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Null Controllability with Vanishing Energy for Discrete-Time Systems in Hilbert Space

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Abstract— **In this paper null controllability with vanishing energy is considered for discrete-time systems in Hilbert space. As in the case of continuous time systems necessary and sufficient conditions in terms of an algebraic Riccati equation are given. Then necessary and sufficient conditions involving the spectrum of the system operator are given. Reachability and controllability with vanishing energy are also considered, and necessary and sufficient conditions for them are given. Finally applications to sampled-data systems, systems with impulse control and periodic systems are discussed.**

I. INTRODUCTION

Consider the linear system

$$\dot{x} = Ax + Bu, \ x(0) = x_0 \in H, \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ in a Hilbert space H, u is a control in some Hilbert space U and $B \in L(U, H)$, the space of bounded linear operators from U into H. For each locally square integrable function $u : [0, \infty) \to U$ define the solution in the mild sense

$$x(t; x_0, u) = S(t)x_0 + \int_0^t S(t - r)Bu(r)dr, \ t \geq 0.$$

We denote by $|\cdot|$ the norm of vectors and by $\sigma(A)$ the spectrum of the operator A. The following definitions are introduced in [10].

Definition 1.1: (a) The system (1) is said to be null controllable with vanishing energy (NCVE for short) if for each initial $x(0) = x_0$ there exists a sequence of pairs (T_N, u_N), $0 < T_N \uparrow \infty$, $u_N \in L_2(0, T_N; U)$ such that $x(T_N; x_0, u_N) = 0$ and

$$\lim_{N\to\infty} \int_0^{T_N} |u_N(t)|^2 dt = 0. \tag{2}$$

(b) The system (1) is said to be exactly controllable with vanishing energy $(ECVE)$ if for any pair (x_0, x_1) of initial and final states there exists a sequence of pairs (T_N, u_N), $0 < T_N \uparrow \infty$, $u_N \in L_2(0, T_N; U)$ such that $x(T_N; x_0, u_N) = x_1$ and (2) holds.

(A,B) is said to be NCVE (ECVE) if the system (1) is NCVE (ECVE). The following theorem gives necessary and sufficient conditions.

Theorem 1.1: (A, B) is NCVE if and only if
(a) it is null controllable on some interval $[0, \tau]$, and
(b) $X = 0$ is the unique solution of the algebraic Riccati equation (ARE)

$$A^*X + XA - XBB^*X = 0$$

in the class of nonnegative operators.

Priola and Zabczyk [10] showed that the condition (b) is necessary and sufficient for NCVE when (A, B) is null controllable on some interval $[0, \tau]$. The necessity of (a) was then shown by van Neerven [9].

Under the following two assumptions Priola and Zabczyk [10] obtained more explicit necessary and sufficient conditions.

Hypothesis 1. There exists a sequence $\{\lambda_n\} \subset \sigma(A)$ such that λ_n is isolated in $\sigma(A)$ and

$$\lim_{n\to\infty} Re(\lambda_n) = s(A) = \sup\{Re(\lambda) : \lambda \in \sigma(A)\}.$$

Hypothesis 2. There exist $S(t)$-invariant subspaces H_s and H_u such that
(a) $H = H_s \oplus H_u$,
(b) A on H_s is exponentially stable, and
(c) the set of all generalized eigenvectors of A contained in H_u is linearly dense in H_u.

Theorem 1.2: Suppose that Hypotheses 1 and 2 hold. Then (A, B) is NCVE if and only if
(a) (A, B) is null controllable on some interval $[0, \tau]$, and
(b) $Re(\lambda) \leq 0$ for any $\lambda \in \sigma(A)$.

Theorem 1.3: Suppose that Hypotheses 1 and 2 hold. Suppose further that $S(t)$ is a strongly continuous group on H. Then (A, B) is ECVE if and only if
(a) (A, B) is exactly controllable on some interval $[0, \tau]$, and
(b) $Re(\lambda) = 0$ for any $\lambda \in \sigma(A)$.

The proof of Theorem 1.1 is based on the theory of optimal quadratic control. For the proof of necessity of Theorem 1.2 the relation between the Riccati equation and the controllability gramian of the pair $(-A, B)$ is used, while for sufficiency the Riccati equation is directly used. Theorem 1.3 is a consequence of Theorem 1.2 and the fact that $(-A, -B)$ is also NCVE.

