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Real Time Solution of Duncan-Mortensen-Zakai Equation Without Memory

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Abstract

It is well known that the nonlinear filtering problem has important applications in both military and commercial industries. The central problem of nonlinear filtering is to solve the DMZ equation in real time and memoryless manner. The purpose of this paper is to show that, under very mild conditions (which essentially say that the growth of the observation $|h|$ is greater than the growth of the drift $|f|$), the DMZ equation admits a unique nonnegative weak solution u which can be approximated by a solution u_R of the DMZ equation on the ball B_R with $u_R\big|_{\partial B_R} = 0$. The error of this approximation is bounded by a function of R which tends to zero as R goes to infinity. The solution u_R can in turn be approximated efficiently by an algorithm depending only on solving the observation-independent Kolmogorov equation on B_R. In theory, our algorithm can solve basically all engineering problems in real time manner. Specifically, we show that the solution obtained from our algorithms converges to the solution of the DMZ equation in L^1-sense. Equally important, we have a precise error estimate of this convergence which is important in numerical computation.

I. Introduction

In 1961, Kalman-Bucy [16] first established the finite dimensional filter for the linear filtering model with Gaussian initial distribution which is highly influential in modern industry. Since then filtering theory has proved useful in Science and Engineering, for example, the navigational and guidance systems, radar tracking, sonar ranging, and satellite and airplane orbit determination. Despite of its usefulness, however, the Kalman-Bucy filter is not perfect. The main weakness is that it is restricted only to the linear dynamical system with Gaussian initial distribution. Therefore there has been tremendous interests in solving the nonlinear filtering problem which involves the estimation of a stochastic process $x = \{x_t\}$ (called the signal or state process) that cannot be observed directly. Information containing x is obtained from observations of a related process $y = \{y_t\}$ (the observation process). The goal of nonlinear filtering is to determine the conditional density $\rho(t, x)$ of x_t given the observation history of $\{y_s \colon 0 \le s \le t\}$. In the late 1960s, Duncan [9], Mortensen [20] and Zakai [29] independently derived the Duncan-Mortensen-Zakai (DMZ) equation for the nonlinear filtering theory which the conditional probability density $\rho(t, x)$ has to satisfy. The central problem of nonlinear filtering theory is to solve the DMZ equation in real time and memoryless way.

In 2000, we [28] proposed a novel algorithm to solve the DMZ equation in real time and memoryless way. Under the assumptions that the drift terms $f_i(x)$ $1 \le i \le n$, and their first and second derivatives, and the observation terms $h_i(x)$, $1 \le i \le m$, and their first derivatives, have linear growth, we showed that the solution obtained from our algorithms converges to the true solution of the DMZ equation. Although the above approach is quite successful, so far it cannot handle the famous cubic sensor in engineering where $f(x) = 0$ and $h(x) = x^3$. It is well known that there is no finite dimensional filter for cubic sensor [27].

The purpose of this paper is to show that under very mild conditions (4.2), (4.5) and (4.13) (which essentially say that the growth of $|h|$ is greater than the growth of $|f|$), the DMZ equation admits a unique nonnegative solution $u \in W_0^{1,1}((0,T) \times \mathbb{R}^n)$ which can be approximated by solutions u_R of the DMZ equation on the ball B_R with $u_R\Big|_{\partial B_R} = 0$. The rate of convergence can be efficiently estimated in L^1 norm. The solution u_R can in turn be approximated efficiently by an algorithm depending only on solving the time independent Kolmogorov equation on B_R. Our algorithm can solve practically all engineering problems including the cubic sensor problem in real time and memoryless way. Specifically we show that the solution obtained from our algorithms converges to the solution of the DMZ equation in L^1 sense. Equally important, we

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have precise error estimate of this convergence which is important in numerical computation.

