P(t_{s_i})$, $x_C(i) := x_C(t_{s_i})$ and $w_f[i] := \{w(t) : t \in [t_{s_i}, t_{s_{i+1}}]\}$. The discrete time model is obtained by applying the jump equations at t_{s_i} and then integrating equations (28)-(29) over one sampling interval $[t_{s_i}, t_{s_{i+1}}]$ starting at $(x_P(i), x_C(i))$.

In [58] it was shown that if a certain (general) dissipation inequality holds for the closed-loop system (25), (26) then a slightly weaker (i.e. semi-global practical) dissipation inequality holds for (30). The discrete time dissipation inequality then can be used to show that an appropriate stability property holds for the discrete-time model (30) of CCS. It is not hard to show under very general conditions that stability of the discrete-time model (30) extends to stability of CCS (28)-(29) - for UGAS and ISS see [73].

[3]We assume that all functions are sufficiently smooth.

Hence, emulation can be used to achieve a range of stability properties that can be cast in terms of dissipation inequalities, such as UGES, UGAS, ISS or L_2 stability. The following result adapted from [58] establishes preservation of general dissipation inequalities under emulation:

Theorem 2: Suppose that there is a differentiable storage function $V(x_P, x_C)$ and a continuous supply rate $s(x_P, x_C, w)$ such that the following holds for all (x_P, x_C) and w for the system (25), (26):

$$\dot{V} = \left\langle \frac{\partial V}{\partial x}, f \right\rangle \le s(x_P, x_C, w) \ ,$$

where $x := (x_P, x_C)$ and $f := (f_P, f_C)$, then for any strictly positive numbers $D > \nu > 0$ there exists $\tau^* > 0$ such that for all $\tau \in (0, \tau^*)$, all (x_P, x_C, w_f) with $|(x_P, x_C)| \le D$, $||w_f||_\infty \le D$ we have that (30) satisfies:

$$\frac{\Delta V}{\tau} \le \frac{1}{\tau} \int_{t_{s_i}}^{t_{s_{i+1}}} s(x_P, x_C, w(t))dt + \nu \ .$$

where $\Delta V := V(F_\tau, G_\tau) - V(x_P, x_C)$ ■

One of the drawbacks of Theorem 2 is that it does not provide explicit and tight estimates of the required τ^* which would be very useful for practitioners[4]. An alternative result that we present next can be used to obtain τ^*. We find it convenient to rewrite the above equations by introducing $e := (y - \hat{y}, u - \hat{u})$ and using the definition of x in Theorem 2. After some manipulations we can write that the closed-loop system (28), (29) in coordinates x, e is:

$$\begin{aligned} \dot{x} &= f(x, e, w) \qquad t \in [t_{s_i}, t_{s_{i+1}}] && (31)\\ \dot{e} &= g(x, e, w) \qquad t \in [t_{s_i}, t_{s_{i+1}}] && (32)\\ e(t_{s_i}^+) &= 0 \ . && (33) \end{aligned}$$

The following result provides conditions under which the CCS (31)-(33) is input-to-state stable (ISS):

Theorem 3: Suppose that the following conditions hold:

1) The system (31) is ISS with linear gain γ from e as the input to x as the output;
2) There exist positive L, c such that for all (x, e) we have $|g(x, e)| \le L|e| + c(|x| + |w|)$;
3) The sampling period τ satisfies:

$$\tau < \tau^* := \frac{1}{L} \ln \left(\frac{L + c\gamma}{c\gamma} \right) \ .$$

Then, the system (31)-(33) is ISS from w to (x, e). Moreover, when $w(t) \equiv 0$ the system is UGAS. ■
Theorem 2 is convenient since it provides an estimate of τ^* for which the CCS is stable, which is useful when implementing the emulated controller. We note that better estimates for τ^* exist in the literature [19].

The first condition in Theorem 3 can be satisfied through the first step of the controller design. The second condition can be relaxed to hold on arbitrarily large sets of states if we are not interested in global stability properties. Finally, the last condition requires sampling to be sufficiently fast in order to have ISS of the CCS with the emulated controller. Results similar to Theorem 2 can be restated for different stability properties, such as L_p stability [68].