If we fix $x_0 = 0$ in (b) of Definition 1.1, (A, B) is said to be reachable with vanishing energy (RVE). It is easy to see that (A, B) is ECVE if and only if it is NCVE and RVE. Suppose that $S(t)$ is a strongly continuous group and let P_T be the controllability operator defined by

$$P_T x = \int_0^T S(t)BB^*S^*(t)x dt.$$

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It is coercive (positive and boundedly invertible) for $T \geq \tau$, if (A, B) is exactly controllable on $[0, \tau]$. The control with minimum norm in $L_2(0, T; U)$ such that $x(T; 0, u) = x_1$ is given by

$$\hat{u}_T = B^* S^*(T-t) P_T^{-1} x_1$$

[10] and its norm by

$$\| \hat{u}_T \|_2 = \langle x_1, P_T^{-1} x_1 \rangle^{\frac{1}{2}}. \tag{3}$$

Lemma 1.1: (A, B) is RVE if and only if
(a) (A, B) is exactly controllable on some interval $[0, \tau]$, and
(b) $P_T^{-1} \to 0$ strongly as $T \to \infty$.

Proof: The proof of necessity of (a) is based on the Baire category theorem and is similar to that of Theorem 1.1, (a) given in [9]. The rest follows from (3). ∎

In this paper we shall establish the discrete-time versions of the theorems above. It is important in its own right but also useful when we consider sampled-data systems with zero-order hold, systems with impulse control and periodic systems. In the discrete-time case the proof of necessity of Theorem 1.2 is more involved since the Riccati equation is more complicated for discrete-time systems. Lemma 2.4 in Section 2 fills this gap and enables us to extend Theorem 1.2. The extension of Theorem 1.3 requires the invertibility of A. It is also useful to introduce reachability with vanishing energy. In Section 2 we give preliminaries concerning necessary notions of discrete-time systems. In Section 3 we consider necessary and sufficient conditions for NCVE and extend Theorem 1.1 and Theorem 1.2. In Section 4 we introduce reachability with vanishing energy and extend Theorem 1.3. Finally in Section 5 we apply NCVE and ECVE results to sampled-data systems, systems with impulse control and periodic systems.

II. Preliminaries

Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \; x(0) = x_0, \tag{4}$$

where $A \in L(H)$, $B \in L(U, H)$, $x \in H$ and $u \in U$. We collect basic definitions and some useful results for (4) as in the finite dimensional case [1].

Definition 2.1: (a) (A, B) is null controllable on $[0, K]$ if for any x_0 there is a sequence of control inputs $u = \{u(0), u(1), ..., u(K-1)\}$ such that $x(K; x_0, u) = 0$.
(b) (A, B) is reachable on $[0, K]$ if for every state x_1 there is a sequence of control inputs $u = \{u(0), u(1), ..., u(K-1)\}$ such that $x(K; 0, u) = x_1$.
(c) (A, B) is exactly controllable on $[0, K]$ if for every pair (x_0, x_1) there is a sequence of control inputs $u = \{u(0), u(1), ..., u(K-1)\}$ such that $x(K; x_0, u) = x_1$.

Lemma 2.1: (a) (A, B) is reachable on $[0, K]$ if and only if it is exactly controllable on $[0, K]$. In this case it is null controllable on $[0, K]$.
(b) If A is invertible and (A, B) is null controllable on $[0, K]$, then (A, B) is exactly controllable on $[0, K]$.

Lemma 2.2: The following statements are equivalent.
(a) (A, B) is null controllable on $[0, K]$.
(b) $R(A^K) \subset R(M_K)$, where $M_K = [B, AB, ..., A^{K-1}B]$ is the reachability operator.
(c) $|M_K^* x| \geq a |(A^*)^K x|$ for some $a > 0$.
If these conditions hold, the operator $\begin{bmatrix} B^* \\ \lambda I - A^* \end{bmatrix}$ is 1 to 1 for any nonzero λ.

Proof: Consider the response of the system (4) with initial condition x_0 and control $u = \{u(0), u(1), ..., u(K-1)\}$. Then

$$x(K; x_0, u) = A^K x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)$$

and the second term of the right hand side lies in $R(M_K)$, the range of M_K. Hence (a) is equivalent to (b). The equivalence of (b) and (c) follows from Corollary 3.5 of [3]. If there exists a nonzero q such that $B^* q = 0$ and $\lambda q = A^* q$, it contradicts to (c) with $x = q$. ∎

Lemma 2.3: Suppose A is exponentially stable [7] i.e., $|A^k| \leq M\rho^k$, $0 < \rho < 1$ and that (A, B) is exactly controllable on $[0, K]$. Then there exists a coercive operator Y such that

$$Y = AYA^* + BB^*.$$

Y is called the controllability gramian of (A, B).