The splitting up method has been used extensively by many authors. This technique is like the Trotter product formula from semigroup theory. Hopkins and Wong [13] used the Trotter product formula to study nonlinear filtering. The approximation method proposed for the DMZ equation, that of operator splitting, has a history going back to Bensoussan, Glowinski, and Rascanu [3], [4]. More recent articles on operator splitting methods in nonlinear filtering are [12], [21], [14], [15]. Rates of convergence and "true" numerical schemes are developed in [10], [14], and [15]. As pointed out by Bensoussan, Glowinski, and Rascanu [3, section 4.3, p. 1431] the method bears the serious limitation that h must be bounded. The numerics of the Kushner–Stratonovitch equations were studied by many people. Two highly competitive classes of methods are "particle methods" (see, for example, [8] and [6]), in which particles move according to the signal dynamics and are weighted, killed, or duplicated according to their likelihood, and "discrete state" approximations (see, for example, [17] and [22]). These methods work nicely under the assumption that h is bounded (cf. [8, p. 348]).

II. Some Basic Concepts

The filtering problem considered here is based on the signal observation model

$$\begin{cases} dx(t) = f(x(t))\,dt + dv(t), x(0) = x_0 \\ dy(t) = h(x(t))\,dt + dw(t), y(0) = 0 \end{cases} \tag{2.1}$$

in which x, v, y and w are respectively $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^m$ valued processes and v and w have components that are independent, standard Brownian processes. We further assume that f and h are C^∞ smooth vector-valued. We shall refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s) \colon 0 \le s \le t\}$. It is well known that $\rho(t,x)$ is given by normalizing a function, $\sigma(t,x)$, which satisfies the following Duncan-Mortensen-Zakai equation:

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)\,dt + \sum\limits_{i=1}^{n} L_i\sigma(t,x)dy_i(t) \\ \sigma(0,x) = \sigma_0 \end{cases} \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2, \tag{2.3}$$

and for $i = 1, \ldots, m$, L_i is the zero degree differential operator of multiplication by h_i. σ_0 is the probability density of the initial point x_0.

Equation (2.2) is a stochastic partial differential equation. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. Davis [7] studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x). \tag{2.4}$$

It is easy to show that $u(t,x)$ satisfies the following time varying partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = L_0 u(t,x) + \sum\limits_{i=1}^{m} y_i(t)[L_0, L_i]u(t,x) \\ \qquad\qquad + \frac{1}{2}\sum\limits_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]u(t,x) \\ u(0,x) = \sigma_0 \end{cases} \tag{2.5}$$

where $[\cdot,\cdot]$ denotes the Lie bracket. It is shown in [28, p. 236]) that the robust DMZ equation (2.5) is of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + (-f(x) + \nabla K(t,x)) \cdot \nabla u(t,x) \\ \qquad + \Big(-\operatorname{div} f(x) - \frac{1}{2}|h(x)|^2 + \frac{1}{2}\Delta K(t,x) \\ \qquad - f(x)\cdot\nabla K(t,x) + \frac{1}{2}|\nabla K(t,x)|^2\Big)u(t,x) \\ u(0,x) = \sigma_0(x). \end{cases} \tag{2.6}$$

where $K = \sum\limits_{j=1}^{m} y_i(t)h_j(x)$, $f = (f_1, \ldots, f_n)$ and $h = (h_1, \ldots, h_m)$.

III. Real Time Solution of the DMZ equation

To simplify our presentation, we introduce the following condition.

Condition (C$_1$):

$$-\frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2 + |f - \nabla K| \le c_1 \quad \forall\ (t,x) \in [0,T]\times\mathbb{R}^n,$$

where c_1 is a constant possibly depending on T.