2) Discrete-time design: In discrete-time design we first discretize the plant model and then design the controller based on the discretized model. This design method does not require fast sampling and the designed controller always stabilizes the plant. In this method we ignore the inter-sample behaviour of the system and, due to the so called inter-sample ripple, satisfactory performance may be hard to achieve. Consider now the nonlinear continuous-time plant[5]

$$\dot{x} = f(x, u) \ , \qquad x(0) = x_\circ \ , \tag{34}$$

where we assume that sampling instants satisfy (27) and control $u(t) = u(i\tau), t \in [t_{s_i}, t_{s_{i+1}}]$. To obtain the discrete time model of the plant, we compute the solution at $t_{s_{i+1}}$ that starts from $x_\circ = x(i)$ and with a constant input $u(i)$:

$$x(i+1) \ = \ x(i) + \int_{i\tau}^{(i+1)\tau} f(x(s), u(i))ds \tag{35}$$

The equation (35) represents the *exact discrete-time model* of the nonlinear sampled-data plant (34). We emphasize that (35) is not known in most cases since this requires an analytic solution of a nonlinear initial value problem. On the other hand, one can easily write down a range of approximate models. For example, the forward Euler approximate model of (34) is given by

$$x(i+1) = x(i) + \tau f(x(i), u(i)) \ . \tag{36}$$

A range of other approximate models (e.g. using Runge-Kutta integration methods) can be found in standard books on numerical analysis [90]. We consider the difference equations corresponding to the exact (35) and approximate (e.g. (36)) discrete-time models of the sampled data system (34) that are denoted respectively as

$$x(i+1) = F_\tau^e(x(i), u(i)) \tag{37}$$

$$x(i+1) = F_\tau^a(x(i), u(i)) \tag{38}$$

and which are parameterized by the sampling period τ.

Most nonlinear sampled-data literature assumes that the exact discrete-time model (37) for the sampled-data plant (34) is known and it is available to the designer. On the other hand, typically the exact discrete-time model can not be analytically computed since it requires solving a nonlinear initial value problem explicitly. Hence, we assume that the exact discrete-time model (37) for the sampled-data system (34) is not known exactly and it is not available to the designer. Therefore, the controller design needs to be carried out using an approximate discrete-time model (38).

In particular, we want to know whether a family of controllers of the form:

$$\begin{aligned} z(i+1) &= G_\tau(z(i), x(i)) && (39)\\ u(i) &= U_\tau(z(i), x(i)) \ , \end{aligned}$$

[4]Such estimates can be deduced from the proofs in [58] but they are typically too conservative.

[5]We assume existence and uniqueness of solutions.

that stabilizes the family of approximate systems (38) for all small sampling periods τ would also stabilize the family of exact systems (37) for all small sampling periods. Note that we want to answer this question without knowing explicitly the exact model (37). We note that examples in [71] show that it is not true that any controller (39) that stabilizes approximate model (38) would also stabilize the exact model (37). Hence, we present conditions under which this is true[6].

Definition 1: **[Equi-Lipschitz Lyapunov function]** Suppose that there exists a Lyapunov function V_τ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that there exists $\tau^* > 0$ such that for all $\tau \in (0, \tau^*)$ and all $\tilde{x}$ where $\tilde{x} := (x, z)$ we have:

$$\alpha_1(|\tilde{x}|) \leq V_\tau(x,z) \leq \alpha_2(|\tilde{x}|) \quad (40)$$

$$\frac{\Delta V^a}{\tau} \leq -\alpha_3(|\tilde{x}|) \ , \quad (41)$$

where $\Delta V^a := V(F_\tau^a(x, U_\tau), G_\tau) - V_\tau(x,z)$. Moreover, suppose that there exists $L > 0$ and $\tau^* > 0$ such that for all $\tau \in (0, \tau^*)$, x_1, x_2 and all z we have:

$$|V_\tau(x_1, z) - V_\tau(x_2, z)| \leq L|x_1 - x_2| \ . \quad (42)$$

If V_τ satisfying (40), (41), (42) exists we call it an *equi-Lipschitz Lyapunov function* for the system (38), (39). ■