Proof: By Lemma 2.1 (A, B) is reachable on $[0, K]$. Hence $M_K = [B, AB, ..., A^{K-1}B]$ is onto and $M_K M_K^* \geq aI$ for some $a > 0$. Define

$$Y = \lim_{k \to \infty} M_k M_k^* = \lim_{k \to \infty} \sum_{j=0}^{k-1} A^j BB^* (A^*)^j.$$

The right hand side converges in the uniform operator topology and $Y \geq M_K M_K^* \geq aI$. Hence Y is coercive. Moreover

$$Y = BB^* + A \sum_{j=0}^{\infty} A^j BB^* (A^*)^j A^* = BB^* + AYA^*.$$

∎

Lemma 2.4: Suppose A is invertible and (A, B) is exactly controllable on $[0, K]$. Then $(A^{-1}, A^{-1}B)$ is exactly controllable on $[0, K]$. If A^{-1} is exponentially stable, then the inverse of its controllability gramian Y exists and satisfies the following algebraic Riccati equation

$$X = A^* XA - A^* XB(I + B^* XB)^{-1} B^* XA. \tag{5}$$

Proof: Since (A, B) is exactly controllable on $[0, K]$, so is $(A^{-1}, A^{-1}B)$. In fact

$$[A^{-1}B, A^{-1}(A^{-1}B), ..., (A^{-1})^{K-1}(A^{-1}B)]$$
$$= (A^{-1})^K [A^{K-1}B, ..., AB, B].$$

Now by definition

$$Y = A^{-1} Y (A^{-1})^* + A^{-1} BB^* (A^{-1})^*,$$

which implies

$$AYA^* = Y + BB^*.$$

By Lemma 2.3 Y is coercive and hence invertible. As in Lemma 3.18 [5], we obtain

$$\begin{aligned} & (A^{-1})^* Y^{-1} A^{-1} \\ = \; & (Y + BB^*)^{-1} \\ = \; & Y^{-1}(I + BB^*Y^{-1})^{-1} \\ = \; & Y^{-1}[I - (I + BB^*Y^{-1})^{-1} BB^* Y^{-1}] \\ = \; & Y^{-1}[I - B(I + B^*Y^{-1}B)^{-1} B^* Y^{-1}] \\ = \; & Y^{-1} - Y^{-1}B(I + B^*Y^{-1}B)^{-1}B^*Y^{-1}, \end{aligned}$$

where for the second equality we have used the equality $Y + BB^* = (I + BB^*Y^{-1})Y$and for the fourth equality the familiar identity $M(I+NM)^{-1} = (I+MN)^{-1}M$ is used. Hence we obtain

$$Y^{-1} = A^*Y^{-1}A - A^*Y^{-1}B(I + B^*Y^{-1}B)^{-1}B^*Y^{-1}A$$

and Y^{-1} is a coercive solution of the ARE (5). ∎

III. Null controllability with vanishing energy

Consider the system (4)

$$x(k+1) = Ax(k) + Bu(k), \; x(0) = x_0.$$

We shall define NCVE for this system.

Definition 3.1: (A, B) is null controllable with vanishing energy if for each x_0 there exists a sequence of pairs (k_N, u_N), k_N a positive integer $\uparrow \infty$, $u_N \in l_2(0, k_N - 1; U)$ such that $x(k_N; x_0, u_N) = 0$ and

$$\lim_{N\to\infty} \| u_N \|_2 = 0,$$

where $l_2(0, k_N - 1; U)$ is the set of vectors $u = \{u(0), u(1), ..., u(k_N - 1)\}$, $u(k) \in U$ with norm

$$\| u \|_2 = (\sum_{k=0}^{k_N - 1} |u(k)|^2)^{\frac{1}{2}}.$$

Lemma 3.1: If (A, B) is NCVE, then (A, B) is null controllable on some interval $[0, K]$

Proof: Based on the Baire category theorem and similar to the proof of Theorem 3.1 [9]. ∎

First we shall prove the following.

Theorem 3.1: (A, B) is NCVE if and only if
(a) (A, B) is null controllable on some interval $[0, K]$, and
(b) $X = 0$ is the unique solution of the ARE (5)

$$X = A^*XA - A^*XB(I + B^*XB)^{-1}B^*XA$$

in the class of nonnegative operators.

We modify Hypotheses 1 and 2 as follows.

Hypothesis 3. There exists a sequence $\{\lambda_n\} \subset \sigma(A)$ such that λ_n is isolated in $\sigma(A)$ and

$$\lim_{n\to\infty} |\lambda_n| = s(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

Hypothesis 4. There exist A-invariant subspaces H_s and H_u such that
(a) $H = H_s \oplus H_u$,
(b) A on H_s is exponentially stable, and
(c) the set of all generalized eigenvectors of A contained in H_u is linearly dense in H_u.

Under Hypotheses 3 and 4 we shall prove the following.

Theorem 3.2: (A, B) is NCVE if and only if
(a) (A, B) is null controllable on some interval $[0, K]$, and
(b) $|\lambda| \leq 1$ for any $\lambda \in \sigma(A)$.