Our main theorems are as follows:

Theorem A. *Consider the filtering model* (2.1). *For any $T > 0$, let u be a solution of the robust DMZ equation (2.6) in $[0,T]\times\mathbb{R}^n$. Assume Condition (C_1) is satisfied.*

Then

$$\sup_{0\le t\le T}\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}}u(t,x)\le e^{(c_1+\frac{n+1}{2})T}\cdot\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}}u(0,x). \quad (3.1)$$

In particular,

$$\sup_{0\le t\le T}\int_{|x|\ge R} u(t,x)\le e^{-\sqrt{1+R^2}}e^{(c_1+\frac{n+1}{2})T}\cdot\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}}u(0,x). \quad (3.2)$$

Theorem A above says that one can choose a ball large enough to capture almost all the density. In fact by (3.2) we have precise estimate of density lying outside this ball .

Theorem B. *Consider the filtering model* (2.1). *For any* $T>0$, *let* u *be a solution of the robust DMZ equation (2.6) in* $[0,T]\times\mathbb{R}^n$. *Assume*

(1) *Condition* (C_1) *is satisfied.*
(2) $-\frac{1}{2}|h|^2-\frac{1}{2}\Delta K-f(x)\cdot\nabla K(t,x)+\frac{1}{2}|\nabla K|^2+12+2n+4|f-\nabla K|\le c_2$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^n$, *where* c_2 *is a constant possibly depending on* T.
(3) $e^{-\sqrt{1+|x|^2}}[12+2n+4|f-\nabla K|]\le c_3$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^n$.

Let $R\ge 1$ *and* u_R *be the solution of the following DMZ equation on the ball* B_R

$$\begin{cases}\dfrac{\partial u_R}{\partial t}=\frac{1}{2}\Delta u_R+(-f+\nabla K)\cdot\nabla u_R\\ \qquad +(-\operatorname{div} f-\frac{1}{2}|h|^2+\frac{1}{2}\Delta K\\ \qquad -f\cdot\nabla K+\frac{1}{2}|\nabla K|^2)u_R\\ u_R(t,x)=0 \qquad \text{for } (t,x)\in[0,T]\times\partial B_R\\ u_R(0,x)=\sigma_0(x).\end{cases} \quad (3.3)$$

Let $v=u-u_R$. *Then* $v\ge 0$ *for all* $(t,x)\in[0,T]\times B_R$ *and*

$$\int_{B_R}\phi v(T,x)\le\frac{e^{c_2T}-1}{c_2}c_3e^{-R}e^{(c_1+\frac{n+1}{2})T}\cdot\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}}u(0,x) \quad (3.4)$$

where $\phi(x)=e^{\frac{|x|^4}{R^3}-\frac{2|x|^2}{R}}-e^{-R}$. *In particular*

$$\int_{B_{\frac{R}{2}}} v(T,x)\le\frac{2(e^{c_2T}-1)}{c_2}c_3e^{-\frac{9}{16}R}e^{(c_1+\frac{n+1}{2})T}\cdot\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}}u(0,x). \quad (3.5)$$

Theorem B above says that we can approximate u by u_R. The approximation is good if R is large enough. In fact we have a precise error estimate of this approximation by (3.5).

Theorem C. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *Let* $F\colon[0,T]\times\Omega\to\mathbb{R}^n$ *be a family of vector fields* C^∞ *in* x *and Holder continuous in* t *with exponent* α *and* $J\colon[0,T]\times\Omega\to\mathbb{R}$ *be a* C^∞ *function in* x *and Holder continuous in* t *with exponent* α *such that the following properties are satisfied*

$$|\operatorname{div} F(t,x)|+2|J(t,x)|+|F(t,x)|\le c \quad \textit{for (t,x)}\in[0,T] \quad (3.6)$$

$$|F(t,x)-F(\bar t,x)|+|\operatorname{div} F(t,x)-\operatorname{div} F(\bar t,x)|+\left|J(t,x)-J(\bar t,x)\right|\le c_1|t-\bar t|^\alpha \quad \textit{for (t,x), }(\bar t,x)\in[0,T] \quad (3.7)$$