Note that conditions (40), (41) imply global asymptotic stability of the approximate closed-loop (38), (39). This alone is not enough to guarantee that exact closed-loop (37), (39) is stable (see examples in [71], [72]). We also need to use the following

Definition 2: **[One-step consistency]**: Suppose that there exists τ^* and $\rho, \alpha_4 \in \mathcal{K}$ such that for all $\tau \in (0, \tau^*)$ and all x, u we have that

$$|F_\tau^e(x,u) - F_\tau^a(x,u)| \leq \tau\rho(\tau)\alpha_4(|(x,u)|) \ . \quad (43)$$

Then, we say that F_τ^e *and* F_τ^a *are on-step consistent.* ■

Definition 3: **[Boundedness of U_τ]** Suppose there exist $\tau^* > 0$ and $\alpha_5 \in \mathcal{K}$ such that for all $\tau \in (0, \tau^*)$ and all $\tilde{x}$ we have that

$$|U_\tau(z,x)| \leq \alpha_5(|\tilde{x}|) \ . \quad (44)$$

Then we say that U_τ *is bounded uniformly in small* τ. ■

Then, we can state the following result:

Theorem 4: Suppose that the following conditions hold:

1) F_τ^e and F_τ^a are one-step consistent.
2) There exists an equi-Lipschitz Lyapunov function for the family of approximate closed-loops (38), (39).
3) U_τ is bounded uniformly in small τ.

Then, there exists $\beta \in \mathcal{KL}$ such that for any positive numbers D, ν there exists $\tau^* > 0$ such that for any $\tau \in (0, \tau^*)$ and any $|\tilde{x}| \leq D$ solutions of the exact closed-loop (37), (39) satisfy the following:

$$|\tilde{x}(i)| \leq \beta(|\tilde{x}(0)|, \tau i) + \nu \qquad \forall k \geq 0 \ .$$

■

More general versions of Theorem 4 are available in the literature [64], [66], [72], [71]. We emphasize that if any of the conditions in Theorem 2 are relaxed, the conclusion does not hold as examples in [71] show.

We note that we do not need to know explicitly F_τ^e to check consistency. This notion is adapted from numerical analysis literature [90]. Hence, all conditions of Theorem 4 can be checked as the continuous-time model (34), approximate model (38) and controller are available to the designer.

The conclusion of Theorem 4 is that the exact closed-loop is semi-globally practically stable in sampling period τ. Hence, Theorem 4 provides a prescriptive framework for controller design via approximate discrete-time models. Indeed, the theorem suggests that the first step of design is to pick a one-step consistent approximation F_τ^a of F_τ^e. Then, one needs to design a controller (39) satisfying items 2 and 3 of the theorem. The theorem does not tell us how to design a controller (39) satisfying items 2 and 3. For particular classes of plants and their approximations one can design controllers satisfying all conditions of the theorem. For instance, backstepping based on the Euler model of strict feedback systems was carried out in [70]; optimization based stabilization was given in [65]; model predictive control based on approximate discrete-time models is given in [27]; observer design in [6], [13]; port controller Hamiltonian systems in [57]; non-holonomic chain of integrators [56], and so on. It is important to emphasize that controllers that satisfy all conditions of Theorem 4 typically perform better in simulations than the emulated controllers for the same sampling period, see [70]. Also, various generalizations of Theorem 4 can be found in the literature: multi-rate versions can be found in [76]; ISS framework is given in [66]; iISS and L_2 stability framework was proved in [64].

B. Networked control systems

CCS presented in the previous section is a very special class of digitally controlled systems. In this section we present an emulation approach for a more general class of networked control systems (NCS) in which communication between the plant and the controller occurs via a digital local area network (LAN). We consider the so called packed based networks in which sensor and actuator values are sent in packets. We will ignore the effects of quantization and concentrate on an emulation controller design method proposed in [95], [96] and further developed in [68], [69], [19].