Proof of Theorem 3.1.
We shall follow the proof of Theorem 1.1 in [10]. We first show necessity. Consider the quadratic cost associated with (4) on $[0, k_N - 1]$

$$J(u; x_0, k_N, Q) = \sum_{k=0}^{k_N - 1} |u(k)|^2 + \langle x(k_N), Qx(k_N) \rangle,$$

where $Q \geq 0$. It is known [8], [12], [13] that the optimal control minimizing the cost function is given by the feedback law

$$\bar{u}(k) = -[I + B^*X(k+1)B]^{-1}B^*X(k+1)Ax(k),$$

where $X(k) = X(k; k_N, Q)$ is the sequence of nonnegative operators defined by the Riccati equation

$$\begin{aligned} X(k) \; &= \; A^*X(k+1)A - A^*X(k+1)B \\ & \quad \times [I + B^*X(k+1)B]^{-1}B^*X(k+1)A, \quad (6) \\ X(k_N) \; &= \; Q. \end{aligned}$$

Moreover,

$$J(\bar{u}; x_0, k_N, Q) = \langle x_0, X(0; k_N, Q)x_0 \rangle.$$

Now we consider the case $Q = qI$, $q > 0$ and let $q \to \infty$. Since (A, B) is null controllable on $[0, K]$, for each x_0 and $k_N \geq K$ there exists a control $u \in l_2(0, k_N - 1; U)$ such that $x(k_N; x_0, u) = 0$. Let u_N be the control with minimum norm among them. Then it is given by $u_N = -\bar{M}_N^*(\bar{M}_N\bar{M}_N^*)^{-1}A^{k_N}x_0$ where $\bar{M}_N = [A^{k_N - 1}B, ..., AB, B]$. Since (A, B) is null controllable, $\lim_{N\to\infty} \| u_N \|_2^2 = 0$ for each x_0 and hence there exists a constant $a > 0$ such that $\| u_N \|_2^2 \leq a|x_0|^2$. Notice that

$$\begin{aligned} J(\bar{u}; x_0, k_N, qI) \; &= \; \langle x_0, X(0; k_N, qI)x_0 \rangle \\ &\leq \; J(u_N; x_0, k_N, qI) \\ &= \; \| u_N \|_2^2 \leq a|x_0|^2, \end{aligned}$$

which yields $X(0; k_N, qI) \leq aI$. Since $X(0; k_N, qI)$ is monotone increasing in q, there exists a limit as $q \to \infty$, denoted by $X(0; k_N)$, i.e., $X(0; k_N) = \lim_{q\to\infty} X(0; k_N, qI)$. Let $\bar{u}_q$ be the optimal control for $J(u; x_0, k_N, qI)$. Then it is uniformly bounded in q. Hence there exists a subsequence q_j such that $\bar{u}_{q_j}$ converges weakly to some limit $\bar{u}_\infty$. Then $x(k_N; x_0, \bar{u}_\infty) = 0$ and $\| \bar{u}_\infty \|_2^2 \leq \langle x_0, X(0; k_N)x_0 \rangle \leq \| u_N \|_2^2$. But u_N is the control with minimum norm and hence $\| \bar{u}_\infty \|_2^2 = \langle x_0, X(0; k_N)x_0 \rangle = \| u_N \|_2^2$. Now suppose that (A, B) is null controllable on $[0, K]$, $K \leq k_N$. Since $X(k; k_N, qI) = X(0; k_N - k, qI)$, the following limit exists:

$$\begin{aligned} \lim_{q\to\infty} X(k; k_N, qI) \; &= \; \lim_{q\to\infty} X(0; k_N - k, qI) \\ &\equiv \; X(k; k_N) \text{ for } k \leq k_N - K. \end{aligned}$$

Moreover, from equation (6) $X(k;k_N)$, $k \le k_N - K$ satisfies the Riccati equation

$$\begin{aligned} X(k) &= A^*X(k+1)A - A^*X(k+1)B \\ &\quad \times [I + B^*X(k+1)B]^{-1}B^*X(k+1)A, \\ X(k_N - K) &= X(k_N - K; k_N). \end{aligned}$$

Since $\langle x_0, X(0;k_N)x_0\rangle = \| u_N \|_2^2$, $X(0;k_N)$ is decreasing in N and has a nonnegative limit

$$X_\infty = \lim_{N\to\infty} X(0;k_N).$$

For $k \le N - K$ we know $X(k;k_N) = X(0;k_N - k)$ and hence $\lim_{N\to\infty} X(k;k_N) = X_\infty$. Letting $N \to \infty$ in the Riccati equation above we see that X_∞ satisfies the ARE (5). Recall that (A,B) is NCVE and hence

$$\langle x_0, X_\infty x_0\rangle \le \langle x_0, X(0;k_N)x_0\rangle = \| u_N \|_2^2 \to 0$$

and $X_\infty = 0$. Now let X be any nonnegative solution of the ARE (5). We shall show that $X \le X_\infty$ to conclude $X = 0$. For this purpose consider the Riccati difference equation (6) with $Q = X$. Then $X(k) = X$ is a solution. Thus

$$\begin{aligned} J(\bar u_X; x_0, k_N, X) &= \langle x_0, Xx_0\rangle \\ &\le J(\bar u_q; x_0, k_N, qI) \\ &= \langle x_0, X(0;k_N,qI)x_0\rangle \end{aligned}$$

for $q \ge \| X \|$, where $\bar u_X$ and $\bar u_q$ denote the optimal controls for the corresponding cost functions. Now passing to the limit $q \to \infty$ and to the limit $N \to \infty$ we obtain $\langle x_0, Xx_0\rangle \le \langle x_0, X(0;k_N)x_0\rangle$ and $\langle x_0, Xx_0\rangle \le \langle x_0, X_\infty x_0\rangle$ respectively. Thus we have shown $X = 0$, which completes the proof of necessity.