Let $u(t,x)$ *be the solution on* $[0,T]\times\Omega$ *of the equation*

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x)=\dfrac{1}{2}\Delta u(t,x)+F(t,x)\cdot\nabla u(t,x)\\ \qquad +J(t,x)u(t,x)\\ u(0,x)=\sigma_0(x)\\ u(t,x)\Big|_{\partial\Omega}=0.\end{cases} \quad (3.8)$$

For any $0\le\tau\le T$, *let* $\mathcal{P}_k=\{0=\tau_0<\tau_1<\tau_2<\cdots<\tau_k=\tau\}$ *be a partition of* $[0,\tau]$ *where* $\tau_i=\frac{i\tau}{k}$. *Let* $u_i(t,x)$ *be the solution on* $[\tau_{i-1},\tau_i]\times\Omega$ *of the following equation*

$$\begin{cases}\dfrac{\partial u_i}{\partial t}(t,x)=\dfrac{1}{2}\Delta u_i(t,x)+F(\tau_{i-1},x)\cdot\nabla u_i(t,x)\\ \qquad +J(\tau_{i-1},x)u_i(t,x)\\ u_i(\tau_{i-1},x)=u_{i-1}(\tau_{i-1},x)\\ u_i(t,x)\Big|_{\partial\Omega}=0.\end{cases} \quad (3.9)$$

Here we use the convention $u_0(t,x)=\sigma(x)$. *Then the solution* $u(t,x)$ *of (3.8) can be computed by means of the solution* $u_i(t,x)$ *of (3.9). More specifically,* $u(\tau,x)=\lim_{k\to\infty}u_k(\tau,x)$ *in* L^1*-sense on* Ω *and the following estimate holds*

$$\int_\Omega|u-u_k|(\tau_k,x)\le\frac{2c_2}{\alpha+1}\frac{T^{\alpha+1}e^{cT}}{k^\alpha} \quad (3.10)$$

where

$$c_2=c_1e^{cT}+c_1\sqrt{\text{Vol }(\Omega)}e^{c^2T}\sqrt{2c^2T\int_\Omega u^2(0,x)+\int_\Omega|\nabla u(0,x)|^2}. \quad (3.11)$$

The right hand side of (3.10) goes to zero as $k \to \infty$.

In case (3.8) and (3.9) are DMZ equations, i.e., $F(t,x) = -f(x) + \nabla K$ and $J(t,x) = -\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2$, by a proposition, which is similar to the Proposition 3.1 of [28]. $u_i(\tau_i, x)$ can be computed by $\widetilde{u}_i(\tau_i, x)$ where $\widetilde{u}_i(t,x)$ for $\tau_{i-1} \le t \le \tau_i$ satisfies the following Kolmogorov equation

$$\begin{cases} \dfrac{\partial \widetilde{u}_i}{\partial t}(t,x) = \dfrac{1}{2}\Delta \widetilde{u}_i(t,x) - \displaystyle\sum_{j=1}^{n} f_j(x) \dfrac{\partial \widetilde{u}_i}{\partial x_j}(t,x) \\ \qquad\qquad -(\operatorname{div} f(x) + \frac{1}{2}\sum_{j=1}^{m} h_j^2(x))\widetilde{u}_i(t,x) \\ \widetilde{u}_i(\tau_{i-1}, x) = \exp\left(\displaystyle\sum_{j=1}^{m} \left(y_j(\tau_{i-1}) - y_j(\tau_{i-2})\right) h_j(x)\right) \\ \qquad\qquad \widetilde{u}_{i-1}(\tau_{i-1}, x). \end{cases} \tag{3.12}$$

In fact

$$u_i(\tau_i, x) = \exp\left(-\sum_{j=1}^{m} y_j(\tau_{i-1}) h_j(x)\right) \widetilde{u}_i(\tau_i, x). \tag{3.13}$$