Suppose that we follow the same emulation steps as in the previous section but in this case we implement the controller via a packed based network. It can be shown that the NCS with the emulated controller can be written in the (x, e) coordinates (see [68] for more details):

$$\dot{x} = f(t,x,e,w) \qquad \forall t \in [t_{s_i}, t_{s_{i+1}}] \quad (45)$$

$$\dot{e} = g(t,x,e,w) \qquad \forall t \in [t_{s_i}, t_{s_{i+1}}] \quad (46)$$

$$e(t_{s_i}^+) = h(i, e(t_{s_i})) \ , \quad (47)$$

$$\epsilon \leq t_{s_{i+1}} - t_{s_i} \leq \tau \ . \quad (48)$$

We adopt terminology from [96] and refer to τ as the *maximum allowable transmission interval* (MATI). There are

[6]The conditions we present are strong but they are easier to state. Weaker alternative conditions can be found in [72], [71] and references cited therein.

two differences between the model (45)-(48) and (31)-(33). First, the sampling times t_{s_i} in (48) are not equidistant as was the case in (27). Second, the jump equation in (33) is a very special case of (47), which models the network protocol. We assume that the network has ℓ "nodes" which can be thought of as groups of sensor and actuator signals that are always transmitted together in one packet so that the protocol gives access to the network at each t_{s_i} to one of the nodes $i \in \{1, 2, \ldots, \ell\}$. Note that if NCS has ℓ nodes, then the error vector can be partitioned as follows $e = [e_1^T \ e_2^T \ \ldots \ e_\ell^T]^T$ where each e_i in the partition corresponds to the group of sensor and actuator signals in the node i. We typically assume that if a node j is transmitted at time t_{s_i} then e_j is reset to zero at time $t_{s_i}^+$, that is $e_j(t_{s_i}^+) = 0$. However, we emphasize that this assumption is not needed in general. If a node j does not transmit at t_{s_i} then we write $e_j(t_{s_i}^+) = e_j(t_{s_i})$.

This model of protocols was introduced in Nešić and Teel in [68]. Numerous protocols can be written in the form (47), as the following examples illustrate.

Example 1: [Sampled-data systems] Suppose that there are $\ell = 1$ nodes and $t_{s_{i+1}} - t_{s_i} = \tau$ where $\tau > 0$ is a constant sampling period. Hence, the protocol transmits all sensor and actuator signals at every transmission instant and in equation (47) we have $h(i, e) = 0$ (in other words we obtain (33)). This is obviously the classical case of sampled-data systems and we refer to this protocol as a sampled-data protocol. ■

Example 2: [Round Robin (RR) protocol] Let there be $\ell \geq 1$ nodes in NCS and let the protocol grant access to the network to the node $i \in \{1, \ldots, \ell\}$ at $t_{s_{i+j\ell}}$, for all $j \in \mathbb{N}$, that is $e(t_{s_{i+j\ell}}^+) = 0, \forall j \in \mathbb{N}$. In this case we can write $h(i, e) = (I - \Delta(i))e$, where $\Delta(i) = \text{diag}\{\delta_1(i)I_{n_1}, \ldots, \delta_\ell(i)I_{n_\ell}\}$, $\sum_{k=1}^{\ell} n_k = n_e$, I_{n_k} are identity matrices of dimension n_k

$$\delta_k(i) := \begin{cases} 1, & \text{if } i = k + j\ell,\ j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

■

Example 3: [Try-Once-Discard (TOD) protocol] This protocol was proposed in [95] and its model proposed in [68]. Suppose that there are ℓ nodes competing for access to the network. The node i with the greatest weighted error at time t_{s_j} will be granted access to network at $t_{s_j}^+$ and hence we have that $e_i(t_{s_j}^+) = 0$. We assume that the weights are already incorporated into the model. If a data packet fails to win access to the network, it is discarded and new data is used at the next transmission time $t_{s_{j+1}}$. If two or more nodes have equal priority, a pre-specified ordering of the nodes is used to resolve the collision. This verbal description can be converted into the model of the form (47) where $h(e) = (I - \Psi(e))e$ and $\Psi(e) := \text{diag}\{\psi_1(e)I_{n_1}, \psi_2(e)I_{n_2}, \ldots, \psi_\ell(e)I_{n_\ell}\}$. I_{n_j} are identity matrices of dimension n_j with $\sum_{j=1}^{\ell} n_j = n_e$

$$\psi_j(e) := \begin{cases} 1, & \text{if } j = \min\left(\arg\max_j |e_j|\right) \\ 0, & \text{otherwise.} \end{cases}$$