To show sufficiency we recall that $\| u_N \|_2^2 = \langle x_0, X(0;k_N)x_0\rangle \to \langle x_0, X_\infty x_0\rangle$. But by condition (b) $X_\infty = 0$ and hence $\| u_N \|_2 \to 0$ and (A,B) is NCVE. ∎

Proof of Theorem 3.2

We shall follow the proof of Theorem 1.2 in [10]. To show necessity we suppose that $|\lambda| > 1$ for some $\lambda \in \sigma(A)$. Then by Hypothesis 3 there exists an isolated element $\mu \in \sigma(A)$ with $|\mu| > 1$. Consider the spectral Riesz projection P_1 associated with μ

$$P_1 x = \frac{1}{2\pi i}\int_\gamma (\lambda I - A)^{-1} x d\lambda, \ x \in H,$$

where γ is a circle containing μ in its interior and $\sigma(A)/\{\mu\}$ in its exterior. Using projections P_1 and $P_2 = I - P_1$, we can split the equation (4) into two subsystems in E_1 and E_2 respectively

$$\begin{aligned} x_1(k+1) &= A_1x_1(k) + B_1u(k), \\ x_2(k+1) &= A_2x_2(k) + B_2u(k), \end{aligned}$$

where $E_i = P_iH$, A_i is the restriction of A to E_i, and $B_i = P_iB$. The subspaces E_i are A-invariant and $H = E_1 \oplus E_2$. Since (4) is null controllable, (A_1, B_1) and (A_2, B_2) are null controllable. Since $\sigma(A_1) = \{\mu\}$, it is invertible and (A_1, B_1) is exactly controllable by Lemma 2.1. Hence $(A_1^{-1}, A_1^{-1}B_1)$ is exactly controllable. Since A_1^{-1} is exponentialy stable, by Lemma 2.3 it possesses a coercive controllability gramian Y

$$Y = A_1^{-1}Y(A_1^{-1})^* + A_1^{-1}B_1B_1^*(A_1^{-1})^*.$$

By Lemma 2.4 $X_1 = Y^{-1}$ is a coercive solution of the ARE

$$X = A_1^*XA_1 - A_1^*XB_1(I + B_1^*XB_1)^{-1}B_1^*XA_1.$$

Then $X = I_HX_1P_1$ is a nontrivial nonnegative solution of the ARE for (5) where I_H is the injection of E_1 into H. This contradicts to Theorem 3.1 and hence $|\lambda| \le 1$ for any $\lambda \in \sigma(A)$.

To show sufficiency let X be any nonnegative solution of the ARE (5). Since $H = H_s \oplus H_u$, it is suficient to show $X = 0$ both on H_s and H_u. As in the proof of Theorem 3.1 consider (6) with $Q = X$ and recall the inequality

$$\begin{aligned} \langle x_0, Xx_0\rangle &= J(\bar u_X; x_0, k_N, X) \\ &\le J(0; x_0, k_N, X) \\ &= \langle A^{k_N}x_0, XA^{k_N}x_0\rangle \to 0, \ x_0 \in H_s. \end{aligned}$$

Hence $Xx_0 = 0$ for any $x_0 \in H_s$. To show $Xx_0 = 0$ for any $x_0 \in H_u$, let $\lambda \in \sigma(A)$ with $|\lambda| \le 1$ which corresponds to an eigenvector p i.e., $Ap = \lambda p$. Then

$$\begin{aligned} &\langle p, Xp\rangle \\ =\ &\langle p, A^*XAp\rangle - \langle p, A^*XB(I + B^*XB)^{-1}B^*XAp\rangle \\ =\ &|\lambda|^2[\langle p, Xp\rangle - \langle p, XB(I + B^*XB)^{-1}B^*Xp\rangle]. \quad (7) \end{aligned}$$