Therefore theoretically to solve the DMZ equation in a real time manner, we only need to compute the following Kolmogorov equation off-line

$$\begin{cases} \dfrac{\partial \widetilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\Delta \widetilde{u}(t,x) - \displaystyle\sum_{j=1}^{n} f_j(x) \dfrac{\partial \widetilde{u}}{\partial x_j}(t,x) \\ \qquad\qquad -(\operatorname{div} f(x) + \dfrac{1}{2}\sum_{j=1}^{m} h_j^2(x))\widetilde{u}(t,x) \\ \widetilde{u}(0,x) = \phi_i(x) \end{cases} \tag{3.14}$$

where $\{\phi_i(x)\}$ is an orthonormal base in $L^2(\mathbb{R}^n)$. The only real time computation here is to express arbitrary initial condition $\phi(x)$ as the linear combination of $\phi_i(x)$. But this can be done by means of parallel computation.

The idea of solving the Kolmogorov equation "off-line" for the elements of an orthogonal basis has a substantial history; see for example [18], and the references therein. In Lototsky, Mikulevicius and Rozovskii [18] approach, they used the Cameron-Martin expansion for the solution of the DMZ equation. Unfortunately to determine the coefficients of the expansion, they need to consider a system of Kolmogorov type equations which is a recursive system. The advantage of our method is that we only need to deal with one Kolmogorov equation.

Theorem D. *Let u_R be the solution of (3.3) the DMZ equation on B_R. Assume that*

(1) *$f(x)$ and $h(x)$ have at most polynomial growth.*

(2) *For any $0 \le t \le T$, there exist positive integer m and positive constants c' and c'' independent of R such that the following two inequalities hold on $\mathbb{R}^n$.*

(a) $\dfrac{m^2}{2}|x|^{2m-2} - \dfrac{m}{2}(m+n-2)|x|^{m-2} - m|x|^{m-2}x \cdot (f - \nabla K) - \dfrac{\Delta K}{2} - \dfrac{1}{2}|h|^2 - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \ge -c'$

(b) $\left|\dfrac{m^2|x|^{2m-2}}{2} - \dfrac{m(m+n-2)}{2}|x|^{m-2} - m|x|^{m-2}(f - \nabla K) \cdot x\right| \le \dfrac{m(m+1)}{2}|x|^{2m-2} + c''$

(3) $-\dfrac{1}{2}|h|^2 - \dfrac{1}{2}\Delta K - \displaystyle\sum_{j=1}^{n} f_j \dfrac{\partial K}{\partial x_j} + \dfrac{1}{2}|\nabla K|^2 \le c_1$ *for all $(t,x) \in [0,T] \times \mathbb{R}^n$ where c_1 is a constant possibly depending on T.*

Then for any $R_0 < R$,

$$\int_{B_{R_0}} (e^{-|x|^m} - e^{-R_0^m}) u_R(T,x) \ge e^{-c'T} \int_{B_{R_0}} (e^{-|x|^m} - e^{-R_0^m}) \sigma_0(x) + \frac{e^{-R_0^m}}{c'}\left(\frac{m(m+1)}{2} R_0^{2m-2} + c''\right)(1 - e^{c'T}) \int_{B_R} \sigma_0(x).$$

In particular, the solution u of the robust DMZ equation on $\mathbb{R}^n$ has the following estimate

$$\int_{\mathbb{R}^n} e^{-|x|^m} u(T,x) \ge e^{-c'T} \int_{\mathbb{R}^n} e^{-|x|^m} \sigma_0(x).$$

In practical nonlinear filtering computation, it is important to know how much density remains within the given ball. Theorem D provides such a lower estimate. In particular, the solution u of the DMZ equation in $\mathbb{R}^n$ obtained by taking $\lim_{R \to \infty} u_R$, where u_R is the solution of the DMZ equation in the ball B_R, is a nontrivial solution.

IV. Existence and Uniqueness of the DMZ Equation

In this section, we give a priori estimation of derivatives of the solution of the DMZ equation up to second order. As a consequence we prove the existence of a weak solution of the DMZ equation. The uniqueness of the weak solution is also shown in this section.