■

We find it very convenient to use h in (47) to introduce an auxiliary discrete time system of the form:

$$e^+ = h(i, e) \qquad i \in \mathbb{N}\ , \tag{49}$$

and refer to it as a *discrete time system induced by the protocol (47)*. Sometimes we abuse the terminology and refer to (49) simply as a protocol. The following class of protocols are used to state the next results:

Definition 4: Let $W : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ be given and suppose that there exist $\rho \in [0, 1)$ and $a_1, a_2 > 0$ such that the following conditions hold for the discrete time system (49) for all $i \in \mathbb{N}$ and all $e \in \mathbb{R}^{n_e}$:

$$a_1 |e| \leq W(i, e) \leq a_2 |e| \tag{50}$$

$$W(i + 1, h(i, e)) \leq \rho W(i, e)\ . \tag{51}$$

Then, we say that the protocol (47) (or equivalently (49)) is *uniformly globally exponentially stable (UGES) with Lyapunov function W*. ■

Definition 4 does not make any reference to the NCS (45), (46) and, hence, it captures the intrinsic properties of the protocol itself. This novel approach to viewing protocols was first proposed in [68]. This approach turned out to be very useful for many other problems in this area. It is a well known fact in the literature that the conditions (50), (51) are equivalent to uniform global exponential stability of the system (49). It was shown in [68] that the protocols we presented in Examples 1-3 are UGES with appropriate Lyapunov functions.

Proposition 1: The sampled-data protocol in Example 1 is UGES with Lyapunov function $W(i, e) = |e|$. In particular, we can take $a_1 = a_2 = 1$ and $\rho = 0$. ■

Proposition 2: The RR protocol (with ℓ nodes) in Example 2 is UGES with the Lyapunov function $W(i, e) := \sqrt{\sum_{k=i}^{\infty} |\phi(k, i, e)|^2}$, where ϕ denotes the solutions of the system (49) induced by the RR protocol. In particular, we can take $a_1 = 1$, $a_2 = \sqrt{\ell}$ and $\rho = \sqrt{\frac{\ell - 1}{\ell}}$. ■

Proposition 3: The TOD protocol (with ℓ nodes) in Example 3 is UGES with the Lyapunov function $W(i, e) := |e|$. In particular, we can take $a_1 = a_2 = 1$ and $\rho = \sqrt{\frac{\ell - 1}{\ell}}$. ■

Then we can state the following result which provides a framework for emulation for general NCS:

Theorem 5: Consider NCS (45)-(48). Suppose that the following conditions hold:

1) System (45) is ISS from (e, w) to x with linear gain γ.
2) $h(i, e)$ in equation (47) is such that inequalities (50), (51) hold, i.e. the protocol (47) is UGES with Lyapunov function W.
3) There exist $L, c \geq 0$ such that for all i, t, e, x, w we have that g in (46) satisfies:
$$\left\langle \frac{\partial W}{\partial e}, g(t, x, e, w) \right\rangle \leq L W(i, e) + c(|x| + |w|)\ . \tag{52}$$
4) MATI in (48) satisfies $\tau \in (\epsilon, \tau^*)$ where
$$\tau^* := \frac{1}{L} \ln \left(\frac{a_1 L + c\gamma}{a_1 \rho L + c\gamma} \right) \tag{53}$$

$\epsilon \in (0, \tau^*)$ is arbitrary, $L, c \geq 0$ come from (52), $a_1 > 0$ comes from (50) and $\rho > 0$ comes from (51).

Then, the NCS is ISS from w to (x, e) with linear gain. In particular, when $w \equiv 0$, the NCS is UGAS. ■

Various versions of Theorem 5 can be stated (see for example [68], [69], [91]). These results provide a framework for emulation of general wireline and wireless NCS to achieve certain stability properties. Note that the result applies to any stable protocols (e.g. UGES) and hence this emulation framework applies to a range of situations. Indeed, it was recently shown [67] that certain quantized control systems can be designed in a similar manner by using an appropriate extension of Theorem 5.

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