If $|\lambda| < 1$, then (7) yields $Xp = 0$. If $|\lambda| = 1$, then it yields $B^*Xp = 0$. In this case we obtain $Xp = \lambda A^*Xp$ from the ARE (5). Hence

$$\begin{bmatrix} B^* \\ \frac{1}{\lambda}I - A^* \end{bmatrix} Xp = 0$$

By Lemma 2.2 the operator above is 1 to 1 and hence $Xp = 0$. Thus for any eigenvector of A we have shown $Xp = 0$. We shall show that $Xq = 0$ for any generalized eigenvector of A, which would then conclude $X = 0$. Now let $q \in N((\lambda I - A)^2)$ i.e., $(\lambda I - A)^2q = 0$. Then $q_1 = (\lambda I - A)q$ satisfies $(\lambda I - A)q_1 = 0$. Repeating the arguments above we conclude $Xq_1 = 0$. Hence $XAq = \lambda Xq$ and from the ARE (5) we obtain

$$\begin{aligned} &\langle q, Xq\rangle \\ =\ &\langle q, A^*XAq\rangle - \langle q, A^*XB(I + B^*XB)^{-1}B^*XAq\rangle \\ =\ &|\lambda|^2[\langle q, Xq\rangle - \langle q, XB(I + B^*XB)^{-1}B^*Xq\rangle]. \end{aligned}$$

This is the same with (7) and hence $Xq = 0$. Repeating this process we conclude $Xq = 0$ for any generalized eigenvector of A satisfying $(\lambda I - A)^kq = 0$. Hence $X = 0$ on H_u. Thus $X = 0$ on H and by Theorem 3.1 (A,B) is NCVE. ∎

In [9] the reproducing kernel Hilbert space associated with the controllability operator was introduced and Theorem 1.1 was extended to the case where H is a Banach space. The extension of Theorem 3.1 to a Banach space is also possible using the Riccati equation directly.

IV. Exact controllability with vanishing energy

First we introduce reachability with vanishing energy (RVE), which is useful to consider ECVE.

Definition 4.1: (A, B) is reachable with vanishing energy if for each x_1 there exists a sequence of pairs (k_N, u_N), $k_N \uparrow \infty$, $u_N \in l_2(0, k_N - 1; U)$ such that $x(k_N; 0, u_N) = x_1$ and

$$\lim_{N\to\infty} \| u_N \|_2 = 0.$$

Lemma 4.1: Suppose (A, B) is RVE. Then $0 \notin \sigma_p(A^*)$. If A is invertible, then (A, B) is RVE if and only if $(A^{-1}, A^{-1}B)$ is NCVE.

Proof: Suppose $0 \in \sigma_p(A^*)$ and $A^*h = 0$ with $|h| = 1$. If (A, B) is reachable on $[0, K]$, then for some sequence $u = (u_j)$

$$\sum_{j=0}^{K-1} A^{K-j-1} B u_j = h.$$

Then

$$\begin{aligned} 1 = \langle h, h\rangle &= \langle h, \sum_{j=0}^{K-1} A^{K-j-1} B u_j\rangle \\ &= \sum_{j=0}^{K-1} \langle B^*(A^*)^{K-j-1} h, u_j\rangle \\ &= \langle B^* h, u_{K-1}\rangle \le |B^* h||u_{K-1}|. \end{aligned}$$

Hence $|u_{K-1}| \ge \dfrac{1}{|B^*h|}$ and (A, B) cannot be RVE. Now assume that A is invertible. Then the system (4) can be written as

$$x(k) = A^{-1}x(k + 1) - A^{-1}Bu(k).$$

Thus if (A, B) is RVE, then redefining u and x we can easily see that

$$\tilde{x}(k + 1) = A^{-1}\tilde{x}(k) + A^{-1}B\tilde{u}(k).$$

is NCVE. The converse is also true since we can reverse the arguments. ∎

From Lemma 4.1 we immediately obtain the following.

Theorem 4.1: Suppose A is invertible and A^{-1} satisfies Hypotheses 3 and 4. Then (A, B) is RVE if and only if
(a) (A, B) is exactly controllable on some interval $[0, K]$, and
(b) $|\lambda| \ge 1$ for any $\lambda \in \sigma(A)$.

Now we are ready to extend Theorem 1.3.

Theorem 4.2: Suppose A and A^{-1} satisfy Hypotheses 3 and 4. Then (A, B) is ECVE if and only if
(a) (A, B) is exactly controllable on some interval $[0, K]$, and
(b) $|\lambda| = 1$ for any $\lambda \in \sigma(A)$.

Proof: Note that (A, B) is ECVE if and only if it is NCVE and RVE. Hence the proof follows from Theorem 3.2 and Theorem 4.1. ∎

V. Applications

In this section we apply our theorems to sampled-data systems, systems with impulse control and periodic systems. First we consider a sampled-data system with zero-order hold [2]

$$\dot{x} = Ax + Bu,$$

where A is the infinitesimal generator of a strongly continuous semigroup $S(t) \in L(H)$, $B \in L(U, H)$ and u is a control given by

$$u(t) = u(k\tau), k\tau \le t < (k + 1)\tau.$$

Then at times $k\tau$ we have the following.

$$\begin{aligned} x((k + 1)\tau) &= S(\tau)x(k\tau) + \int_0^\tau S(r)Bdru(k\tau) \\ &\equiv A_d x(k\tau) + B_d u(k\tau). \end{aligned}$$