Existence and uniqueness of solutions to the robust DMZ equation (2.6) have been treated by many well-known authors, including Pardoux [23], [24], Chaleyat-Maurel, Michel, and Pardoux [5], Rozovskii [26], Bensoussan [2], Fleming and Mitter [11], Sussmann [27], Michel [19], and Baras, Blankenship, and Hopkins [1]. They all obtained important estimates on the DMZ equation under special conditions. For example, Fleming and Mitter [11] treated the case where f and ∇f are bounded, while Michel [19] analyzed regularity properties of solutions to DMZ equations with bounded f and h. Pardoux's earlier paper [23] treated the case f, h bounded using arguments based on coercivity. It also contains many other interesting ideas. Pardoux [24] has also treated nonlinear filtering problems with unbounded coefficients (f, h have linear growth). Starting with methods somewhat like those used by [25], Baras, Blankenship, and Hopkins also obtained important results on existence, uniqueness, and asymptotic behavior of solutions to a class of DMZ equations with unbounded coefficients. However, they focused on only one spatial dimension and their result cannot cover the linear case. The Sobolev space setup in this section is quite standard in partial differential equations and has been used by many people (see, for example, [23].).

To begin with we need a priori estimation of zero, first and second derivatives of the solution of the robust DMZ equation on $[0,T]\times B_R$.

Theorem E. *Consider the robust DMZ equation (3.3) on $[0,T]\times B_R$, where $B_R=\{x\in\mathbb{R}^n\colon |x|\le R\}$ is a ball of radius R,*

Let $C_1=\max\limits_{0\le t\le T}\left[\sum\limits_{i=1}^m |y_i(t)|^2\right]^{\frac{1}{2}}$ be the smallest constant such that

$$|\nabla K(t,x)|\le C_1|\nabla h(x)| \text{ for } (t,x)\in[0,T]\times B_R, \tag{4.1}$$

where $|\nabla h|^2=\sum\limits_{i=1}^m|\nabla h_i(x)|^2$.

Suppose that there exists a constant $C>0$ such that for any $r\ge 0$

$$\min_{|x|=r}\frac{|h|^2+\operatorname{div} f+C}{\sqrt{|f|^2+|h|^2+\operatorname{div} f+C}+|f|}-C_1\max_{|x|=r}|\nabla h|\ge 0. \tag{4.2}$$

Let $g(x)$ be a positive radial symmetric function on $\mathbb{R}^n$ (i.e., $g=g(r)$ where $r=|x|=(\sum\limits_{i=1}^n x_i^2)^{\frac{1}{2}}$) such that

$$|g'(r)|\le\min_{|x|=r}\frac{|h|^2+\operatorname{div} f+C}{\sqrt{|f|^2+|h|^2+\operatorname{div} f+C}+|f|} -C_1\max_{|x|=r}|\nabla h|. \tag{4.3}$$

Then, for $0\le t\le T$,

$$\int_{B_R} e^{2g}u_R^2(t,x)\le e^{ct}\int_{B_R}e^{2g}\sigma^2(x). \tag{4.4}$$

Theorem F. *Consider the robust DMZ equation (3.3) on $[0,T]\times B_R$, where $B_R=\{x\in\mathbb{R}^n\colon |x|<R\}$ is a ball of radius R. Assume that*

$$\sqrt{\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2+\frac{C}{2}} -\big|f\big|-\big|\nabla K\big|\ge 0, \tag{4.5}$$

where C is the constant in Theorem E. Choose a nonnegative function $\widetilde{g}$ so that

$$|\nabla\widetilde{g}|\le \sqrt{\tfrac{1}{2}|h|^2+\operatorname{div} f-\tfrac{1}{2}\Delta K+f\cdot\nabla K-\tfrac{1}{2}|\nabla K|^2+\tfrac{C}{2}} -\big|f\big|-\big|\nabla K\big| \tag{4.6}$$

and

$$e^{2\widetilde{g}}\left|\nabla\left(\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right)\right|^2 \le e^{2g} \tag{4.7}$$

where g is chosen as in Theorem E. Then

$$\int_{B_R}e^{2\widetilde{g}}|\nabla u_R|^2(T,x)+\frac{1}{2}\int_0^T\int_{B_R}e^{2\widetilde{g}}(\Delta u_R)^2(t,x) \le\int_{B_R}e^{2\widetilde{g}}|\nabla u_R|^2(0,x)+T\int_{B_R}e^{2g}\sigma^2(x). \tag{4.8}$$

Using Theorem E and Theorem F above, we can establish the existence and uniqueness of a weak solution for the DMZ equation.

Definition 1. *We denote $W^1(\mathbb{R}^n)$ the space of functions $\phi(x)$ such that $\phi(x)\in L^2(\mathbb{R}^n)$ and $\dfrac{\partial\phi}{\partial x_i}\in L^2(\mathbb{R}^n)$ for $i=1,\dots,n$ with the scalar product*

$$(\phi_1,\phi_2)_1:=\int_{\mathbb{R}^n}\phi_1(x)\phi_2(x)\,dx+\int_{\mathbb{R}^n}\sum_{i=1}^n\frac{\partial\phi_1}{\partial x_i}\frac{\partial\phi_2}{\partial x_i}\,dx. \tag{4.9}$$

We shall denote by $W^{1,1}(Q)$ the space of functions $v(t,x)$ for which $v(t,x)\in L^2(Q)$, $\dfrac{\partial v(t,x)}{\partial x_i}\in L^2(Q)$ $(i=1,\dots,n)$ and $\dfrac{\partial v(t,x)}{\partial t}\in L^2(Q)$, with the scalar product

$$(v_1,v_2)_{1,1}:=\iint_Q v_1(t,x)v_2(t,x)\,dt\,dx+ \iint_Q\left(\sum_{i=1}^n\frac{\partial v_1}{\partial x_i}\frac{\partial v_2}{\partial x_i}+\frac{\partial v_1}{\partial t}\frac{\partial v_2}{\partial t}\right)dx\,dt \tag{4.10}$$

It is known that $W^1(\mathbb{R}^n)$ and $W^{1,1}(\mathbb{R}^n)$ are complete. The norms in $L^2(Q)$, $W^1(\mathbb{R}^n)$, and $W^{1,1}(Q)$ will be written $\|v\|_0$, $\|v\|_1$ and $\|v\|_{1,1}$ respectively.

Definition 2. *The subspace of $W^1(\mathbb{R}^n)$ consisting of functions that have compact supports in $\mathbb{R}^n$ is written $W_0^1(\mathbb{R}^n)$, and the subspace of $W^{1,1}(Q)$ consisting of functions $v(t,x)$ which have compact supports in $\mathbb{R}^n$ for any t is written $W_0^{1,1}(Q)$.*

Theorem G. *Under the hypothesis of Theorem* F *the robust DMZ equation (2.6) on $[0,T]\times\mathbb{R}^n$ with initial condition $\sigma_0(x)\in W_0^1(\mathbb{R}^n)$ has a weak solution.*

Theorem H. *Let $Q=(0,T)\times\mathbb{R}^n$. Assume that for some $c>0$,*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^n} e^{cr}u^2(t,x)\,dx<\infty \tag{4.11}$$

$$\int_0^T\int_{\mathbb{R}^n} e^{cr}|\nabla u(t,x)|^2\,dx\,dt<\infty \tag{4.12}$$

where $r=\sqrt{x_1^2+\cdots+x_n^2}$. Suppose that there exists a finite number α such that

$$\left|\frac{c}{2}\nabla r+f-\nabla K\right|^2-2\left(\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K\right.$$
$$\left.+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right)\le\alpha. \tag{4.13}$$

Then the weak solution $u(t,x)$ of the robust DMZ *equation on Q is unique.*

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