The sampled-data system is said to be NCVE (ECVE) if it is NCVE (ECVE) in the sense of Definition 1.1 with $T_N = N\tau$. Note that the sampled-data system is NCVE (ECVE) if and only if (A_d, B_d) is NCVE (ECVE). Hence, if A_d satisfies Hypotheses 3 and 4, then by Theorem 3.2 the sampled-data system is NCVE if and only if
(a) (A_d, B_d) is null controllable on some interval $[0, K]$, and
(b) $|\lambda| \le 1$ for any $\lambda \in \sigma(A_d)$.
If $S(t)$ is a group and $S(\tau)^{-1}$ satisfies Hypotheses 3 and 4, then the sampled-data system is ECVE if and only if
(a) (A_d, B_d) is exactly controllable on some interval $[0, K]$, and
(b) $|\lambda| = 1$ for any $\lambda \in \sigma(A_d)$.

Next we consider the system (1) with impulse control $u(k - 1)\delta(t - k\tau)$ at time $k\tau$, $k \ge 1$. Then the state $x(k\tau)$ after the impulse $u(k - 1)\delta(t - k\tau)$ satisfies

$$x((k + 1)\tau) = S(\tau)x(k\tau) + Bu(k).$$

Lemma 5.1: The system (2) with impulse control is NCVE if and only if $(S(\tau), B)$ is NCVE.

Lemma 5.2: Suppose $S(t)$ is a group and $S(\tau)^{-1}$ satisfies Hypotheses 3 and 4. Then the system (2) with impulse control is ECVE if and only if $(S(\tau), B)$ is ECVE.

Proof: Note that the system (2) with impulse control is ECVE if and only if it is NCVE and RVE. Let $K\tau \le T < (K + 1)\tau$ and consider the controllability operator

$$\begin{aligned} P_T^{imp} &= S(T - K\tau)(\sum_{j=1}^{K} S(\tau)^{K-j} BB^*(S(\tau)^*)^{K-j}) \\ &\qquad \times S^*(T - K\tau) \\ &= S(T - K\tau)P_{K\tau}^{imp} S^*(T - K\tau). \end{aligned}$$

Hence $(P_T^{imp})^{-1} \to 0$ strongly if and only if $(P_{K\tau}^{imp})^{-1} \to 0$ strongly and as in Lemma 1.1 the assertion follows. ∎

Now we have the following.

Theorem 5.1: (1) The system (2) with impulse control is NCVE if and only if
(a) $(S(\tau), B)$ is null controllable on some interval $[0, K]$,

and
(b) $|\lambda| \le 1$ for any $\lambda \in \sigma(S(\tau))$.
(2) Suppose $S(t)$ is a group and $S(\tau)^{-1}$ satisfies Hypotheses 3 and 4. Then the system (2) with impulse control is ECVE if and only if
(a) $(S(\tau), B)$ is exactly controllable on some interval $[0, K]$, and
(b) $|\lambda| = 1$ for any $\lambda \in \sigma(S(\tau))$.

Finally consider the T-periodic system

$$\dot{x} = A(t)x + B(t)u, \; x(t_0) = x_0, \; 0 \le t_0 < T \qquad (8)$$

where $A(t)$ is T-periodic and generates an evolution operator $S(t, s)$ and $B(t)$ is T-periodic and strongly continuous. Then

$$\begin{aligned} & x((k+1)T + t_0) \\ = \; & S(T + t_0, t_0)x(kT + t_0) \\ & \quad + \int_{t_0}^{T+t_0} S(T + t_0, r)B(r)u(k, r)dr \\ \equiv \; & S(T + t_0, t_0)x(kT + t_0) + B_d u(k), \end{aligned} \qquad (9)$$

where we have used the property $S((k+1)T+t_0, kT+r) = S(T+t_0, r)$, and $u(k, r) = u(kT + r)$, for $t_0 \le r < t_0 + T$, $u(k) = u(k, \cdot) \in L_2(t_0, t_0 + T; U)$ and B_d is a bounded linear operator in $\mathcal{L}(L_2(t_0, t_0 + T; U), H)$. Notice that the periodic system is NCVE if and only if $(S(T+t_0, t_0), B_d)$ is NCVE. Then by Theorem 3.2 the periodic system is NCVE if and only if
(a) it is null controllable on some interval $[t_0, \tau]$, and
(b) $|\lambda| \le 1$ for any $\lambda \in \sigma(S(T + t_0, t_0))$.

Suppose $S(t, s)$ is a two-parameter group so that $S(T + t_0, t_0)$ is boundedly invertible.

Lemma 5.3: The periodic system (8) is ECVE if and only if the discrete-time system (9) is ECVE.

Proof: Consider the controllability operator

$$P_L x = \int_{t_0}^{L} S(L, r)B(r)B(r)^* S^*(L, r)x dr.$$

Let $KT + t_0 \le L < (K + 1)T + t_0$. Then $\alpha P_{(K+1)T} \ge P_L \ge \beta P_{KT}$ for some $\alpha > 0$ and $\beta > 0$. Hence $P_L^{-1} \to 0$ strongly if and only if $(P_{KT})^{-1} \to 0$ strongly. By a periodic version of Lemma 1.1 the assertion follows. ∎

Lemma 5.4:

$$S(T + t_0, t_0) = S(t_0, 0)S(T, 0)S(t_0, 0)^{-1}$$

and $\sigma(S(T + t_0, t_0)) = \sigma(S(T, 0))$.

Suppose further $S(T+t_0, t_0)^{-1}$ satisfies Hypotheses 3 and 4. Then from Theorem 4.2 we obtain the following.

Theorem 5.2: The periodic system (8) is ECVE if and only if
(a) it is exactly controllable on some interval $[t_0, \tau]$, and
(b) $|\lambda| = 1$ for any $\lambda \in \sigma(S(T, 0))$.

VI. An Example

The linearized equations of relative motion of a satellite with respect to another in an elliptical orbit are known as Tschauner-Hempel equations, and the in-plane motion is given by

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dot{\theta}^2 + 2\mu/R_0^3 & -2\dot{R}_0\dot{\theta}/R_0 & 0 & 2\dot{\theta} \\ 2\dot{R}_0\dot{\theta}/R_0 & \dot{\theta}^2 - \mu/R_0^3 & -2\dot{\theta} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \qquad (10)$$

where μ is the gravitational parameter of the Earth, R_0 the distance from the center of the Earth to the reference satellite, θ the true anomaly, and they satisfy

$$\begin{aligned} \ddot{R}_0 - R_0\dot{\theta}^2 &= -\mu/R_0^2, \\ 2\dot{R}_0\dot{\theta} + R_0\ddot{\theta} &= 0. \end{aligned}$$

It is shown in [11] that the monodromy matrix $S(T, 0)$ has a quadruple eigenvalue 1 and that the system (10) is NCVE. Using this property, feedback controllers with small L_1 norm are designed for the relative orbit transfer problem. When the reference orbit is circular, Tschauner-Hempel equations are reduced to Hill-Clohessy-Wiltshire equations which are time-invarant. In this case the relative orbit transfer problem by impulse control is considered in [6]. Using the NCVE property, feedback controllers with small l_1 norm are designed.

As for an infinite dimensional example, we refer to [4], where a strongly damped wave equation with Neumann boundary condition and a periodic damping coefficient is considered. It is shown that all eigenvalues of $S(T, 0)$ have modulus less than 1.

References

[1] F. M. Callier and C. A. Desoer, *Linear System Theory*, Springer-Verlag, Berlin, 1991.
[2] T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*, Springer-Verlag, London, 1995.
[3] R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear Systems Theory*, Lecture Note in Control and Information Sciences, vol.8, Springer-Verlag, Berlin, 1978.
[4] G. Da Prato and A. Lunardi, Floquet exponents and stabilizability in time-periodic parabolic systems, *Applied Math. Optim.*, vol. 22, 1990, pp. 91-113.
[5] A. Ichikawa and H. Katayama, *Linear Time Varying Systems and Sampled-data Systems*, Lecture Note in Control and Information Sciences, vol.265, Springer-Verlag, London, 2001.
[6] Y. Ichimura and A. Ichikawa, Optimal impulsive relative orbit transfer along a circular orbit, *J. Guidance, Control, Dynamics*, vol. 31, 2008, pp. 1014-1027.
[7] C. S. Kubrusly, Mean square stability for discrete bounded linear systems in Hilbert space, *SIAM J. Control Optim.*, vol. 23, 1985, pp. 19-29.
[8] K. Y. Lee, S. N. Chow and R. O. Barr, On the control of discrete-time distributed parameter systems, *SIAM J. Control*, vol. 10, 1972, pp. 361-376.
[9] J. M. A. M. Van Neerven, Null controllability and the algebraic Riccati equation in Banach spaces, *SIAM J. Control Optim.*, vol. 43, 2005, pp. 1313-1327.
[10] E. Priola and J. Zabczyk, Null controllability with vanishing energy, *SIAM J. Control Optim.*, vol. 42, 2003, pp. 1013-1032.
[11] M. Shibata and A. Ichikawa, Orbital rendezvous and flyaround based on null controllability with vanishing energy, *J. Guidance, Control, Dynamics*, vol. 30, 2007, pp. 934-945.
[12] J. Zabczyk, Remarks on the control of discrete-time distributed parameter systems, *SIAM J. Control*, vol. 12, 1974, pp. 721-735.
[13] J. Zabczyk, On optimal stochastic control of discrete-time systems in Hilbert space, *SIAM J. Control*, vol. 12, 1975, pp. 1217-1234.